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Vacuum degeneracy of chiral spin states in compactified space

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A chiral spin state is not only characterized by the T and P order parameter $E_{123} = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$, it is also characterized by an integer k. In this paper we show that this integer k can be determined from the vacuum degeneracy of the chiral spin state on compactified spaces. On a Riemann surface with genus g the vacuum degeneracy of the chiral spin state is found to be $2k^g$. Among those vacuum states, some k^g states have $\langle E_{123} \rangle > 0$, while other k^g states have $\langle E_{123} \rangle < 0$. The dependence of the vacuum degeneracy on the topology of the space reflects some sort of topological ordering in the chiral spin state. In general, the topological ordering in a system is classified by topological theories.

Recently, Witten[1] (also see Ref. 2) studied the quantization of many topological theories, including the quantization of the non-Abelian Chern-Simons theory in $1+2$ dimensions described by

$$L = \int d^3x \frac{k}{8\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\lambda - \tfrac{1}{3} A_\mu A_\nu A_\lambda)\epsilon^{\mu\nu\lambda} . \quad (1)$$

It is found that the Hilbert space of (1) has finite dimensions and the number of dimensions depends on the topology of the (compactified) two-dimensional space.

In some studies of the high-T_c superconductors, it is shown that the vacuum of the frustrated-spin model may be a chiral spin state.[3,4] The fluctuations around a chiral spin state are described by the following low-energy effective action:[4]

$$L_{\text{eff}} = \int d^3x \left[\frac{k}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} + \frac{1}{g^2}(f_{\mu\nu})^2 \right] + \cdots , \quad (2)$$

where the ellipsis represents higher-order terms and a_μ is the dynamically generated $U(1)$ gauge field.[5] Note the charge coupled to a_μ is chosen to be one. Equation (2) should be regarded as a compact $U(1)$ gauge theory.

The effective theory (2) is not a topological theory, but when g^2 is large, all local excitations have gaps of order g^2. At energy scales much lower than g^2, only the global excitations are allowed and the model is described by the topological theory

$$L_{\text{top}} = \frac{1}{4\pi} \int d^3x \, k a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} . \quad (3)$$

Using renormalization-group language, we may say (3) is the infrared fix point of (2).

Because the topological theory (3) contains only linear time derivative terms, the Hamiltonian is identically zero, and all the quantum states of (3) have zero energy. Note that (3) is a scaleless theory containing no dimensional parameters. The quantum states cannot have nonzero energies. Therefore, the number of dimensions of the Hilbert space of (3) is equal to the number of the vacuum degeneracy of the model (2) (defined on compactified two-dimensional space). We may use the method in the topological theory to calculate the vacuum degeneracy of systems described by (2).

The studies in Ref. 4 suggest that the frustrated Heisenberg model may support T- and P-symmetry-breaking vacua—chiral spin states. The T and P order parameter is given by a three-spin operator

$$E_{123} = (\mathbf{S}_1 \times \mathbf{S}_2) \cdot \mathbf{S}_3 . \quad (4)$$

Under T and P, $E_{123} \rightarrow -E_{123}$. The T- and P-symmetry-breaking properties of chiral spin states are characterized by the nonzero vacuum expectation value (VEV) of E_{123}. Furthermore, a mean-field study of chiral spin states suggest that chiral spin (liquid) states are not only characterized by the nonzero VEV of E_{123}, but also characterized by an integer (Fig. 1). The integer is nothing but the integer k appearing in front of the Chern-Simons term in the effective action[4] of chiral spin states. Now the question is whether there is a direct and a physical way to measure the integer k which characterized a chiral spin state. In the following we will show that the integer k can be measured directly by measuring the vacuum degeneracy of a chiral spin state on a compactified space such as a torus. We find that on a torus the vacuum degeneracy of a chiral spin state is equal to $2|k|$ if $k \neq 0$. Thus a chiral spin state is characterized both by the T and P order parameter E_{123} and by its vacuum degeneracy on a torus.

Since the low-energy excitations of chiral spin states are described by (3), to calculated the vacuum degeneracy of the chiral spin states on a torus, we only need to calculate the vacuum degeneracy of (3) on the same torus.

Now let us first quantize (3). Following Refs. 1 and 2 we may quantize (3) in the gauge[6]

$$a_0 = 0 .$$

The equation of motion for a_0 still needs to be taken into

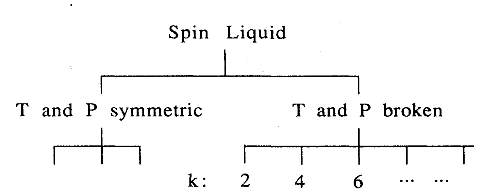

FIG. 1. A classification of spin-liquid states.

account and serves as a constraint. That is,

$$\frac{\delta}{\delta a_0} L_{\text{top}} = \epsilon^{0ij} f_{ij} = 0 \,, \tag{5}$$

which means f_{12} vanishes on the torus. The gauge potential satisfying (5) is parametrized by two real parameters corresponding to the constant gauge potential:

$$a_1 = a_1(t) \,, \quad a_2 = a_2(t) \,. \tag{6}$$

Due to residual (time-independent) gauge symmetries, different a_i may correspond to gauge-equivalent configurations. Actually, if the torus (denoted at T_L) is given by a rectangle $L_1 \times L_2$ with periodic boundary conditions, then the following two gauge configurations a_i and a_i' are gauge equivalent:

$$(a_1,a_2) \sim (a_1',a_2') = \left[a_1 + \frac{2\pi n}{L_1}, a_2 + \frac{2\pi m}{L_2} \right] , \tag{7}$$

where n,m are integers. The gauge transformation that relates a_i and a_i' ($a_i' = a_i - iU^{-1}\partial_i U$) is given by

$$U - \exp\left[2\pi i \left(\frac{nx^1}{L_1} + \frac{mx^2}{L_2} \right) \right] .$$

Note that U is single valued on the torus when and only when m and n are integers. Thus, the gauge inequivalent configurations are given by a point on a torus (denoted as T_a) of size $2\pi/L_1 \times 2\pi/L_2$. The dynamics of a_i are described by the Lagrangian (taking $a_0 = 0$),

$$L = k\frac{\pi}{A} \int dt (\dot{a}_1 a_2 - \dot{a}_2 a_1) \,, \tag{8}$$

where A is the area of the torus T_a, $A = 4\pi^2/L_1 L_2$. Equation (8) is obtained by substituting (6) into (3).

To quantize (8), it is convenient to introduce (x,y) such that

$$(a_1,a_2) = \left[\frac{2\pi x}{L_1}, \frac{2\pi y}{L_2} \right] . \tag{9}$$

Thus (x,y) and $(x+n,y+m)$ are equivalent points and (x,y) parametrize a torus. Using the new variables, (8) becomes

$$L = k\pi \int dt (\dot{x} y - \dot{y} x) \,. \tag{10}$$

It is also convenient to add a small mass term to (10) and to write it as

$$L = \int dt \left[k\pi(\dot{x}y - \dot{y}x) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2) \right] . \tag{11}$$

Later we will let m go to zero. If we start with (2), the mass term is actually generated by the Maxwell term with $m \propto g^{-2}$.

The Lagrangian (11) describes a unit charged particle moving on a torus parametrized by (x,y). The first term in (11) indicates that there is a uniform "magnetic" field $B = 2\pi k$ on the torus. The appearance of the magnetic field indicates that an Abelian gauge structure is induced in the gauge configuration space.[7] The total flux passing through the torus is equal to $2\pi k$. The Hamiltonian of (11) is given by

$$H = \frac{1}{2m} [-(\partial_x - iA_x)^2 - (\partial_y - iA_y)^2] \,. \tag{12}$$

This Hamiltonian has been studied in detail by Haldane and Rezayi.[8] Choosing the gauge

$$A_x = 0 \,, \quad A_y = Bx = 2\pi k x \,, \tag{13}$$

it is found that the ground state of H in Eq. (11) is k-fold degenerate. The wave function of the ground states is given by

$$\psi_l(x,y) = \left[\sum_n \exp\left[2\pi(x+iy)(nk+l) - \frac{(nk+l)^2}{k}\pi \right] \right] e^{-(1/2)Bx^2} , \tag{14}$$

where $l = 0,1,\ldots,k-1$. The expression within the square brackets is a Θ function. The wave functions satisfy the boundary conditions

$$\psi(x+1,y) = e^{i2\pi k y}\psi(x,y) \,, \quad \psi(x,y+1) = \psi(x,y) \,, \tag{15}$$

which is consistent with the gauge choice (13). All the excited states have energies of order $1/m$ and can be ignored in the $m \to 0$ limit.

The k-fold-degenerate ground states that we find for the Hamiltonian (12) correspond to k ground states of the chiral spin state. $\langle E_{123} \rangle$ have the same sign in these k ground states, say $\langle E_{123} \rangle > 0$. There are other k-fold-degenerate ground states with $\langle E_{123} \rangle < 0$. Thus the total degeneracy of the ground states of the chiral spin state is $2k$. In other words, the chiral spin states have twofold degenerate vacua on uncompactified space, one with $\langle E_{123} \rangle > 0$, another with $\langle E_{123} \rangle < 0$. The low-energy effective action for each vacuum is the topological gauge theory (3) with coefficient $\pm k$. When we compactify the space into, say, a torus, each vacuum in the uncompactified space generates k-fold-degenerate vacua.

The chiral spin state studied in Ref. 4 has flux π per plaquette. The integer k is found to be equal to two. Thus, the π-flux chiral spin state has fourfold degenerate vacua on a torus. In general, one may have a chiral spin state with flux $2\pi(p/q)$ per plaquette, where q is an even integer. In this case,[4] k is found to be equal to q and the vacuum of such a chiral spin state has $2q$-fold degeneracy on a torus.

Haldane[9] has studied the chiral spin state on a torus using a generalized Laughlin wave function for spin-$\frac{1}{2}$ electrons and using the Θ function method. He found that the chiral spin state studied by Kalmeyer and Laughlin[3] has twofold degeneracy on a torus. The same result has also been obtained by Laughlin.[10] This result agrees with the result in Ref. 4 and in this paper that the Kalmeyer-Laughlin state is a $k=2$ chiral spin state. In order to obtain more general chiral spin states, one needs to use wave functions involving higher Landau levels or hierarchial fractional-quantum-Hall-effect (FQHE) wave functions. The degeneracy of the chiral spin states is closely related to the degeneracy of the FQHE states on a torus, as studied by Haldane and Rezayi.[8]

The vacuum degeneracy of general two-dimensional

(2D) spin-$\frac{1}{2}$ systems has been studied in Ref. 11. Affleck[11] gives an argument that a 2D spin-$\frac{1}{2}$ system must have either vacuum degeneracy or gapless excitations. Haldane,[11] using the nonlinear σ-model, further demonstrates that a 2D gap full spin-liquid state of half-odd-integer spins should have fourfold-degenerate ground states, which agrees with our result obtained here. However, as we have seen from the previous discussions in this paper, the vacuum degeneracy may come from the global excitations. In this case, some degenerate ground states appear to be "accidental" and have nothing to do with the broken symmetries. As we will see later, the global excitations, as well as the vacuum degeneracy, depend on boundary conditions and topology of the two-dimensional lattices. The vacuum degeneracy discussed in Ref. 11 does not always imply broken discrete symmetries. Sometimes it may imply the existence of some topological order in the spin-liquid states.

We may compactify the space into a Riemann surface with higher genus as well. Again, each vacuum in the uncompactified space generated degenerate vacua in the compactified space. The degeneracy is different than that on a torus. To calculate the vacuum degeneracy of the chiral spin state on a Riemann surface Σ_g with genus g, we first need to parametrize all the gauge configurations satisfying $f_{12}=0$. Let A_a, B_a $(a=1,\ldots,g)$ be the canonical one cycles on Σ_g (Fig. 2), and w_a, η_a be the closed one forms on Σ_g. One can choose w_a and η_a such that[12]

$$\int_{A_a} w_b = \delta_{ab}\,,\quad \int_{B_a} w_b = 0$$
$$\int_{A_a} \eta_b = 0\,,\quad \int_{B_a} \eta_b = \delta_{ab}\,. \tag{16}$$

The flat gauge connection (satisfying $f_{12}=0$) is given by a closed one form which can be written as

$$a = a_1 dx^1 + a_2 dx^2 = 2\pi(x^a w_a + y^a \eta_a)\,. \tag{17}$$

Thus, the flat gauge connection is parametrized by $2g$ real parameters, x^a and y^a. Two gauge connections a and a' are gauge equivalent if

$$x'^a = x^a = \text{integer}\,,\quad y'^a - y^a = \text{integer}\,. \tag{18}$$

Using the relation

$$\int_{\Sigma_g} w_a \wedge \eta_b l = \delta_{ab}\,,\quad \int_{\Sigma_g} w_a \wedge w_b = \int_{\Sigma_g} \eta_a \wedge \eta_b = 0\,, \tag{19}$$

and substituting (17) into (3), we find that the Chern-Simons Lagrangian reduces to

$$L = \int dt \left[k\pi(\dot{x}^a y^a - \dot{y}^a x^1) + \frac{m}{2}\left[(\dot{x}^a)^2 + (\dot{y}^a)^2 \right] \right]\,, \tag{20}$$

where a small mass regulation term is included. Equation (20) is just g copies of the system described by (10). Thus, the ground states of (20) are k^g-fold degenerate. We conclude that the chiral spin state on Σ_g has k^g vacua with $\langle E_{123}\rangle > 0$ and k^g vacua with $\langle E_{123}\rangle < 0$.

In the above, we have calculated the vacuum wave functions of a chiral spin state in terms of the effective gauge potential. One may ask what are the ground-state wave functions in terms of the original spin variable? Formally, one may write the ground-state wave functions

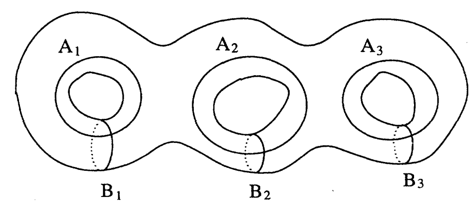

FIG. 2. The canonical one cycles A_a and B_a on Riemann surface Σ_g $(g=3)$.

as

$$|\Phi_l\rangle = \int da_1\, da_2\, \psi_l(a_1,a_2)\, |a_1,a_2\rangle\,, \tag{21}$$

where ψ_l is given in (14) (assuming the space is a torus) and $|a_1,a_2\rangle$ is the spin state corresponding to the effective gauge potential (a_1,a_2). Motivated by the mean-field approach of chiral spin states,[4] the spin state $|a_1,a_2\rangle$ may be constructed in the following way. Consider the following mean-field Hamiltonian defined on a torus:

$$H = \sum \chi_{ij} c_j^\dagger c_i \exp i \int_i^j \mathbf{a}\cdot d\mathbf{x}\,. \tag{22}$$

H contains both next-neighbor and second-neighbor hopping terms. χ_{ij} in (22) characterizes the mean-field chiral spin state and is given in Ref. 4. $\mathbf{a}=(a_1,a_2)$ is the constant gauge potential corresponding to the quantum fluctuations around the mean-field vacuum. H in (22) can be diagonalized and we can obtain the N-electron ground state $|a_1,a_2\rangle_{\text{mean}}$ (N is the number of the lattice sites). The spin state $|a_1,a_2\rangle$ can be obtained by doing Gutzwiller projection on the mean-field state $|a_1,a_2\rangle_{\text{mean}}$,

$$|a_1,a_2\rangle = P_G\, |a_1,a_2\rangle_{\text{mean}}\,. \tag{23}$$

We stress that the above construction of the ground-state wave function is only an approximate construction, in the sense that the wave functions $|\Phi_l\rangle$ obtained are not the exact ground states of simple Hamiltonians, e.g., the frustrated Heisenberg model. The same thing is true for the Laughlin's wave function of FQHE. However, the wave functions we constructed are expected to contain correct topological structure and given rise to correct quantum numbers for the quasiparticle excitations. Although we are unable to write down the exact ground-state wave functions, the vacuum degeneracy discussed above is strictly correct.

We would like to remark that the vacuum degeneracy of chiral spin states discussed in this paper is exact only in the thermodynamic (large volume) limit. On a finite lattice the would-be vacuum states have small energy differences which vanish when the lattice size goes to infinity. Our effective theory for chiral spin states (2) is exact only in an infinity long-wavelength limit. Thus, in general, one expects the finite-size effects to lift the degeneracy of the vacuum states.

There is another subtlety related to the lattice. In order to use computers to test the vacuum degeneracy, one must perform calculations on so-called unfrustrated lattices. Naively speaking, the ground state on the unfrustrated lattice represents the true vacuum of the Hamiltonian under consideration. The ground state on the frustrated lattice may contain some topological excitations such as domain wall, soliton, etc. For example, consider an antiferromagnetic Ising model defined on an $N_1 \times N_2$ lattice with a periodic boundary condition. If one of N_i is odd, the lattice is frustrated. The ground states contain a domain wall and they have a large degeneracy corresponding to different positions of the domain wall. Only in the unfrustrated lattices (with both N_i even) do the ground states have twofold degeneracy, which corresponds to the true vacuum degeneracy of the Ising model. However, without any knowledge of the vacuum property of a system, how does one know which lattice is frustrated and which is unfrustrated? Here we propose a model-independent definition of an unfrustrated lattice. We first pick a small energy $\epsilon > 0$ which is much less than the typical energy scale of the system under consideration and pick a large integer I. We define a lattice as unfrustrated when the number of the states with energy less than ϵ (measured from the ground state) is less than I. When ϵ is less than the energy gap and I is larger than the vacuum degeneracy of the system, the above definition unambiguously defines the unfrustrated lattice in the large lattice size limit. Let I_N be the number of states with energy less than ϵ in an unfrustrated lattice of N sites. For small enough ϵ and large enough I, we have three different cases: (i) There is no unfrustrated lattice with size $N > N_0(\epsilon, I)$, where $N_0(\epsilon, I)$ is a function of ϵ and I. This means that the system has gapless quasiparticle excitation. (ii) The unfrustrated lattices exist for arbitrary large N and $I_N \rightarrow I_\infty$ as $N \rightarrow \infty$. If ϵ is small enough, all the I_∞ states below ϵ have zero energy. We may say the system has I_∞-fold degenerate vacua. This can be regarded as a definition of the vacuum degeneracy of a lattice system. (iii) Unfrustrated lattices exist for arbitrary large N but the limit $I_N |_{N \rightarrow \infty}$ does not exist. For example, I_N may alternatively keep taking several different values as $N \rightarrow \infty$. In this case, the vacuum property of the system has a strong dependence on lattice size. The system may not have a well-defined continuum limit.

The main purpose of this paper is to address the question of characterization of chiral spin states. A chiral spin state is not only characterized by its T and P breaking property, but also characterized by an integer which can be determined from the vacuum degeneracy of the chiral spin state on Riemann surfaces. The integer measures the strength of circulation of spins in the chiral spin state. We stress that the vacuum degeneracy studied in this paper is not a consequence of symmetries. The appearance of the additional vacuum degeneracy and the dependence of the vacuum degeneracy on the topology of the compactified spaces suggest that chiral spin states contain nontrivial topological structures which we may call the topological order in chiral spin states. Measuring the vacuum degeneracy for different spaces is one of the simplest ways to probe the topological order in a system. A more complete characterization of the topological order in chiral spin states will appear elsewhere.[13]

Generally speaking, sometimes the vacuum states of a system are not completely characterized by order parameters. The vacuum states may have additional topological ordering. If all quasiparticle excitations above the vacuum states have finite energy gaps, from the above example we see that the topological order in the vacua is classified by various topological theories. In other words, the infrared effective theory may be trivial even when all quasiparticles have finite energy gaps. The effective Lagrangian of the system may flow to a topological theory at low energies which support nontrivial global excitations. It would be interesting to see whether various topological theories can be realized as the low-energy theories for different condensed-matter systems.

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