Memory effect of waves in disordered systems: A real-space approach

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(Received 23 November 1988)

We present a real-space treatment of the memory effect which was recently proposed by Feng *et al.* [Phys. Rev. Lett. **61**, 834 (1988)] for propagation of waves through a disordered system. The real-space approach provides physical insight for the effect. We extend the theory to include absorption and show that the memory effect is enhanced. The role of boundary conditions on the memory effect is also discussed.

Correlation effects of optical waves through random media have received much attention recently.[1–13] In particular, the reflected- and transmitted-intensity correlation functions for static[1–4] and dynamic[5–12] disorder have been the subject of many recent studies. Very recently Feng *et al.*[13] predicted a surprising correlation effect, which they called the "memory effect," for wave transmission through static disordered media. This effect correlates between the change in the incident direction of the wave and the resulting change in direction of the transmitted wave through a slab. The memory effect has recently been confirmed experimentally by Freund, Rosenbluh, and Feng.[14]

In this paper, we provide a real-space treatment of the memory effect which reveals more explicitly its physical origin. Our results for the transmitted and reflected correlation functions for the memory effect coincide with the results obtained diagrammatically by Feng *et al.*[13] by use of the so-called factorized diagrams. Moreover, we extend the memory effect to include the effect of absorption and find that the memory effect is *enhanced* by increasing absorption. Namely, the "memory" correlation function falls off more *slowly* as the degree of absorption increases. We have also investigated the effect of different boundary conditions on the memory effect. When a continuous injection of photons is properly taken into account, we find that the backscattered memory correlation function falls off much more rapidly whereas the transmitted correlation function falls off more slowly.

Real-space formulation for dynamics[5–10] and static[4] correlation functions were shown[11] to correspond to the diagrammatic approaches which rely on the factorization approximation in which the intensity-intensity correlation function is given[1,2] by the square of the electric-field–electric-field correlation functions. The real-space theories not only describe accurately the diagrammatic and experimental correlation functions in the weak-disorder limit but also lead to an insight into the physical origins of these correlations. Recent numerical simulations[15] confirmed the validity of the factorization approximation in the weak-disorder limit.

We here apply the real-space theory to study the memory effect. In Fig. 1 we show schematically this correlation effect as predicted by Feng *et al.*[13] Here $(\mathbf{q}_a,\mathbf{q}_{a'})$ are the transverse incident wave vectors and $(\mathbf{q}_b,\mathbf{q}_{b'})$ are the corresponding transmitted ones. We seek for the intensity-intensity correlation function $\langle \delta I(\mathbf{q}_a,\mathbf{q}_b)\delta I(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle$, where $\langle \cdots \rangle$ denotes an ensemble average over the uniform distribution of the scatterer position. Thus, if $\mathbf{q}_a$ is changed to $\mathbf{q}_{a'}$ the question in hand is to what extent the transmitted intensity at $\mathbf{q}_b$ correlates with the transmitted intensity at angle $\mathbf{q}_{b'}$. In the diagrammatic-factorization approximation, this is given by

$$\langle \delta I(\mathbf{q}_a,\mathbf{q}_b)\delta I(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle = |\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle|^2 . \quad (1)$$

In the real-space representation, a wave incident at point l with an initial value of the electric field E_0 is transmitted at point n after undergoing multiple elastic scattering with N random steps. The wave acquires a random phase $\phi_{n,l,N}$. The amplitude of the wave at point n is $P_{n,l,N}$ and $P^2_{n,l,N}=W_{n,l,N}$ is its random-walk probability,[16-18] which is given below. In this representa-

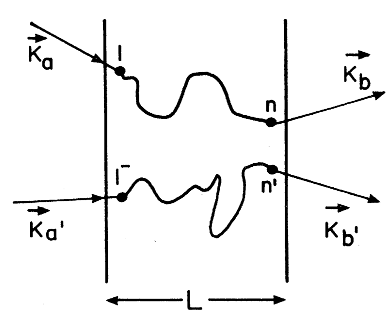

FIG. 1. Two photon trajectories which after summing and averaging as described in Eq. (2) describe the electric-field–electric-field correlation function.

tion, the electric-field–electric-field correlation function is given by

$$\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle = E_0^2 \sum_{\substack{N,l,n,\\ N',l',n'}} \langle P_{n,l,N}P_{n',l',N'}\rangle \langle \exp(-i\mathbf{q}_a\cdot\mathbf{R}_l + i\mathbf{q}_{a'}\cdot\mathbf{R}_{l'} + i\mathbf{q}_b\cdot\mathbf{R}_n - i\mathbf{q}_{b'}\cdot\mathbf{R}_{n'})\rangle \langle \exp[i(\phi_{n,l,N}-\phi_{n',l',N'})]\rangle . \quad (2)$$

Since $\phi_{n,l,N}$ and $\phi_{n',l',N'}$ are random variables, only diagonal terms survive the ensemble average and (2) becomes

$$\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle = E_0^2 \sum_{n,l,N} W_{n,l,N} \exp[-i\mathbf{R}_l(\mathbf{q}_a-\mathbf{q}_{a'}) + i\mathbf{R}_n(\mathbf{q}_b-\mathbf{q}_{b'})] , \quad (3)$$

where $W_{n,l,N}$ is the solution of the diffusion equation for a slab geometry with the appropriate boundary conditions and is given by

$$A_{n,l} = (\pi D L)^{-1} \sum_m \sin\frac{m\pi Z_n}{L} \sin\frac{m\pi Z_l}{L} K_0\left[\frac{m\pi}{L}|R_n - R_l|\right] , \quad (4)$$

where $A_{n,l} = \sum_N W_{n,l,N}$ and $K_0(x)$ is a Bessel function.

Using the continuum approximation, we get

$$\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle = C\int\int d^2R\, d^2R'\, A(|\mathbf{R}-\mathbf{R}'|) e^{-i\Delta\mathbf{q}_a\cdot\mathbf{R}} e^{i\Delta\mathbf{q}_b\cdot\mathbf{R}'} , \quad (5)$$

where $\Delta\mathbf{q}_a = \mathbf{q}_a - \mathbf{q}_{a'}$ and $\Delta\mathbf{q}_b = \mathbf{q}_b - \mathbf{q}_{b'}$. From (5), by changing variables it is apparent that the correlation effect is a *geometrical* effect *independent* of the phases that the wave acquired by performing the multiple-scattered trajectories. The correlation function (5) depends only on the properties of the diffusion probability $A(|\mathbf{R}-\mathbf{R}'|)$ as given by (4). Moreover, from (5) it follows that

$$\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle = CA(\Delta q_a)\delta_{\Delta\mathbf{q}_a,\Delta\mathbf{q}_b} . \quad (6)$$

Thus, a nonzero correlation is obtained only when the transverse momentum is "conserved," i.e., $\Delta\mathbf{q}_a = \Delta\mathbf{q}_b$. The correlation function falls off with increasing $\Delta\mathbf{q}_a$ since it is proportional to the square of the Fourier transform $A(\Delta q_a)$. The final result in the case of transmission is

$$|\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle|^2 \propto \delta_{\Delta\mathbf{q}_a,\Delta\mathbf{q}_b} \frac{(\Delta q_a L)^2}{\sinh^2(\Delta q_a L)} \quad (7)$$

and agrees with the result obtained by Feng *et al.*[13] The mean free path, l, does *not* appear in the correlation function (7) since the functional form of $A(|\mathbf{R}-\mathbf{R}'|)$ does not depend on l. The L dependence enters from the *range* of $A(|\mathbf{R}-\mathbf{R}'|)$ which is of order L.

We now calculated the memory effect for the reflected waves and obtain

$$|\langle E(\mathbf{q}_a,\mathbf{q}_b)E^*(\mathbf{q}_{a'},\mathbf{q}_{b'})\rangle|^2 \propto \delta_{\Delta\mathbf{q}_a,\Delta\mathbf{q}_b} \frac{L \sinh\Delta q_a(L-l)\sinh\Delta q_a l}{(L-l)l\Delta q_a \sinh\Delta q_a L} , \quad (8)$$

which again coincides with the diagrammatic calculation.[14] Here, unlike the correlation for the transmission case the memory effect is *not* universal but depends on the degree of disorder via l. This l dependence enters because the typical range of $A(|\mathbf{R}-\mathbf{R}'|)$ for reflection is a few mean free path and therefore the "form factor" $A(\Delta q_a)$ must depend on l.

Thus, for a *shorter* range of $A(|\mathbf{R}-\mathbf{R}'|)$, we expect to get a *slower* falloff of $A(\Delta q_a)$ and hence a stronger memory effect. We therefore expect that for real samples in which absorption takes place the range of $A(|\mathbf{R}-\mathbf{R}'|)$ will be reduced and the memory effect will be *more* apparent. We now include the effect of the absorption in our real-space theory by replacing $W_{n,l,N}$ by $W_{n,l,N}\exp(-Nl/L_a)$, where L_a is the absorption length. This leads to a modification of Eq. (4) and instead of $K_0(m\pi/L\,|R_n-R_l|)$ we get $K_0([(m\pi/L)^2+(1/L_a)^2]^{1/2}|R_n-R_l|)$. The memory effect for transmission in the presence of absorption is modified to

$$C_T(\mathbf{q}_a,\mathbf{q}_{a'},\mathbf{q}_b,\mathbf{q}_{b'}) = \langle T_{ab}\rangle\langle T_{a'b'}\rangle \delta_{\Delta\mathbf{q}_a\Delta\mathbf{q}_b} \left[\frac{\sinh^2(L/L_a)(\Delta\tilde{q}_a)^2 L_a^2}{\sinh^2(\Delta\tilde{q}_a L)}\right] , \quad (9)$$

where $\Delta\tilde{q}_a = (L_a^{-2}+\Delta q_a^2)^{-1/2}$ and $\langle T_{ab}\rangle$ is the transmitted intensity given by

$$\langle T_{ab}\rangle = \langle T_{ab}^0\rangle (L/L_a)/\sinh(L/L_a)$$

and $\langle T_{ab}^0\rangle$ is the transmitted intensity without absorption. The transmitted intensity $\langle T_{ab}\rangle$ is, of course, reduced in the presence of absorption and therefore the absolute value of the intensity-intensity correlation function is reduced. However, the *normalized* correlation function falls off much *slower*. This is demonstrated in Fig. 2 for different values of L_a/L. We see that as L_a/L becomes smaller the correlation function falls off more slowly. Moreover, Eq. (9) suggests that the memory correlation function in the presence of absorption can be obtained from the memory effect *without* absorption by replacing everywhere Δq_a by $(\Delta q_a^2 + L_a^{-2})^{1/2}$. Similar behavior was found for the coherent-backscattering peak.[16]

For reflected light, in the presence of absorption, we obtained the following memory correlation function:

$$C_R(\mathbf{q}_a,\mathbf{q}_b,\mathbf{q}_{a'}\mathbf{q}_{b'}) = \langle R_{ab}\rangle\langle R_{a'b'}\rangle\delta_{\Delta\mathbf{q}_a,\Delta\mathbf{q}_b}\left[\frac{\sinh[\Delta\tilde{\mathbf{q}}_a(L-l)]\sinh(\Delta\tilde{\mathbf{q}}_a L)\sinh(L/L_a)}{L_a\Delta\tilde{\mathbf{q}}_a\sinh(\Delta\tilde{\mathbf{q}}_a L)\sinh[(L-l)/L_a]\sinh(l/L_a)}\right]^2 \tag{10}$$

and the reflected intensity in the presence of absorption is given by

$$\langle R_{ab}\rangle = \langle R_{ab}^0\rangle\frac{L\sinh[(L-l)/L_a]\sinh(l/L_a)}{(L-l)(l/L_a)\sinh(L/L_a)}$$

and $\langle R_{ab}^0\rangle = 1-\langle T_{ab}^0\rangle$.

We have also used the diagrammatic approach of Feng *et al.*[13] and have included the effect of absorption by correcting the Ladder propagator, which in real space in the diffusion approximation is a solution of the following equation:

$$(-\nabla^2+L_a^{-2})L(r-r') = (3/l^2)\delta(r-r'), \tag{11}$$

where $L(r-r')$ must vanish at the boundaries $z=0$ and L. This leads to exactly the same correlation functions both for transmission as in Eq. (9) and reflection as in Eq. (10).

Finally, we discuss the role of boundary conditions on the memory effect. In the continuous-injection boundary condition[17] the photon can be scattered at *any* distance from *both* boundaries. When we use this continuous-injection boundary condition[17,18] we obtain for transmitted waves an *enhanced* memory effect. The enhancement of the memory effect results from an effective *shorter* sample for this boundary condition. This net effect in the correlation functions (7) is to replace L by $L-a$ where a is of the order of the mean free path. This is plotted in Fig. 3(a). Therefore, the influence of the boundary conditions will be most noticeable for thin slabs or for backscattering. For the reflected wave the influence of the boundary condition on the correlation function (8) is to replace l, which represents the distance at which the photon begins to diffuse in the slab, by l', which is the *averaged* distance, $l'\simeq 3l$. In Fig. 3(b) we show that the memory correlation function under the continuous boundary condition falls off more steeply than the fixed boundary condition. This figure also shows that the exact correlation function in this case can be obtained by replacing l by $2.7l$. It is interesting to note that the role of the continuous boundary condition, which we believe is more physical, changes the memory correlation function for transmitted and reflected light in *opposite* ways.

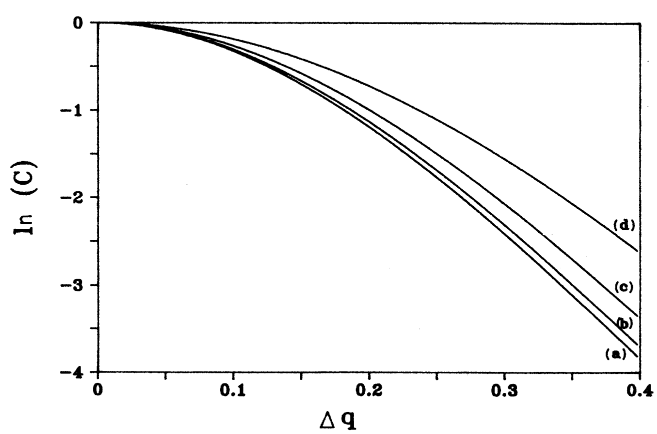

FIG. 2. The transmission correlation function $C_T(\mathbf{q}_a,\mathbf{q}_{a'},\mathbf{q}_b,\mathbf{q}_{b'})$ plotted for different values of the absorption length L_a. a, $L_a=\infty$ (no absorption); b, $L_a=L$; c, $L_a=L/2$; and d, $L_a=L/4$. The slab thickness $L=10l$.

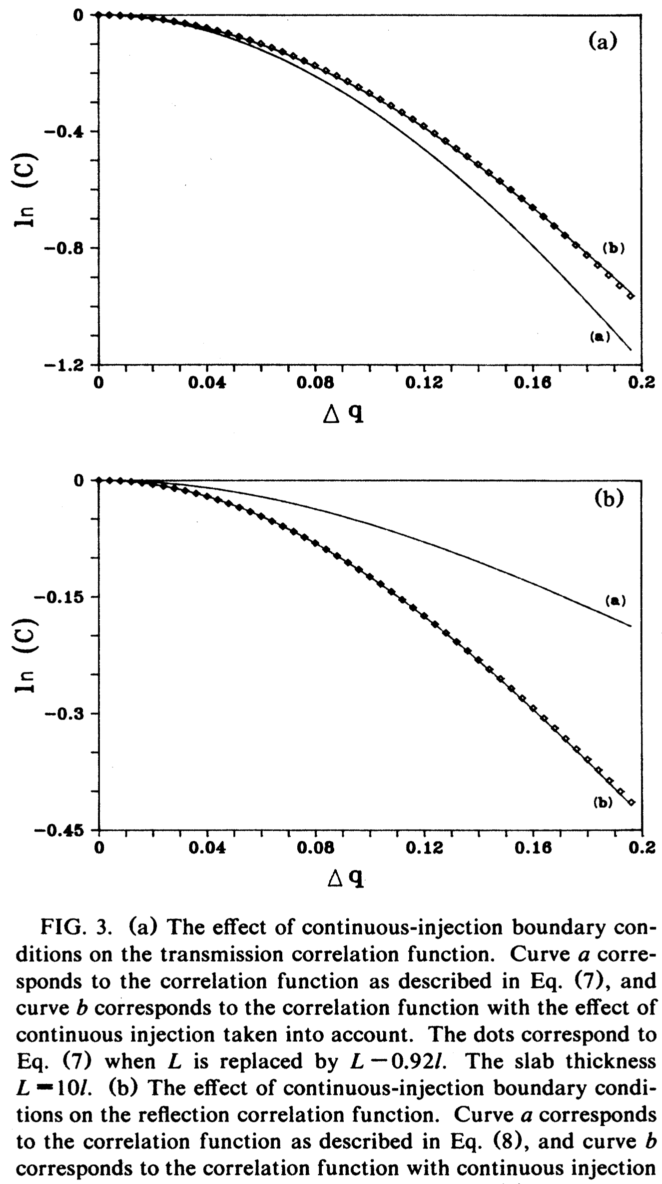

FIG. 3. (a) The effect of continuous-injection boundary conditions on the transmission correlation function. Curve *a* corresponds to the correlation function as described in Eq. (7), and curve *b* corresponds to the correlation function with the effect of continuous injection taken into account. The dots correspond to Eq. (7) when L is replaced by $L-0.92l$. The slab thickness $L=10l$. (b) The effect of continuous-injection boundary conditions on the reflection correlation function. Curve *a* corresponds to the correlation function as described in Eq. (8), and curve *b* corresponds to the correlation function with continuous injection taken into account. The dots correspond to Eq. (8) when l is replaced by $2.7l$.

In summary, we have presented a real-space treatment of the memory effect. The physical origin of the effect is demonstrated by including in the theory the effect of absorption which enhances the memory effect and the role of continuous boundary conditions which is important for reflected light.

We are grateful to I. Freund and M. Rosenbluh for discussions regarding this paper. We acknowledge the Israel–U.S. Binational Science Foundation and the Academy of Science and Humanities for supporting this research.

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