Central-force models which exhibit a splay-rigid phase

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Two models, one random the other periodic, are described which exhibit splay rigidity but are not rigid with respect to compression. The random model is based on a periodic lattice of rhombuses whose sides consist of central-force springs, which is perturbed in the following way: rhombuses can have diagonal central force struts with probability y or they can have one of the horizontal springs removed with probability x. For $x,y \ll 1$ we are led to consider a long-ranged anisotropic percolation process which is solved exactly on a Cayley tree. We show that for y/x near 2 the compressional rigidity of this system is zero but the Frank elastic constant, K, describing splay rigidity is nonzero. This is the first example of a percolation model for which this phenomenon, suggested earlier, is conclusively established. For $y/x \gtrsim 2\sqrt{2}$ the system has nonzero bulk and shear moduli. We also study the excitation spectrum for a periodic model which possesses only splay rigidity and obtain a libron dispersion relation $\omega = c_S q$, where q is the wave vector and $c_S \sim (K/\rho)^{1/2}$, where ρ is the mass density. These results are generalized to obtain a scaling form for c_S and the density of states of the random model which is valid when the correlation length for compressional rigidity becomes large.

I. INTRODUCTION

Recently there has been great interest in the randomly diluted lattice of central-force springs on a two-dimensional triangular lattice. The conventional picture is that as the concentration, p, of springs present in the lattice is increased beyond a threshold value, p_{cf} (which is[1,2] significantly larger than the classical percolation threshold at p_c) the system passes from a totally nonrigid phase into a phase where the bulk modulus and the shear modulus both grow continuously from zero for p larger than p_{cf}. Indeed, there has been quite some controversy as to the values of the critical exponents associated with this elastic threshold.[1-5] This model has several obvious similarities[6] with the analogous electrical network problem[7] obtained by replacing each central-force spring present in the diluted lattice by a conductance σ between the lattice sites. Because of this similarity[8,9] it is clearly desirable to adopt the field theoretic formulation of the randomly diluted resistor network[10,11] to this elastic problem. In approaching this question we[12] have developed series expansions for a quantity, the splay-resistance susceptibility, which was formulated to be the analog of the order-parameter susceptibility developed by Stephen[10,11] for the random resistor network. The virtue of this approach is that the resistive susceptibility reduces in the limit $\sigma \to \infty$ to the usual percolation susceptibility. In this theory, therefore, the exponents associated with the conductance threshold appear as crossover exponents[10,11] associated with "turning on" σ^{-1}. The relevant conclusion from our previous work[12] was that a major role was played by the angular displacement $\theta(b)$, of the bond b. It was shown that there could exist clusters which could be compressed or sheared with no restoring force, but for which distant bonds when twisted with respect to one another had a restoring force. This

bond-angle, or splay, rigid phase is analogous to the bond-angle rigid or hexatic phase in a two-dimensional liquid.[13] Furthermore, in the limit where the spring constants become infinite, this splay-resistance susceptibility defines a new percolation problem, i.e., one describing the percolation of splay rigidity.

Although this analogy is very satisfying, it remained to establish whether the threshold for splay rigidity could be distinct from that for total rigidity, as suggested.[12] In fact, Tremblay et al.[14] and later Marshall and Harris[15] have shown that these thresholds appear to coincide for the diluted central-force model on a triangular lattice. While it seems clear that these two thresholds can be made different by including appropriate bond-angle forces, the purpose of the present work is to give a central-force model in which the two thresholds can be established to be different. This demonstration not only shows that such a splay-rigid phase is possible, but it also indicates that the usual threshold for rigidity should be considered to be a multicritical point at which the thresholds for splay rigidity and bulk rigidity coincide. We also study the nature of the elementary excitations in the splay-rigid phase by analyzing the phonon spectrum of a periodic system which is rigid only with respect to splay. Here we find that the low-frequency modes are librons with a dispersion relation $\omega = c_S q$, where ω is the frequency and q the wave vector. Using the result we obtain a scaling form for c_S and the density of phonon states for the random system by a simple ansatz for the dependence of the velocity c_S on the size of the rigid regions.

Briefly, this paper is organized as follows. In Sec. II we define a type of local dilution of the central-force model in two spatial dimensions. In Sec. III we give bounds for the thresholds for the propagation of splay rigidity and compressional rigidity in terms of a long-ranged percolation model whose exact solution is given in

the Appendix. In Sec. IV we study the elementary excitations to be expected in the splay-rigid phase, first by the exact solution of a periodic splay-rigid system and then by scaling arguments using the properties of large totally rigid clusters. Our conclusions are summarized in Sec. V.

II. DEFINITION OF THE MODEL

We start from a periodic lattice of rhombuses with central-force springs of spring constant k between nearest-neighboring sites, for which the Hamiltonian is

$$H = \tfrac{1}{2}k \sum_{\langle x,x' \rangle} \{ [u(x) - u(x')] \cdot \hat{n}_{x,x'} \}^2 , \tag{1}$$

where $u(x)$ is the displacement of the node at site x, $\langle x,x' \rangle$ indicates that the sum is over pairs of nearest-neighboring sites, and $\hat{n}_{x,x'}$ is a unit vector along the line connecting sites x and x'. We will perturb this system randomly in the following way. We first label every other plaquette as in Fig. 1, so that each bond belongs to one and only one plaquette. Each labeled plaquette can randomly be in one of four states, as shown in Fig. 2: (1) with probability y it has a diagonal central-force spring (of spring constant k) inserted to connect the lower node to the upper node, (2) with probability x, the bottom horizontal spring is removed, (3) with probability x, the top horizontal spring is removed, and finally (4) with probability $1 - 2x - y$ it is not modified at all. In this model therefore, the vertical springs are always present, some horizontal springs are removed, and some diagonal struts are added. As will be seen below, adding struts increases the tendency of this system to be splay rigid, and to a lesser extent, totally rigid, while taking out horizontal bonds decreases the bulk modulus, and to a lesser extent, the splay rigidity modulus. Thus we find that for small x and y, where we can analyze the system conclusively, there is a regime around $y = 2x$ for which the system is splay rigid but not totally rigid.
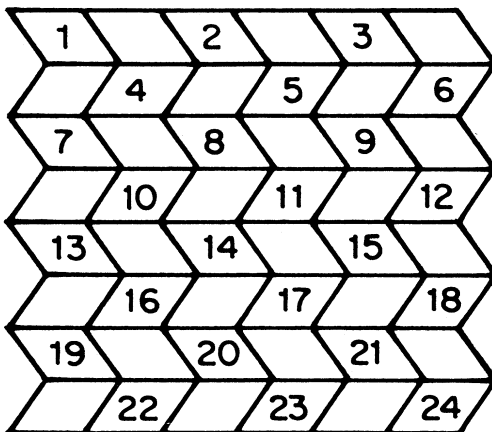

FIG. 1. Labeling of plaquettes on the periodic lattice of rhombuses.

FIG. 2. The four possible states of a plaquette with their respective probabilities.

III. A NEW PERCOLATION PROBLEM

In this section we will study a new percolation problem based on this model in which we consider simultaneously "strut" percolation and "vacancy" percolation. We will first show that when "strut" percolation occurs, the system is splay rigid. By an argument that one may view as dual to the above, we will next show that when "vacancy" percolation occurs, the bulk modulus is zero. The final step is then to show that for a regime of x and y both types of percolation occur simultaneously. Normally in two dimensions one would not expect two competing types of connectedness to percolate simultaneously. The reason why we find such a result is because the percolation process we are led to consider becomes long ranged when x and y are small. It is this feature that also enables us to analyze the situation conclusively.

We first consider "splay-rigidity" percolation. One can easily establish that pairs of opposite sides of a rhombus are splay rigid. That is to say, if one applies forces to twist opposite sides of the rhombus as shown in Fig. 3, there is a finite response. A formal definition of splay rigidity may be given as follows. For the system of N particles in any random configuration, C, we consider the elastic Green's function, $G(E,C)$, which is a $2N \times 2N$ matrix with rows and columns labeled by (i,α), where i labels the site and $\alpha = x,y$ the component and E is the energy. We have

$$G(E,C) = (V - EI)^{-1} , \tag{2}$$

where I is the unit matrix, V the matrix of coefficients of the potential energy implied by Eq. (1), and the limit $E \to 0$ is implicit. Generalized displacements are $2N$ component vectors denoted $|q\rangle$. We are especially interested in generalized displacements corresponding to rotation of a bond b clockwise through an angle θ about its midpoint and we will denote such a generalized dis-
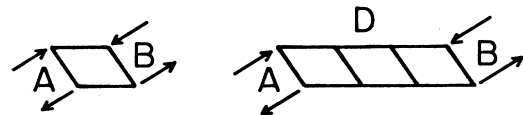

FIG. 3. Propagation of splay rigidity along a line of rhombuses. When torques of opposite orientation are applied to opposite sides (A,B) of a line of rhombuses the response is finite. In contrast, when torques of opposite sense are applied to sides A and D, the response is infinite, indicating that these two sides are not in the same splay-rigid cluster.

placement by $|\theta_b\rangle$. Bonds b and b' are splay rigid with respect to one another if, for $E \to 0$,

$$\langle \theta_b - \theta_{b'} | G | \theta_b - \theta_{b'} \rangle \equiv C_{bb'} < \infty , \qquad (3)$$

where $|\theta_b - \theta_{b'}\rangle$ denotes the generalized displacement $|\theta_b\rangle - |\theta_{b'}\rangle$. Physically, $C_{bb'}$ is the angular response in the coordinate $|\theta_b - \theta_{b'}\rangle$ to a unit conjugate generalized torque. When $C_{bb'}$ is finite, there is a nonzero restoring torque against rotating the two bonds in opposite senses.

Alternatively, if $|\phi_i\rangle$, $i = 1, 2, \ldots$ are the zero eigenvectors of V, then bonds b and b' are splay rigid if

$$\langle \theta_b - \theta_{b'} | \phi_i \rangle = 0 \quad \text{for } i = 1, 2, \ldots . \qquad (4)$$

We can show that splay-rigidity obeys the cluster properties: We will assume that b and b' are splay-rigidly connected, and that b' and b'' are splay-rigidly connected. With these assumptions we will show that b and b'' are also splay-rigidly connected. (This property may seem trivial, but it does not hold if one defines two bonds as rigidly connected if there is a nonzero restoring force associated with moving their centers towards one another.) We have

$$\langle \theta_b - \theta_{b''} | \phi_i \rangle = \langle \theta_b - \theta_{b'} | \phi_i \rangle + \langle \theta_{b'} - \theta_{b''} | \phi_i \rangle . \qquad (5)$$

By assumption, each term on the right-hand side of this equation vanishes for all i. Therefore, the left-hand side vanishes for all i, also. Thus splay rigidity obeys the cluster property one expects for a percolation process and which is necessary if one wishes to identify clusters.

In particular, splay rigidity will propagate through a linear array of such rhombuses, as illustrated in Fig. 3. Now consider the situation when a rhombus has a strut, as shown in Fig. 4. Now all four sides of this rhombus are splay rigid with respect to one another, in contrast to the situation without the strut, where only opposite sides are splay-rigidly connected. One sees that at a strut splay rigidity will branch out into both transverse and longitudinal directions. If we ignore vacancies for the moment, one sees that the splay-rigid structure will be obtained by connecting all struts in the same row or column. But it is also clear that the presence of a vacancy will interrupt this propagation of splay rigidity or "strut percolation." The question is now whether or not splay rigidity can propagate arbitrarily far when the effects of interruption by vacancies compete with the presence of the struts.
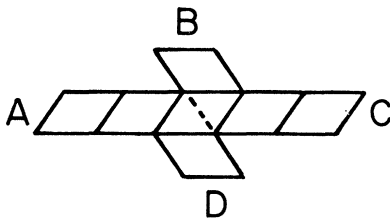
This situation is illustrated schematically in Fig. 5. In summary, the system exhibits long-range splay rigidity if "strut" percolation occurs. As noted previously[12] compressional rigidity propagates differently than splay rigidity. Opposite sides of a rhombus are rigid with respect to compression (although not, of course, with respect to shear). This compressional rigidity will also propagate along a line of rhombuses. But at a strut, the compressional rigidity does *not* propagate transversely.

When x and y are small the struts and vacancies are far apart and the path of strut percolation is topologically equivalent to a Cayley tree of coordination number 4. (At each strut there emanate 4 pathways of rigidity.) One can see that from a strut, splay rigidity will percolate to a neighboring strut to the right of it in the same horizontal row if the path is *unobstructed by a vacancy*. The probability $p_h^{(\text{strut})}$ that this path is unobstructed is given by

$$p_h^{(\text{strut})} = \sum_{k=0}^{\infty} (1-x)^{2+2k}(1-2x-y)^k y$$

$$= \frac{(1-x)^2 y}{1-(1-x)^2(1-2x-y)} , \qquad (6)$$

where the kth term is the probability that the two struts are separated by k steps having no intervening vacancies or previous struts. In constructing this expression we naturally took account of the state of the plaquettes in the rows adjacent to the struts as well as those in the same row as the struts. Also struts are allowed only in numbered plaquettes which alternate with unnumbered plaquettes. Thus $p_h^{(\text{strut})}$ is the probability for horizontal propagation of splay rigidity. Likewise the probability $p_v^{(\text{strut})}$ that from a given strut, splay rigidity will percolate to a neighboring strut below it in the same vertical column *unobstructed by a vacancy* is given by

$$p_v^{(\text{strut})} = \sum_{k=0}^{\infty} (1-2x-y)^k y = \frac{y}{2x+y} . \qquad (7)$$

We are thus led to consider a long-ranged anisotropic percolation problem in the low density limit for which the Cayley tree provides an asymptotically correct model.

FIG. 4. Effect of a strut. In the absence of the diagonal strut, splay rigidity percolates from A to C, but not from A to either B or D. With the strut present sides A, B, C, and D are all splay rigid with respect to one another.
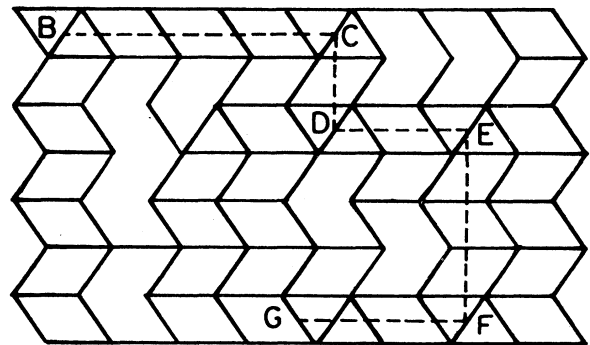

FIG. 5. Percolation of splay rigidity via struts. Note that splay rigidity propagation is interrupted by vacancies. Thus, splay rigidity propagates from D to G via E and F but not directly from D to G, since a vacancy intervenes.

The condition for percolation on a Cayley tree of coordination number z is[16] $p(z-1)>1$, where p is the probability that a bond is occupied. Here we have two sets of probabilities, and in the Appendix we show that the condition for percolation here is

$$4p_h^{(\text{strut})}p_v^{(\text{strut})}>(1-p_v^{(\text{strut})})(1-p_h^{(\text{strut})}) \ . \tag{8}$$

In the limit of small x and y, this condition is

$$y^2>2x^2 \ , \tag{9}$$

in which case the lattice is splay rigid. This analysis is expected to be exact in the limit $x,y \to 0$.

Now we consider the effect of vacancies. We will show that vacancy percolation implies that the lattice is completely compressible with respect to vertical stress. In the following discussion we therefore will assume that the bottom row of the system is undisplaced and we will show that when vacancy percolation occurs, the top row can be displaced vertically with no restoring force. For the moment, let us ignore the effect of struts. If two vacancies occur in the same row at the edges of the sample, then it is possible to displace the region above the line connecting the vacancies downward. The displacements which allow this to occur are shown in Fig. 6. In this simple example one sees that the downward *vertical* component of the displacement is 1 for a node above the line, $\frac{1}{2}$ for a node on the line, and 0 for a node below the line. (For this calculation we assume the acute angle of the rhombuses to be 60°.) However, it is clear that such displacements cannot take place if there are struts adjacent to the line connecting the two vacancies. The situation when struts are also present is shown in Fig. 7(a). There it is seen that to have zero compressional rigidity we must be able to trace a path of vacancies from one side of the system to the other. This path is constructed by connecting vacancies which are both either in the same row or in the same column and such that this path is *unobstructed by a strut*. The vertical component of the displacement of a node not on the path is n, where n is the net number of lines in the path going to the right below the node, where a line to the right is counted as $+1$ and one to the left as -1. This construction is illustrated schematically in Fig. 7(b). For a node on the path the

vertical component of the displacement is given by the average value of n as calculated for regions just above and just below the line. The horizontal components of the displacement (for nodes on the path) are those needed to keep all springs unstretched.

We now carry out the same analysis based on the Cayley tree for "vacancy" percolation. The probability $p_h^{(\text{vac})}$ that a vacancy will have another vacancy in the same row unobstructed by a strut is
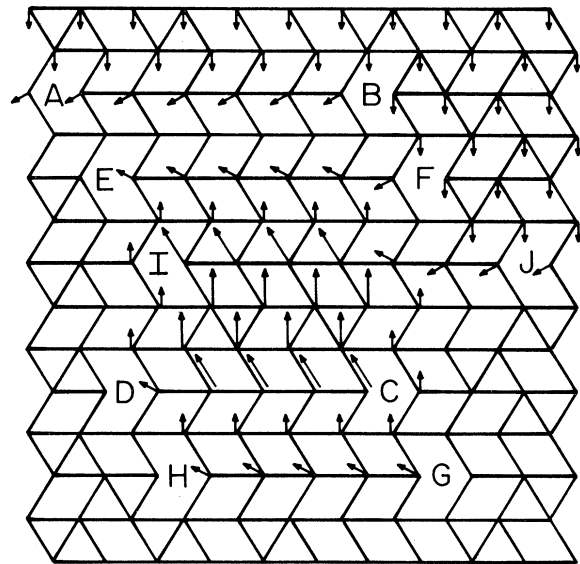
(a)
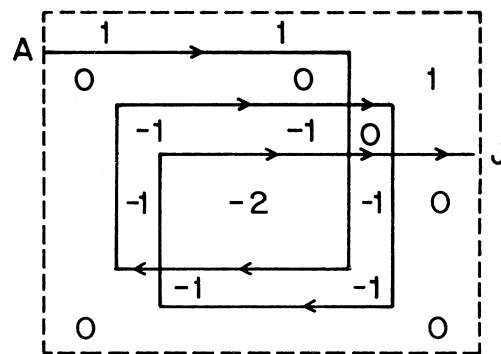
(b)

FIG. 7. (a) Percolation of vacancies across the system via the path *A-B-C-D-E-F-G-H-I-J*. Struts are not allowed to be adjacent to horizontal lines connecting vacancies or to intersect vertical lines connecting vacancies. The deformation shown indicates that the system has zero compressional modulus against vertical stress. (b) Schematic representation of the displacements associated with the path of propagation of vacancies of panel (a). The path defines regions each characterized by a uniform vertical displacement. This vertical displacement is equal to the net number of lines going to the right below the region, as discussed in the text.
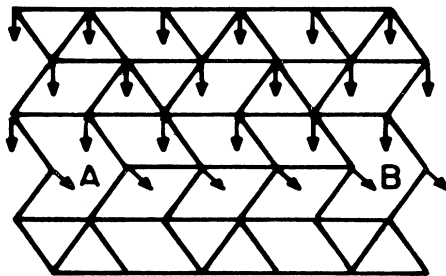

FIG. 6. Pattern of displacements involving no restoring forces which can be made above a line connecting two vacancies in the same horizontal row. No struts are allowed to be adjacent to the line connecting A and B, but they can be anywhere in the uniformly displaced regions.

$$p_h^{(vac)} = \sum_{k=0}^{\infty} (1 - x - y)^k x = \frac{x}{x+y} \qquad (10)$$

and the analogous probability for a column is

$$p_v^{(vac)} = \sum_{k=0}^{\infty} (1 - 2x - y)^k (2x) = \frac{2x}{2x+y} . \qquad (11)$$

Here the factor $2x$ comes from the fact that each plaquette has two horizontal bonds each of which may be the neighboring vacancy. In this case the condition for percolation obtained from Eq. (8) is

$$y^2 < 8x^2 \qquad (12)$$

for x and y small. Thus the compressional rigidity is zero for $y < 2\sqrt{2}x$ in the Cayley tree approximation.

Because of the horizontal rigidity of this anisotropic model, it does not yet form an example of a system with zero bulk elastic constants having nonzero splay-rigidity modulus, K. To give such an example we should put together domains consisting of regions like those treated here, some unrotated and some rotated by 90°. In summary, for $\frac{1}{2}y < \sqrt{2}x < y$ the lattice is splay rigid but not compressionally rigid.

IV. LIBRONS IN THE SPLAY-RIGID PHASE

It is of interest to study the frequency spectrum of small vibrations of the splay-rigid phase of randomly diluted central-force springs. Numerical work[17] shows that near a rigidity threshold there is critical behavior in the
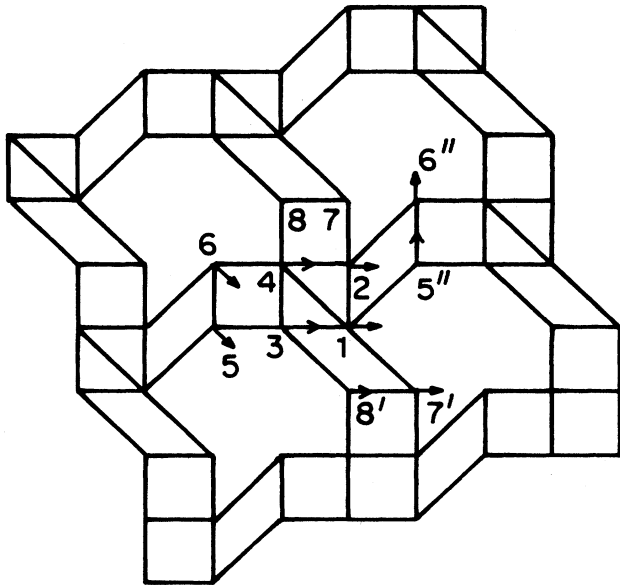

FIG. 8. Periodic lattice which is rigid with respect to splay but not to compression or shear. The unit cell contains particles numbered 1,2,...,8. The square formed by particles 1–4 can be translated in either the x or y directions by a localized distortion (see Table I) in which the connections between squares flex like an accordion. The pattern of localized displacements for translating the square in the x direction is indicated by arrows. Particles numbered with primes are in another unit cell and those with double primes are in yet another unit cell.

associated density of states, but the form of the singular behavior is not yet understood. Since no work has yet been done on excitations in the presence of long-range splay rigidity we consider a periodic model of such a system of central-force springs. This model is shown in Fig. 8, where each bond represents a central-force spring connecting adjacent particles with spring constant, k, as in Eq. (1). That this system is nonrigid with respect to shear and compression can be seen as follows. The square in Fig. 8 consisting of sites 1, 2, 3, and 4 can be rigidly displaced in either the x or y directions with a localized zero-energy distortion. The associated displacements of the particles are listed in Table I and are shown in Fig. 8 for a displacement in the x direction. Since we can form wavelike linear combinations of these distortions, we conclude that for each value of wave vector q there will be two zero-energy "phonons."

To study the excitation spectrum we choose the unit cell to contain particles 1–8 in Fig. 8. Thus the spectrum of small oscillations has 16 branches. To obtain these we write the elastic energy in terms of the *dimensionless* dynamical matrix as

$$E = \tfrac{1}{2}k \sum_{\alpha,\beta,\tau,\tau',q} D_{\tau\tau'}^{\alpha\beta}(\mathbf{q}) u_\tau^\alpha(\mathbf{q}) u_{\tau'}^\beta(-\mathbf{q}) , \qquad (13)$$

where

TABLE I. Displacements of various zero-energy modes for the splay-rigid system of Fig. 8. We label the modes as follows: Q_x is the localized mode in which the square 1–4 is translated in the positive x direction, and similarly for Q_y. Q_L is the libron mode at zero wave vector for which the displacements are repeated throughout all unit cells. For Q_L we give only the displacements of particles 1–8. Q_L has not been orthogonalized to the uniform translational modes.

	Q_x	Q_y	Q_L
$u_{1,x}$	1	0	1
$u_{1,y}$	0	1	1
$u_{2,x}$	1	0	−1
$u_{2,y}$	0	1	1
$u_{3,x}$	1	0	1
$u_{3,y}$	0	1	−1
$u_{4,x}$	1	0	−1
$u_{4,y}$	0	1	−1
$u_{5,x}$	1	0	1
$u_{5,y}$	−1	0	1
$u_{6,x}$	1	0	−1
$u_{6,y}$	−1	0	1
$u_{7,x}$	0	1	1
$u_{7,y}$	0	1	1
$u_{8,x}$	0	1	1
$u_{8,y}$	0	1	−1
$u_{5'',x}$	0	0	
$u_{5'',y}$	1	1	
$u_{6'',x}$	0	0	
$u_{6'',y}$	1	1	
$u_{7',x}$	1	−1	
$u_{7',y}$	0	0	
$u_{8',x}$	1	−1	
$u_{8',y}$	0	0	

$$u_\tau^\alpha(\mathbf{q}) = (N_{uc})^{-1/2} \sum_{\mathbf{R}} \exp[i\mathbf{q}\cdot(\mathbf{R}+\mathbf{r}_\tau)]u_\tau^\alpha(\mathbf{R}) , \qquad (14)$$

where $u_\tau^\alpha(\mathbf{R})$ is the α component of the displacement of the particle at location $\mathbf{r}_\tau$ in the unit cell at $\mathbf{R}$ and N_{uc} is the number of unit cells in the system. The dimensionless dynamical matrix, D, is given in Table II. The first Brillouin zone is defined by $-\pi/a_0 < q_\xi, q_\eta < \pi/a_0$, where $a_0 = \sqrt{10}a$, $q_\xi = q_x\cos\theta_0 + q_y\sin\theta_0$, $q_\eta = q_y\cos\theta_0 - q_x\sin\theta_0$, with $\tan\theta_0 = \frac{1}{3}$ and a is the length of a side of the squares in Fig. 8. By sampling over the first Brillouin zone, using the linear extrapolation method of Gilat and co-workers,[18] we obtained the density of states $\rho(\omega)$ [normalized by $\int\rho(\omega)d\omega = 1$] as

$$\rho(\omega) = \tfrac{1}{8}\delta(\omega) + \tfrac{7}{8}\hat\rho(\omega) . \qquad (15)$$

The δ-function contribution to $\rho(\omega)$ represents the localized zero-energy distortions described in Table I. In actual calculations it is convenient to find the ω_i^2 rather than the ω_i themselves. Thus in Fig. 9 we show $F(\omega^2) \equiv \tfrac{1}{2}\hat\rho(\omega)/\omega$. The extreme irregularity of the spectrum is due to the large number of Van Hove singularities for the large unit cell in our model.

In addition to the two zero-energy modes which occur for each value of the wave vector $\mathbf{q}$, there is also for $q = 0$ a zero-energy libronlike mode whose displacements within a unit cell are shown in Fig. 10 and given in Table I. (These displacements are repeated in all unit cells, since $q = 0$.) For small q our numerical results gives the libron dispersion relation as

$$m\omega^2 = \tfrac{4}{15}ka^2q^2 . \qquad (16)$$

We have verified this result by a perturbative calculation in powers of $\mathbf{q}$.

We may generalize this result heuristically in a way which may be applied to random systems. We assume the libron mode to consist primarily of coupled librations of rigid units of size ξ_R, where ξ_R is the correlation length for total rigidity. ξ_R is finite in the splay-rigid regime, but diverges as the threshold for total rigidity is approached. For the periodic model of Fig. 8, $\xi_R \sim a$. We write the Lagrangian per unit area as

$$L = \tfrac{1}{2}nI\dot\theta(r)^2 - \tfrac{1}{2}K[\nabla\theta(r)]^2 , \qquad (17)$$

where n is the number of clusters per unit area, I is a typical moment of inertia associated with a rigid region, and θ is the angular coordinate assigned to each rigid region. Since $I \sim mr^2$, we write the first term in Eq. (17) in terms of a sum over clusters α as $\tfrac{1}{2}\sum_\alpha n_\alpha r_\alpha^2 \dot\theta(r)^2$. The cluster average of r_α^2 which appears in this formulation is

TABLE II. Dynamical matrix for the splay-rigid periodic lattice of Fig. 8. Here X denotes $\exp(iq_x a)$, Y denotes $\exp(iq_y a)$, $\bar{X}$ denotes $\exp(-iq_x a)$, and $\bar{Y}$ denotes $\exp(-iq_y a)$. The first eight rows and columns refer to the x coordinates, the second eight to the y coordinates, in each case ordered 1–8 according to the particle numbering in Fig. 8.

|															
$\frac{5}{2}$	0	$-X$	$-\frac{X\bar{Y}}{2}$	$-\frac{\bar{X}Y}{2}$	0	$-\frac{\bar{X}Y}{2}$	0	$-\frac{1}{2}$	0	0	$\frac{X\bar{Y}}{2}$	$-\frac{\bar{X}Y}{2}$	0	$\frac{\bar{X}Y}{2}$	0
0	$\frac{3}{2}$	0	$-X$	0	$-\frac{\bar{X}Y}{2}$	0	0	0	$\frac{1}{2}$	0	0	0	$-\frac{\bar{X}Y}{2}$	0	0
$-\bar{X}$	0	$\frac{5}{2}$	0	$-X$	0	0	$-\frac{\bar{X}Y}{2}$	0	0	$-\frac{1}{2}$	0	0	0	0	$\frac{\bar{X}Y}{2}$
$-\frac{\bar{X}Y}{2}$	$-\bar{X}$	0	$\frac{5}{2}$	0	$-X$	0	0	$\frac{\bar{X}Y}{2}$	0	0	$-\frac{1}{2}$	0	0	0	0
$-\frac{X\bar{Y}}{2}$	0	$-\bar{X}$	0	$\frac{3}{2}$	0	0	0	$-\frac{X\bar{Y}}{2}$	0	0	0	$\frac{1}{2}$	0	0	0
0	$-\frac{X\bar{Y}}{2}$	0	$-\bar{X}$	0	$\frac{3}{2}$	0	0	0	$-\frac{X\bar{Y}}{2}$	0	0	0	$\frac{1}{2}$	0	0
$-\frac{X\bar{Y}}{2}$	0	0	0	0	0	$\frac{3}{2}$	$-X$	$\frac{X\bar{Y}}{2}$	0	0	0	0	0	$-\frac{1}{2}$	0
0	0	$-\frac{X\bar{Y}}{2}$	0	0	0	$-\bar{X}$	$\frac{3}{2}$	0	0	$\frac{X\bar{Y}}{2}$	0	0	0	0	$-\frac{1}{2}$
$-\frac{1}{2}$	0	0	$\frac{X\bar{Y}}{2}$	$-\frac{\bar{X}Y}{2}$	0	$\frac{\bar{X}Y}{2}$	0	$\frac{5}{2}$	$-\bar{Y}$	0	$-\frac{X\bar{Y}}{2}$	$-\frac{\bar{X}Y}{2}$	0	$-\frac{\bar{X}Y}{2}$	0
0	$\frac{1}{2}$	0	0	0	$-\frac{\bar{X}Y}{2}$	0	0	$-Y$	$\frac{5}{2}$	0	0	0	$-\frac{\bar{X}Y}{2}$	$-\bar{Y}$	0
0	0	$-\frac{1}{2}$	0	0	0	0	$\frac{\bar{X}Y}{2}$	0	0	$\frac{3}{2}$	$-\bar{Y}$	0	0	0	$-\frac{X\bar{Y}}{2}$
$\frac{\bar{X}Y}{2}$	0	0	$-\frac{1}{2}$	0	0	0	0	$-\frac{\bar{X}Y}{2}$	0	$-Y$	$\frac{5}{2}$	0	0	0	$-\bar{Y}$
$-\frac{X\bar{Y}}{2}$	0	0	0	$\frac{1}{2}$	0	0	0	$-\frac{X\bar{Y}}{2}$	0	0	0	$\frac{3}{2}$	$-\bar{Y}$	0	0
0	$-\frac{X\bar{Y}}{2}$	0	0	0	$\frac{1}{2}$	0	0	0	$-\frac{X\bar{Y}}{2}$	0	0	$-Y$	$\frac{3}{2}$	0	0
$\frac{X\bar{Y}}{2}$	0	0	0	0	0	$-\frac{1}{2}$	0	$-\frac{X\bar{Y}}{2}$	$-Y$	0	0	0	0	$\frac{3}{2}$	0
0	0	$\frac{X\bar{Y}}{2}$	0	0	0	0	$-\frac{1}{2}$	0	0	$-\frac{X\bar{Y}}{2}$	$-Y$	0	0	0	$\frac{3}{2}$
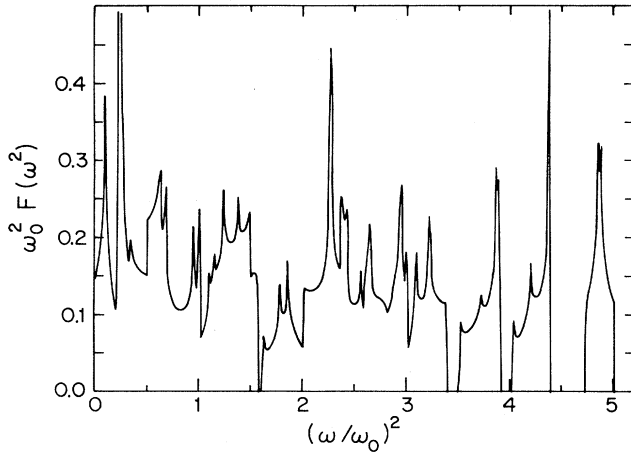
FIG. 9. Reduced normalized density of states $\frac{1}{2}\rho(\omega)/\omega \equiv F(\omega^2)$ for the splay-rigid periodic system of Fig. 8. We plot the dimensionless quantities $\omega_0^2 F(\omega^2)$ vs $(\omega/\omega_0)^2$, where $\omega_0^2 = k/m$. As indicated in Eq. (15) the δ-function contribution is not represented.

known[19] to be given by $r^2 \sim \xi_R^{2-\beta/\nu}$. Also we identify the scaling of K to be similar to that of the superconducting exponent, s, in the resistor network of randomly mixed resistive and nonresistive elements.[20-22,9] We denote this exponent by s_{el}, since we expect it to be similar, but not identical to that for the resistor network. (The equivalence shown in Ref. 9 only holds for the unusual anisotropic model considered there.) Thus we have

$$L = c_1 \xi_R^{2-\beta/\nu} \dot{\theta}(r)^2 - c_2 \xi_R^{s_{el}/\nu} [\nabla \theta(r)]^2 , \qquad (18)$$

where c_1 and c_2 are constants. From this form we obtain the dispersion relation for librons as

$$\omega^2 \sim (\xi_R^{\beta/\nu+s_{el}/\nu-2})q^2 \equiv c_S^2 q^2 , \qquad (19)$$
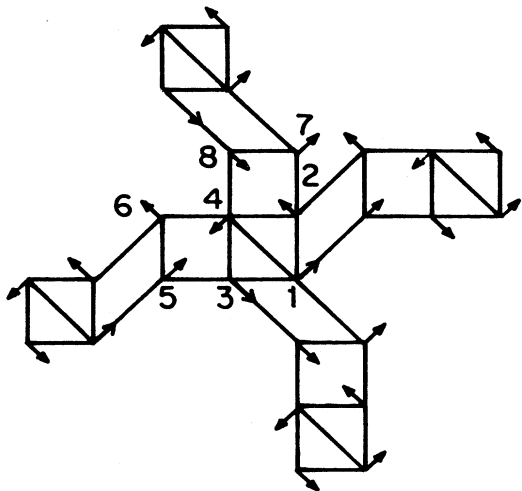

FIG. 10. Displacements for the zero-wave-vector libron mode for the system of Fig. 8.

where the exponents are those for *elastic* percolation, as the totally rigid phase is approached from within the splay-rigid phase. These exponents have not yet been investigated. If we note the existence of zero-frequency modes associated with translating rigid clusters, we obtain for d spatial dimensions

$$\rho(\omega) = c_1' g(\xi_R)\delta(\omega)$$
$$+ c_2' \omega^{-1}(\omega a/c_S)^d \Theta(c_S/\xi_R - \omega)/K , \qquad (20)$$

where $\Theta(x)$ is unity for $x > 0$ and vanishes otherwise, c_1' and c_2' are constants. The restriction on ω in Eq. (20) is caused by the short wavelength cutoff at $q\xi_R = 1$ implied by the rigidity of the fundamental units. The density of zero-frequency modes will have a regular part and a singular part. The former corresponds to the existence of small rigid elements whose density varies smoothly through the rigidity threshold, in loose analogy with the percolation free energy[20-22] which describes the number of clusters. For the singular part of $g(\xi_R)$ we assume that $g(\xi_R) \sim (a/\xi_R)^d$. Considering only the singular part, we write the result in Eq. (20) in scaling form as

$$\rho(\omega) = \omega^{-1}(a/\xi_R)^d f(\omega\xi_R/c_S) , \qquad (21)$$

with $f(x) = c_1'x\delta(x) + c_2'x^d\Theta(1-x)$. It is not clear how this scaling relation should be modified to treat the compressionally rigid phase.

V. DISCUSSION AND CONCLUSION

It is important to discuss several modifications of the model which might make it physically more realistic. One might treat a system of unlinearized or "real" Hooke's Law springs for each of which $E = \frac{1}{2}k(r-r_0)^2$, where r is the distance between particles whose equilibrium positions are r_i and r_j, so that $\mathbf{r} = \mathbf{r}_i + \mathbf{u}_i - \mathbf{r}_j - \mathbf{u}_j$. One could imagine two subcases depending on whether or not $r_0^2 = (\mathbf{r}_i - \mathbf{r}_j)^2$. If this equality does not hold, i.e., if the springs are under tension, say, then dilution would randomly perturb the equilibrium positions of the particles. In this case there would occur transverse terms not included in our analysis. Even if the system is not under tension, motion perpendicular to the equilibrium direction of the spring gives rise to anharmonic terms which limit transverse displacements.[23] These effects would eventually limit the range of amplitude over which libration of large rigid units could be considered to be harmonic, as we have done. Obviously, the situation is potentially quite complex. To avoid the myriad of complications and also to clarify the properties of this widely studied model we have restricted our analysis to models described by Eq. (1).

We may summarize our conclusions as follows. (1) We have constructed a random model of central-force springs in two dimensions which has a splay-rigid phase in which the compressional rigidity vanishes, but which has a nonzero Frank elastic constant, K, describing the stiffness of the system against non-uniform twist. (2) As more cross links are added, there is a transition to a conventional solid phase where the elastic constants are all nonzero and K is infinite. (3) By considering a periodic

splay-rigid lattice, we have also identified the libron dispersion relation as $\omega = c_S q$, and the scaling behavior of c_S as the solid phase is approached is given in Eq. (19).

ACKNOWLEDGMENTS

APPENDIX: ANISOTROPIC PERCOLATION ON THE CAYLEY TREE

We start from the origin of a Cayley tree labeling outgoing bonds $1, 2, \ldots, z$, as in Fig. 11. We iterate this labeling, making sure that all z bonds intersecting a given site carry distinct labels. We now consider percolation in which the probability that a bond labeled i is actually present is p_i. In the limit $z = 2d \to \infty$ this model is equivalent to fully anisotropic percolation on a d-dimensional hypercubic lattice.

To locate the percolation threshold, we let P_i be the probability that bond of type i does not lead outward from the origin to points infinitely far away. Then the percolation threshold in $(p_1, p_2, \ldots, p_z)$ space is the boundary which separates the regime where $P_i = 1$ for all i from that where $P_i < 1$ for all i. The recursion relation for P_i is

$$P_i = \prod_{k \neq i} (1 - p_k + p_k P_k) , \tag{A1}$$

which expresses the fact that bond i is finitely connected only if it leads to branches all of which are either missing or are finitely connected. We write Eq. (A1) as $\Phi_i(\{P\}) = 0$. Clearly, $P_i = 1$ for all i is a solution to Eq. (A1). It joins onto a nontrivial solution when

$$\det \left| \frac{\partial \Phi_i}{\partial P_j} \right| = \det |\delta_{ij} - (1 - \delta_{ij}) p_j| = 0 , \tag{A2}$$

where δ_{ij} is the Kronecker delta and the determinant is evaluated when $P_i = 1$ for all i. This relation is $\det |M| = 0$, where

$$M = M_0 - |\psi\rangle\langle\phi| , \tag{A3}$$

where $(M_0)_{ij} = (1 + p_i)\delta_{ij}$, $\langle\psi| = (p_1, p_2, \ldots, p_z)$, and $\langle\phi| = (1, 1, \ldots, 1)$. Thus
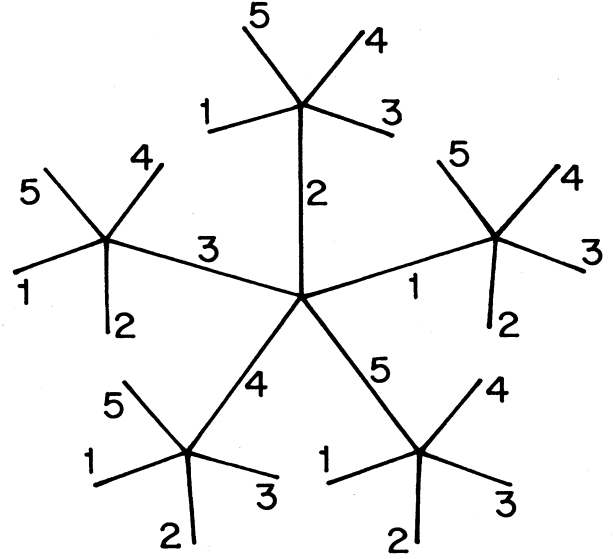

FIG. 11. Labeling of bonds for anisotropic percolation on a Cayley tree. From each node there emanate z bonds, the ith of which has probability of occupation, p_i. Here the case $z = 5$ is shown.

$$\ln \det M = \mathrm{Tr} \ln M$$

$$= \mathrm{Tr} \ln M_0 - \sum_{k=1}^{\infty} k^{-1} \mathrm{Tr}[M_0^{-1} |\psi\rangle\langle\phi|]^k$$

$$= \mathrm{Tr} \ln M_0 - \sum_{k=1}^{\infty} k^{-1} \langle\phi|M_0^{-1}|\psi\rangle^k$$

$$= \sum_{i=1}^{z} \ln(1 + p_i) + \ln[1 - \langle\phi|M_0^{-1}|\psi\rangle] . \tag{A4}$$

Thus

$$\det M = \left[1 - \sum_{i=1}^{z} \frac{p_i}{1 + p_i} \right] \prod_{i=1}^{z} (1 + p_i) \tag{A5}$$

and the condition $\det M = 0$ for the percolation threshold is

$$1 = \sum_{i=1}^{z} \frac{p_i}{1 + p_i} . \tag{A6}$$

For $p_i = p$ this gives the usual result, $p_c = (z - 1)^{-1}$. For coordination number $z = 4$ with two probabilities, p_a and p_b, we write $\langle\psi| = (p_a, p_a, p_b, p_b)$ and Eq. (A6) is

$$4 p_a p_b = (1 - p_a)(1 - p_b) . \tag{A7}$$

Percolation takes place when the left-hand side of this equation exceeds the right-hand side, as written in Eq. (8) of the text.

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