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$1/f$ noise, distribution of lifetimes, and a pile of sand

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A connection between the distribution of lifetimes and the power spectrum is derived. It is shown that the flow of sand down the slope in the cellular automaton model, considered recently by Bak, Tang, and Wiesenfeld [Phys. Rev. Lett. **59**, 381 (1987)], has a $1/f^2$ power spectrum in one and two dimensions. The flow over the rim of the system behaves similar to the transport in a real sand pile as measured by Jaeger, Liu, and Nagel [Phys. Rev. Lett. **62**, 40 (1989)].

Recently Bak, Tang, and Wiesenfeld[1] introduced a new very appealing concept which they denoted *self-organized criticality*. Some dissipative many-body systems are believed to evolve into a self-organized critical state with no characteristic time or length scales. It is expected that due to scaling of certain lifetime distribution functions the dynamics of the critical state exhibits $1/f$ power spectra. Bak, Tang, and Wiesenfeld used a cellular automaton to illustrate the basic idea of self-organized criticality. This simple numerical model was believed to describe qualitatively the flow of avalanches in a pile of sand. Hence, the applicability of the concept of self-organized criticality became somewhat unclear[2,3] when Jaeger, Liu, and Nagel[4] reported an experiment on the flow of a sand pile in which they did not find power-law distributions or $1/f$ noise.

In this Rapid Communication we give a detailed discussion of the connection between the distribution of a set of random signals and the power spectrum of the total signal obtained by linear superposition of the elementary signals. We find that the so-called *distribution of weighted lifetimes* used by Bak, Tang, and Wiesenfeld is irrelevant for the power spectrum of the sand flow and should be replaced by another distribution function, given below, which has no simple power-law behavior.

We further note that it is important to distinguish between the flow of sand down the slope of the sand pile and flow of sand over the rim of the system. The first signal has a Lorentzian power spectrum in the model of Bak, Tang, and Wiesenfeld. This signal was not considered in the experiment by Jaeger, Liu, and Nagel.[4] We demonstrate that the flow over the rim in the numerical model has a power spectrum similar to the one measured in the sand-pile experiment.[4]

Lifetimes and power spectra. Consider a set of uncorrelated time signals (labeled by the index α) with time profiles given by $f_\alpha(t)$. Let $F(t)$ be the signal produced by a superposition of the signals $f_\alpha(t)$ started at random times with a total rate ν. The probability that a signal f_α is started in a time interval dt is constant and given by $P(\alpha)dt$. Using the notation from Ref. 5 we can write

$$F(t)=\sum_\alpha \sum_{t_r=-\infty}^{t_r=t} f_\alpha(t-t_r)p_r^\alpha , \qquad (1)$$

where p_r^α is 1 or 0, depending on whether an elementary signal $f_\alpha(t)$ was started in the time interval $[t_r,t_r+\delta]$. By a straightforward generalization of Campbell's theorem[5] we obtain the autocorrelation function Ψ_F of the total signal $F(t)$

$$\Psi_F(\tau)=\nu\int d\alpha P(\alpha)I_\alpha(\tau) ,$$

where (2)

$$I_\alpha(\tau)=\int_0^\infty dt f_\alpha(t)f_\alpha(t+|\tau|) .$$

In the case of the sand pile $f_\alpha(t)$ is the amount of sand sliding at time t in an avalanche labeled by α. The precise profile $f_\alpha(t)$ differs from avalanche to avalanche but is well approximated by a box function

$$f_{S,T}(t)=\begin{cases} S/T & \text{if } 0<t<T , \\ 0 & \text{otherwise} , \end{cases} \qquad (3)$$

where S is the total time-integrated amount of sliding during the lifetime T of the avalanche. This form of $f_{S,T}(t)$ leads through Eq. (2) to an autocorrelation function

$$\Psi_F(\tau)=\nu\int_{|\tau|}^\infty dT\int_0^\infty dS P(S,T)\left[\frac{S}{T}\right]^2(T-|\tau|) , \qquad (4)$$

where $P(S,T)$ is the joint probability distribution. Partial integration of the Wiener-Khintchine theorem[5]

$$S(f)=2\int_0^\infty d\tau\Psi_F(\tau)\cos(2\pi f\tau) \qquad (5)$$

gives the following expression for the power spectrum

$$S(f)=\frac{\nu}{(\pi f)^2}\int_0^\infty d\tau G(\tau)\sin^2(\pi f\tau) , \qquad (6)$$

where

$$G(T)=\frac{1}{T^2}\int_0^\infty dS P(S,T)S^2 . \qquad (7)$$

A form like $G(T)\propto T^a\exp(-T/T_0)$ in an interval $T\in[t_0,\infty]$ will in the region $f\in[1/T_0,1/t_0]$ give a power spectrum

$$S(f)\propto\begin{cases} f^{-(3+a)} & \text{when } a<-1 , \\ f^{-2} & \text{when } a>-1 . \end{cases} \qquad (8)$$

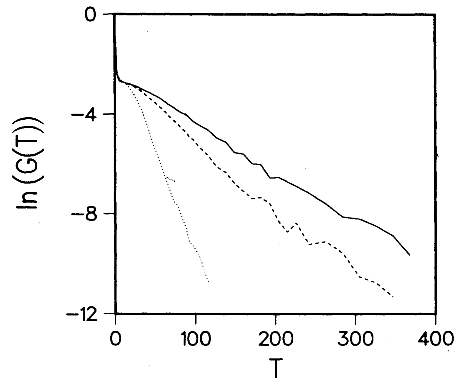

FIG. 1. Semilog plot of the distribution function $G(T)$ defined in Eq. (7) for three different system sizes: dotted 20×20, dashed 50×50, and solid 75×75.

For $f < 1/T_0$ the power spectrum becomes constant and $S(f)$ falls off like $1/f^2$ for $f > 1/t_0$.

Cellular automaton. Let us now analyze the cellular automaton used by Bak, Tang, and Wiesenfeld. To be specific we consider the open-boundary model where "sand" can leave the system over the right edge.[6] We first consider flow down the slope of the pile, i.e., $F(t)$ is the total number of slides at time t. In one dimension the model can easily be solved analytically. The distribution $G(T)$ is constant ($=1/L$, where L is the size of the system) for $T \in [0,L]$ and zero outside this interval. Hence, sand randomly (white noise) added on to the system will give rise to a $1/f^2$ power spectrum of the flow down the slope.[7] In the two- and three-dimensional case we have to measure $P(S,T)$, and calculate the weighted distribution $G(T)$ from Eq. (7). We show in Fig. 1 $G(T)$ for two-dimensional systems of different sizes. This semilog plot shows that $G(T)$ approximately decreases exponentially with $T_0 \approx 12$, 39, and 51 for size 20×20, 50×50, and 75×75, respectively; indicating a linear increase in T_0 with the linear size of the system. A pure exponential form of $G(T)$ will, according to Eq. (6), produce a Lorentzian power spectrum $S(f) = 2vT_0^3/[1+(2\pi T_0 f)^2]$.

Figure 2(a) shows the power spectrum of $F(t)$, where $F(t)$ is obtained according to Eq. (1), by linear superimposing signals from individual avalanches started at random times. Figure 2(b) shows the spectrum obtained by inserting the measured $G(T)$ into Eq. (6). The small de-

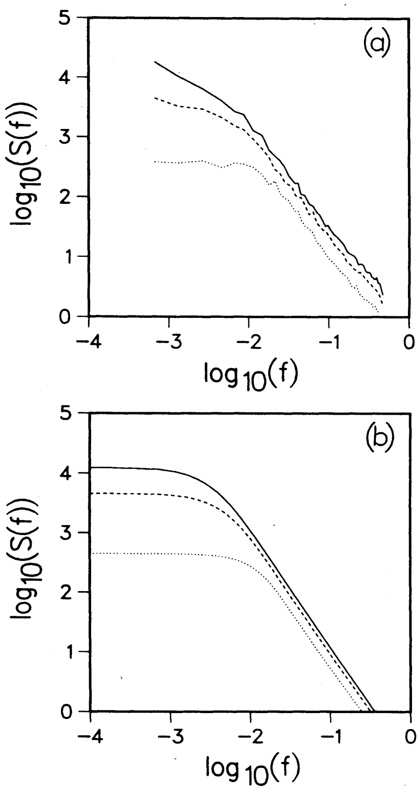

FIG. 2. (a) Power spectrum of the flow down the slope. Individual avalanche signals started at random times are superimposed linearly and the resulting time sequence is Fourier transformed. (b) Power spectrum obtained by inserting the measured $G(T)$ from Fig. 1 into Eq. (6). The rate v is equal to 0.5. The dotted, the dashed, and the solid curves are for system size 20×20, 50×50, and 75×75, respectively.

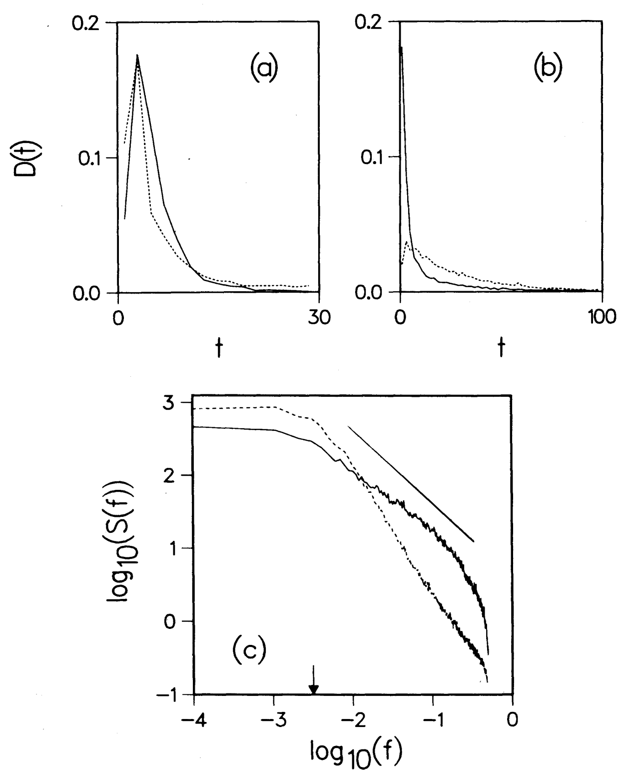

FIG. 3. Distribution of the intervals Δt between avalanches (dashed line) and the duration of avalanches τ (solid line) for the flow (a) over the rim and (b) down the slope, in a continuously driven automaton. The probability for adding sand at a site is 0.05 per time step. The associated power spectra are shown in (c). Solid line corresponds to flow over the rim (the data are multiplied by 10^2), dashed line to flow down the slope. The straight line indicates $1/f$ behavior. The arrow locates the frequency $2\pi/51$—the characteristic frequency in the Lorentzian corresponding to $G(T)$ in Fig. 1.

viation between the two figures is probably due to numerical inaccuracies in using the discrete Fourier transform. The slight enhancement of $S(f)$ in Fig. 2 at high frequencies is due to aliasing.

Flow over the rim. As our final topic we demonstrate that the power spectrum of the flow over the rim in the numerical model under consideration behaves qualitatively similar to the flow measured in a real sand pile.[4] Instead of linearly superimposing individual avalanche signals we simply continuously add sand randomly (in time and space) at a constant rate and let the model evolve according to the updating algorithm. The flow down the slope as well as the flow over the rim is analyzed. Once in a while the addition of sand leads to slides. The time interval between two pulses, Δt, and the duration of these pulses, τ, are monitored. We show in Fig. 3(a) the distributions of the two quantities for the flow over the rim and in Fig. 3(b) the same distributions for the flow down the slope.

The power spectra of the flow signals are shown in Fig. 3(c). The spectrum for the flow down the slope is identical to those obtained above. The spectrum and distribution of Δt and τ for the flow over the rim should be compared to the corresponding quantities measured in the experiment by Jaeger, Liu, and Nagel.[4] One notices that the distribution of Δt and τ in both cases are narrow. The power spectrum of the numerical model is qualitatively similar to the one measured experimentally when a vibration is applied to the system.

In conclusion, we have shown that the power spectrum of the cellular automaton model, discussed by Bak, Tang, and Wiesenfeld,[1] behaves as $1/f^2$ in one and two dimensions. Furthermore, we have discussed the difference between flow down the slope of a sand pile and flow over the rim of the system. Comparison of the second signal can be made between the numerical model and the experiment of Jaeger, Liu, and Nagel.[4] One finds qualitative agreement. As far as we know, the power spectrum of the flow down the slope of a sand pile has not been measured yet. Hence, it is still an open question to what extent the power-law behavior predicted from the cellular automaton model exists in real sand piles.

[1](a) P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); (b) Phys. Rev. A **38**, 364 (1989); (c) K. Wiesenfeld, C. Tang, and P. Bak, J. Stat. Phys. **54**, 1441 (1989).

[2]P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **62**, 110 (1989).

[3]A self-organized lattice-gas model with 1/*f* power spectra and close connection to systems such as flux flow, highway traffic, etc., is discussed by H. J. Jensen (to be published).

[4]H. M. Jaeger, Chu-heng Liu, and Sidney R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).

[5]D. K. C. MacDonald, *Noise and Fluctuations: An Introduction* (Wiley, New York, 1962).

[6]The model is defined in Ref. 1(b). Sand is "added" by use of Eq. (3.1) in this paper. Updating is done by Eq. (3.2) and the boundary conditions are Eq. (3.12) and $z(0,j)=z(i,0)=0$. The ratio between the number of additions and the number of resulting excitations on the open rim sites is on average equal to $\frac{3}{2}$. Balance between in and out flow is obtained if we interpret the addition rule as representing the addition of two grains of sand to the system and the rule for updating the open rim sites as representing $\frac{4}{3}$ grains leaving the system. This somewhat artificial bookkeeping has its origin in the use of a "scalar slope" Ref. 1, a more realistic description should make use of a real gradient [Leo P. Kadanoff, Sidney R. Nagel, Lei Wu, and Su-min Zhou, Phys. Rev. A **39**, 6524 (1989)].

[7]In Ref. 1(b) it is stated that the resulting sand flow has a $1/f^0$ spectrum. This is the spectrum of the flow over the *rim* of the one-dimensional system.