Griffiths singularities in random magnets: Results for a soluble model

A. J. Bray and Deng Huifang*
Department of Theoretical Physics, The University, Manchester M13 9PL, England
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A soluble, but nontrivial, model of a dilute Ising ferromagnet is studied, with infinite-range interactions but finite average connectivity c. The density of (Yang-Lee) zeros of the partition function in the complex $z=\exp(-2H)$ plane (where H is the external magnetic field in units of the temperature) is calculated explicitly in the high-temperature phase for large but finite c and small $|H|$. The density of zeros on the unit circle $H=i\theta$ has the form $\rho(\theta)\sim\exp\{-[cf(K)/|\theta|]\ln(1/|\theta|)\}$ for $|\theta|\to 0$. The function $f(K)$ $(K=J/T)$ vanishes at the critical coupling $K_C(c)$. Heuristic arguments are given for the form of $\rho(\theta)$ expected for systems with short-range interactions.

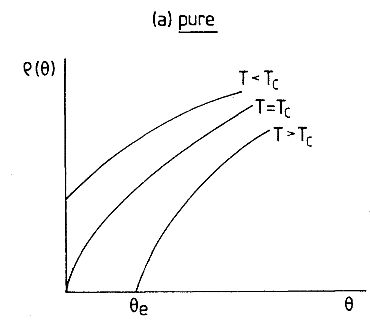
I. INTRODUCTION

In 1969 Griffiths[1] showed that the free energy of a dilute Ising ferromagnet is nonanalytic as a function of the external magnetic field for all temperatures below the critical temperature (called T_G hereafter) of the corresponding nondilute (or pure) system. The signature of the nonanalyticity which Griffiths explored was the distribution of the zeros of the partition function Z (the Yang-Lee zeros) in the complex magnetic field plane. In practice, it is convenient to use the variable $z=\exp(-2H)$ (with the usual $1/T$ factor absorbed into H) since Lee and Yang showed, in their famous theorem,[2] that the zeros of Z lie on the unit circle $H=i\theta$ in the complex z plane. The Griffiths singularities are associated with the appearance of Yang-Lee (YL) zeros arbitrarily close to the point $z=1$ (i.e., $H=0$) below T_G, implying a zero radius of convergence for expansions of thermodynamic functions in powers of H.

For the pure Ising ferromagnet, the temperature at which the distribution of YL zeros pinches the real axis is just the critical temperature, i.e., the temperature T_C at which long-range order sets in. For $T<T_C$ the density of zeros at $z=1$ is proportional to the spontaneous magnetization.[2] For the dilute Ising model, by contrast, there are zeros arbitrarily close to the real axis for all $T<T_G$,[1] but the density at $z=1$ strictly vanishes for $T>T_C$, the critical temperature of the dilute system. This difference between pure and dilute system is illustrated schematically in Fig. 1.

In this paper, we are interested in the temperature regime $T_C<T<T_G$, which has elsewhere been termed the Griffiths phase.[3] There has been much recent interest in the dynamics of this phase, in which relaxation is slower than exponential due to clustering effects.[3,4] Here only equilibrium thermodynamic properties are considered. We introduce a soluble model with infinite-range interactions but finite average connectivity, i.e., the ferromagnetic analog of the Viana-Bray model of dilute spin glasses.[5] This model has the advantage of exact solubility (in the sense that it may be reduced to a self-consistent single-site problem) combined with the physically desirable feature of finite connectivity. It turns out, in fact, that the model is closely related to a randomly diluted Bethe lattice. It avoids, however, the (notorious) difficulties of the latter model which are associated with the fact that, for a finite lattice, a nonzero fraction of the sites are at the surface.

The model is described in detail in Sec. II, while the general connection between the density of YL zeros and the thermodynamic functions is briefly reviewed in Sec. III. Explicit results for the density of YL zeros near the



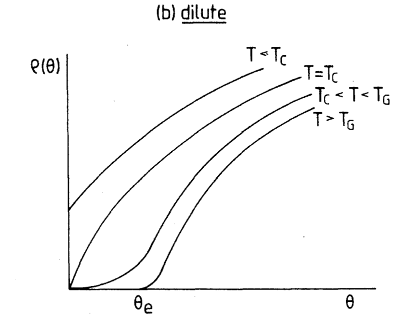


FIG. 1. Density (schematic) of Yang-Lee zeros in the complex $z=\exp(-2H)$ plane, with $H=i\theta$ for (a) pure and (b) dilute Ising ferromagnets. For $T>T_C$ (pure) or $T>T_G$ (dilute), there are no zeros below the Yang-Lee edge θ_e. Since $\rho(-\theta)=\rho(\theta)$, only positive θ is shown.

real z axis are derived in Secs. IV and V, in the limit that the average site coordination number c is large. For c strictly infinite (Sec. IV) one recovers mean-field theory, which does not contain Griffiths singularities. The large (but finite) c behavior, discussed in Sec. V, contains the leading nontrivial correction to mean-field theory. The result has the form $\rho(\theta) \sim \exp\{-[cf(K)/|\theta|]\ln(1/|\theta|)\}$, for $|\theta| \to 0$. Note that this is nonperturbative in $1/c$: it is associated in the present model with the existence of rare, highly coordinated sites. The amplitude $f(K)$ in the exponent vanishes, as expected, at the onset of ferromagnetic long-range order: for $T < T_C(c)$ there is nonvanishing density of YL zeros at $H=0$, proportional to the spontaneous magnetization.

The results are discussed and summarized in Sec. VI. In particular, heuristic arguments for models with short-range interactions yield a slightly different form for $\rho(\theta)$ in which the logarithmic factor in the exponent is absent.

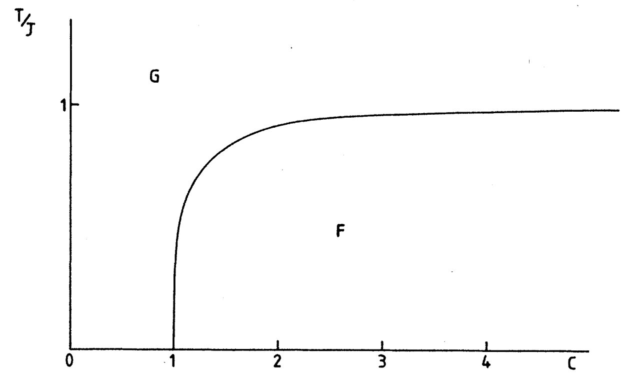


FIG. 2. Phase diagram for the model considered in this paper, showing ferromagnetic (F) and Griffiths (G) phases. The phase boundary is given by $1=c\tanh(J/cT)$. The Griffiths phase G extends to infinite temperature.

II. THE MODEL

The model is described by the Ising Hamiltonian for spins $S_i = \pm 1$ $(i=1,\dots,N)$

$$\mathsf{H} = -\sum_{(i,j)} J_{ij} S_i S_j - h\sum_i S_i \ , \quad (1)$$

where the sum is over all distinct pairs of sites. The bonds $J_{ij}(=J_{ji})$ are independent random variables with distribution

$$P(J_{ij}) = (c/N)\delta(J_{ij}-J/c) + (1-c/N)\delta(J_{ij}) \ . \quad (2)$$

The $1/c$ scaling of the nonzero bonds ensures a sensible large-c limit.

The geometrical properties of the network generated by (2) have been discussed in detail by Bray and Rodgers.[6] For $c<1$, the system breaks up into disconnected finite clusters, which have a branched (or treelike) structure. For $c>1$ an infinite cluster appears, in addition to finite clusters, i.e., $c=1$ is the percolation threshold of the model. The cluster size distribution for the finite clusters can be calculated for general c,[6] but we will not require it here.

The analysis of the Ising model defined by (1) and (2) is straightforward, and follows a well-trodden path.[5,7] The disorder-averaged free energy may be calculated using the replica method, via

$$-[F]_{av}/T = [\ln Z]_{av} = \lim_{n\to 0}([Z^n]_{av}-1)/n \ ,$$

where $[\dots]_{av}$ indicates an average with respect to the distribution (2). The sites may then be decoupled by introducing auxiliary variables $q_\alpha, q_{\alpha\beta}, q_{\alpha\beta\gamma}\dots$, conjugate to $\sum_i S_i^\alpha, \sum_i S_i^\alpha S_i^\beta, \sum_i S_i^\alpha S_i^\beta S_i^\gamma \dots$, respectively, where $\alpha,\beta,\gamma\dots$ are replica indices running from 1 to n. These auxiliary variables are the order parameters of the theory.

In the high-temperature phase, all order parameters vanish for external field $h=0$. The phase boundary for the onset of ferromagnetic order is given by the equation[5]

$$1 = c\tanh(K/c) \ , \quad (3)$$

where $K=J/T$. For $c<1$, the system is below the percolation threshold, and there is no long-range order at any nonzero temperature. For $c \gg 1$, $\tanh(K/c)$ can be replaced by its argument, and the mean-field result $T_c = J$ is recovered. The phase diagram is shown in Fig. 2. Note that, in the present model, the Griffiths phase extends to infinite temperature, since the corresponding pure system, in which all bonds are nonzero, has a critical temperature which diverges in the thermodynamic limit (the exchange interaction would need to be scaled by $1/N$ to leave the critical temperature of the pure system finite).

Since the Hamiltonian is ferromagnetic, and we consider only the high-temperature phase (with $h \neq 0$), we can certainly assume that the order parameters are replica symmetric, i.e., $q_\alpha = q_1$, $q_{\alpha\beta} = q_2 \dots$, independent of the replica indices. The order parameters then have the following physical significance:

$$q_r = [\langle S_i \rangle^r]_{av} \ , \quad (4)$$

where $\langle \dots \rangle$ indicates a canonical average for given disorder. In particular, $q_1 = m$, the average magnetization per site.

The order parameters $\{q_r\}$ obey an infinite set of coupled equations. Considerable progress, however, can be made by introducing an order function $P(x)$, defined as the probability distribution of the variable $x = \tanh^{-1}(\langle S_i \rangle)$. According to (4), the moments of $P(x)$ are the order parameters:

$$q_r = \int_{-\infty}^{\infty} dx\, P(x)(\tanh x)^r \ . \quad (5)$$

It can be readily shown that $P(x)$ satisfies the nonlinear integral equation

$$P(x) = e^{-c}\int_{-\infty}^{\infty}(dy/2\pi)\exp\left[-iy(x-H) + c\int_{-\infty}^{\infty} dz\, P(z)\exp\{iy\tanh^{-1}[\tanh(K/c)\tanh z]\}\right] \ , \quad (6)$$

where $H=h/T$ and $K=J/T$. Equation (6) is a straightforward generalization to $H\neq 0$ of the result derived in Ref. 7. While this equation is difficult to solve for general c, significant progress is possible for $c\gg 1$. This limit is explored in detail in sec. IV. First, however, it is necessary to discuss the relationship between the density of YL zeros and the thermodynamic functions. This is the subject of the following section.

III. DENSITY OF YANG-LEE ZEROS

In this section we review briefly the argument[2] relating the density of YL zeros of the partition function to the *real part* of the magnetization in the presence of an *imaginary* magnetic field. The first step is to note that the partition function $Z(K,H)$ can be written as a sum of terms with fixed magnetization $M=\sum_{i=1}^{N} S_i$,

$$\begin{aligned} Z(K,H) &= \sum_{M=-N}^{N} \exp(MH)\mathrm{Tr}_{\{\sum_i S_i=M\}} \exp\left[\sum_{(ij)} J_{ij}S_iS_j/T\right] \\ &= \sum_{M=-N}^{N} \exp(MH)a_M(K) \\ &= a_{-N}(K)\exp(NH)\prod_{i=1}^{N}[z-z_i(K)] \ , \end{aligned}$$

where $z=\exp(-2H)$. Note that the sums on M in the above are over the values $M=-N,-(N-2),\ldots,(N-2),N$. Thus the free energy f and magnetization m *per site* are give, respectively, by

$$\begin{aligned} -f/T &= (1/N)\ln Z \\ &= (1/N)\ln a_{-N}+H+(1/N)\sum_{i=1}^{N}\ln(z-z_i) \ , \\ m &= (\partial/\partial H)(-f/T)=(1/N)\sum_{i=1}^{N}(z_i+z)/(z_i-z) \ . \end{aligned} \tag{7}$$

So far $z[=\exp(-2H)]$ is a general complex number. According to the Yang-Lee theorem,[2] however, the zeros z_i of Z all lie on the unit circle, corresponding to a pure imaginary magnetic field. Therefore we write $H=i\theta+\epsilon$, where θ is real and ϵ is a real infinitesimal whose role will become clear below. Similarly, we put $z_i=\exp(-2i\phi_i)$ with ϕ_i real. Then $-\pi/2<\phi_i,\theta\leq\pi/2$. Replacing $(1/N)\sum_{i=1}^{N}\ldots$ in (7) by $\int_{-\pi/2}^{\pi/2} d\phi\,\rho(\phi)\ldots$ gives

$$m=-i\int_{-\pi/2}^{\pi/2} d\phi\,\rho(\phi)\cot(\theta-\phi-i\epsilon) \ .$$

Thus

$$\mathrm{Re}m=\pi\,\mathrm{sgn}(\epsilon)\rho(\theta) \ , \tag{8}$$

i.e., the density of YL zeros on the unit circle is simply related to the real part of the magnetization induced by an imaginary magnetic field. The infinitesimal real part of the field determines the sign of Rem. From Fig. 1 we note that, for the pure system above T_C, Rem vanishes for small θ. In the Griffiths phase of the dilute system, however, Rem is nonzero for any $\theta\neq 0$. In the ferromagnetic phase, $\rho(0)$ is nonzero and equal to π^{-1} times the spontaneous magnetization.

IV. THE LARGE-c LIMIT

Returning to Eq. (6), we expand the integrand in powers of $1/c$ as follows:

$$\begin{aligned} P(x) &= e^{-c}\int_{-\infty}^{\infty}(dy/2\pi)\exp\left[-iy(x-H)+c\int_{-\infty}^{\infty}dz\,P(z)[1+(iyK/c)\tanh z+O(1/c^2)]\right] \\ &= \int_{-\infty}^{\infty}(dy/2\pi)\exp[-iy(x-H-Km)+O(1/c)] \\ &= \delta(x-H-Km)+O(1/c) \ . \end{aligned} \tag{9}$$

In the above we used

$$m=\int_{-\infty}^{\infty} dz\,P(z)\tanh z \ ,$$

which follows from (4) and (5). Thus m obeys the equation

$$m=\tanh(H+Km)+O(1/c) \ . \tag{10}$$

For $c=\infty$, therefore, the model is equivalent to the infinite-range Kac model [for which the Hamiltonian is $\mathrm{H}=-(J/N)\sum_{(ij)}S_iS_j$], and conventional mean-field results are recovered. The density of YL zeros for $c=\infty$ is easily derived from (10). Setting $H=i\theta$ and $m=m_1+im_2$, and equating real and imaginary parts, yields

$$m_1=\tanh(Km_1)\sec^2(\theta+Km_2)/D(m_1,m_2) \ , \tag{11}$$

$$m_2=\tan(\theta+Km_2)\mathrm{sech}^2(Km_1)/D(m_1,m_2) \ , \tag{12}$$

$$D(m_1,m_2)=1+\tanh^2(Km_1)\tan^2(\theta+Km_2) \ . \tag{13}$$

For $K<1$ (i.e., $T>T_C=J$), and θ small, Eqs. (11)–(13) only allow $m_1=0$. Linearizing (11) in m_1 and setting $m_1=0$ in (12) and (13) gives the critical values (Yang-Lee edges) $\theta=\pm\theta_e$ for the appearance of solutions with nonzero m_1:

$$\theta_e=\tan^{-1}\{[(1-K)/K]^{1/2}\}-[K(1-K)]^{1/2} \ . \tag{14}$$

For $1-K\ll 1$, one obtains $\theta_e\simeq\frac{2}{3}(1-K)^{3/2}$. For $K\to 0$, $\theta_e\to\pi/2$. Equations (11)–(13) can be solved numerically for general $K<1$ to obtain $m_1(\theta)$, and hence $\rho(\theta)$ via Eq. (8). The form of the solution is shown schematically in Fig. 3. Near the edges, $\rho(\theta)$ vanishes with a square-root

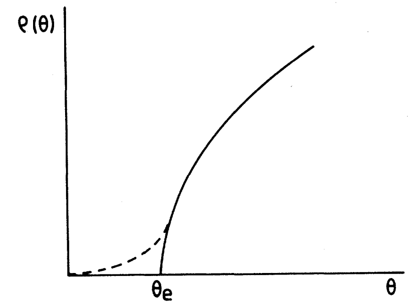


FIG. 3. Density (schematic) of Yang-Lee zeros in the $z=\exp(-2H)$ plane, with $H=i\theta$, for the soluble model considered in this paper. Only positive θ is shown. The continuous curve indicates the mean-field ($c=\infty$) result, obtained from Eqs. (11)–(13), with θ_e the mean-field Yang-Lee edge. The dashed curve shows the contribution (18) (magnified for clarity) from rare, highly coordinated sites.

singularity, $\rho(\theta)\simeq A(K)(|\theta|-\theta_e)^{1/2}$.

We conclude that at the level of mean-field theory (i.e., for $c=\infty$) there are no Griffiths singularities. It is easy to show that order by order in $1/c$ this result is qualitatively unchanged: There remains a well-defined Yang-Lee edge and a vanishing density of YL zeros for small $|\theta|$. For example, to order $1/c$ one obtains for $|\theta|-\theta_e$ small

$$\rho(\theta)\simeq A(K)(|\theta|-\theta_e)^{1/2}+c^{-1}B(K)(|\theta|-\theta_e)^{-1/2}$$
$$\simeq A(|\theta|-\theta_e+2c^{-1}B/A)^{1/2} ,$$

i.e., the naive divergence of the order $1/c$ term at the unperturbed edge can be absorbed by a shift in the position of the edge. While this reduces perturbatively the size of the gap in $\rho(\theta)$, order by order in perturbation theory a gap remains.

These results are reminiscent of those obtained for the eigenvalue density for the exchange matrix, i.e., the matrix whose elements are J_{ij}.[8] For $c=\infty$, the Wigner semicircular distribution is obtained, with sharp band edges. Order by order in $1/c$, the band edges remain sharp, i.e., there is no evidence for Lifshitz tails in the distribution. The latter are associated with *nonperturbative* (in $1/c$) contributions, associated with rare, highly coordinated sites.[8] In Sec. V we show that similar nonperturbative terms are responsible for the Griffiths singularities (which are the analogs for the magnetic problem of the Lifshitz tails of the eigenvalue problem) and that these fill in the gap in $\rho(\theta)$, as indicated by the dashed curve sketched in Fig. 3.

V. THE NONPERTURBATIVE TERM

We return to the fundamental equation (6). For large c we can replace $P(z)$ on the right-hand side by its $c=\infty$ form (9) to obtain

$$P(x)=e^{-c}\int_{-\infty}^{\infty}(dy/2\pi)\exp[-iy(x-H)+c\exp(iyKm_0/c)] ,$$

where m_0 satisfies the mean-field equation (10), i.e., $m_0=\tanh(H+Km_0)$. Expanding the exponential of the exponential as a sum of exponentials,

$$\exp[c\exp(iyKm_0/c)]=\sum_{r=0}^{\infty}(c^r/r!)[\exp(iryKm_0/c)] ,$$

and integrating term by term yields

$$P(x)=e^{-c}\sum_{r=0}^{\infty}(c^r/r!)\delta(x-H-rKm_0/c) .$$

Finally the magnetization m is given by

$$m=\int_{-\infty}^{\infty}dx\,P(x)\tanh x$$
$$=e^{-c}\sum_{r=0}^{\infty}(c^r/r!)\tanh(H+rKm_0/c) . \quad (15)$$

This result has a simple physical interpretation. The probability for a given site to be connected to precisely r other sites is

$$p_r={}^{N-1}C_r(c/N)^r(1-c/N)^{N-1-r}$$
$$\rightarrow e^{-c}c^r/r!$$

for $N\rightarrow\infty$, i.e., the coordination number of a randomly chosen site has a Poisson distribution with mean c. Equation (15) can now be recognized as

$$m=\sum_{r=0}^{\infty}p_r\tanh H_r ,$$

where $H_r=H+rKm_0/c$ is the local field at an r-coordinated site, and m_0, satisfying (10), is the magnetization of each of the neighboring spins.

For $c\gg1$, the sum in (15) is naively dominated by values of r close to c, where p_r has a sharp maximum (of width $\sqrt{c}$). Replacing r by c inside the tanh function, and using $\sum_r p_r=1$, one recovers (10). This would be the desired result for real H. We have seen, however, that for imaginary field $H=i\theta$, (10) gives Re$m=0$ for $|\theta|<\theta_e$. In this regime the dominant contribution to Rem is obtained from rare, highly coordinated sites with $r\gg c$. To see this, we note that for $H=i\theta$, with θ small, the argument of the tanh function is predominantly imaginary, the imaginary part being $\theta+rKm_{02}/c$, where m_{02} is given by Eqs. (11)–(13) for m_2. Since $r\gg c$ will dominate the sum for Rem, the explicit θ dependence can be dropped—an implicit dependence remains though the θ dependence of m_{02}. The real part of the argument of tanh in (15) is infinitesimal for $\theta\rightarrow0$. In this limit, therefore, we can write

$$H+rKm_0/c\rightarrow irKm_{02}/c+\epsilon=ir\alpha+\epsilon ,$$

where $\alpha=Km_{02}/c$ and ϵ is infinitesimal. Substituting into (15) gives

$$\text{Re}m=\text{Re}\,e^{-c}\sum_{r=0}^{\infty}(c^r/r!)\tanh(ir\alpha+\epsilon)$$
$$=e^{-c}\sum_{r=0}^{\infty}\frac{c^r}{r!}\epsilon\frac{1+\cot^2(r\alpha)}{\epsilon^2+\cot^2(r\alpha)} .$$

In the limit $\epsilon \to 0$, the ϵ-dependent terms yield a series of Dirac δ functions at the zeros of $\cot(r\alpha)$. Hence, taking $\epsilon > 0$ without loss of generality,

$$\mathrm{Re}m = \pi e^{-c} \sum_{r=0}^{\infty} (c^r/r!) \sum_{s=1}^{\infty} \delta[r\alpha - (2s-1)\pi/2] . \quad (16)$$

Making an integral representation for each δ function and evaluating the sum over r gives

$$\mathrm{Re}m = \pi e^{-c} \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} (dy/2\pi) \sum_{r=1}^{\infty} (c^r/r!) \exp iy[r\alpha - (s-\tfrac{1}{2})\pi]$$

$$= \pi e^{-c} \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} (dy/2\pi) \exp[-i\pi(s-\tfrac{1}{2})y + c\exp(i\alpha y)] .$$

The change of variable $y = z/\alpha = (c/Km_{02})z$ yields

$$\mathrm{Re}m = (c/2Km_{02}) e^{-c} \sum_{s=1}^{\infty} \int_{-\infty}^{\infty} dz \exp[c g_s(z)] , \quad (17)$$

$$g_s(z) = -(\pi/Km_{02})(s-\tfrac{1}{2}) iz + \exp(iz) . \quad (18)$$

For large c, the integral over z can be evaluated by the method of steepest descents. The function $g_s(z)$ has a saddle point $z_s = -i \ln[\pi(s-\frac{1}{2})/Km_{02}]$, obtained from $(\partial g_s/\partial z)_{z=z_s} = 0$. Deforming the integration contour to pass over the saddle point and integrating away from the saddle point along the line of steepest descent using

$$(\partial^2 g_s/\partial z^2)_{z=z_s} = -[\pi(s-\tfrac{1}{2})/Km_{02}] ,$$

yields

$$\mathrm{Re}m \simeq (c/2Km_{02})^{1/2} e^{-c} \sum_{s=1}^{\infty} (s-\tfrac{1}{2})^{-1/2} \exp\{-c[\pi(s-\tfrac{1}{2})/Km_{02}] \ln[\pi(s-\tfrac{1}{2})/Km_{02}e]\} .$$

For large c, $s=1$ is the dominant term in the sum. For $\theta \to 0$, linearizing (10) in m and H gives $m_{02} \simeq \theta/(1-K)$. Finally, Eq. (8) gives

$$\rho(\theta) \simeq \pi^{-1}(c[1-K]/K\theta)^{1/2} \exp - c[1 + (b/\theta)\ln(b/e\theta)] , \quad (19)$$

$$b = \pi(1-K)/2K , \quad (20)$$

for $\theta \to 0+$. [For $\theta \to 0-$, θ should be replaced by $|\theta|$ in (19).] Now we can justify the claim that sites with $r \gg c$ give the dominant contribution to Rem for $\theta \to 0$. The δ function in (16) with $s=1$ picks out $r = \pi/2\alpha = (\pi/2Km_{02})c \gg c$ for $m_{02} \to 0$ (i.e., $\theta \to 0$).

Before concluding this section, we should mention a technical point which was glossed over in the derivation of (19). The observant reader will have noticed that, due to the periodicity of $\exp(iz)$, the function $g_s(z)$ has additional saddle points at $z = z_s + 2n\pi$, where n is any integer. If the contributions from all the saddle points are included, one obtains precisely zero, unless $(2s-1)\pi/2\alpha$ is an integer for some s, when the result is infinite. Nevertheless, (19) is still correct when suitably interpreted. The reason for these apparently bizarre results follows directly from (16). Unless $(2s-1)\pi/2\alpha$ is an integer, the δ function vanishes for all r; if it is an integer, the δ function gives infinity for one r. This simply reflects the fact that the YL zeros occur at a discrete set of points. In order to define a physically sensible density of zeros, a coarse graining of the density is required. Alternatively, the δ functions may be "smeared out" by, for example, keeping ϵ nonzero in the equation above (16). Then the contributions from the saddle points with $n \neq 0$ are exponentially suppressed for c large and ϵ fixed. Taking the limit $\epsilon \to 0$ *after* the limit $c \to \infty$, one recovers (19).

Equation (19) represents our final result. It gives the leading large-c behavior for the density of YL zeros as $\theta \to 0$. The dependence on θ reveals the expected essential singularity at $\theta = 0$. The temperature dependence of the result is also interesting. The amplitude b in the exponential vanishes at $K=1$, signaling the onset of magnetic long-range order at that temperature. By contrast, b diverges for $K \to 0$ (i.e., $T \to \infty$). This is consistent with the expectation that Griffiths singularities disappear for $T \to T_{G-}$, and the identification $T_G = \infty$ for the present model.

VI. DISCUSSION AND SUMMARY

An exact and explicit result [Eq. (19)] has been obtained for the density of Yang-Lee zeros, in the model defined by (2), for small imaginary magnetic field $H = i\theta$ and large mean coordination number c. The θ dependence has the form of an essential singularity at $\theta = 0$, as expected.[9,10] The amplitude of the essential singularity vanishes at the onset of magnetic long-range order and diverges at the upper-temperature limit of the Griffiths phase, in this case $T = \infty$. These features are characteristic of the expected behavior of $\rho(\theta)$ for dilute Ising models with *short-range* interactions. The detailed form of the essential singularity, $\rho(\theta) \sim \exp\{-[cf(K)/|\theta|]\ln(1/|\theta|)\}$, is, however, slightly different from what we expect for short-range (SR) interactions. Specifically, in the SR case we expect (see below)

$$\rho(\theta)\sim\exp[-A(T)/|\theta|]\ , \qquad (21)$$

with no logarithmic correction, where $A(T_C)=0$ and $A(T_G)=\infty$.

The difference between (19) and (21) reflects the different type of statistical fluctuations responsible for Griffiths singularities: For the infinite-range interactions of the present model, we have seen that rare, highly coordinated sites give the dominant contribution to $\rho(\theta)$ for small $|\theta|$; for SR interactions, large regions characteristic of the ordered phase at the given temperature will in general dominate.[3,4]

The form (21) can be derived explicitly in one part of the phase diagram, namely at very low temperatures below the percolation threshold, where the system consists of isolated finite clusters. For infinitesimal T, each cluster has significant statistical weight only for the two states in which all spins in the cluster are aligned parallel, or all are aligned antiparallel, to the external field (assumed weak). Then a cluster of r spins has mean magnetization per spin $m_r=\tanh(rH)$. But the probability that a randomly chosen site belongs to a cluster of r spins is, for large r, $p_r\sim\exp(-ar)$, up to an algebraic (in r) prefactor, where a vanishes at the percolation threshold.[11] Thus the mean magnetization per site is

$$m=\sum_r p_r m_r\sim\sum_r \exp(-ar)\tanh(rH)\ .$$

The same manipulations which lead from (15) to (19) now yield, for $H=i\theta$,

$$\mathrm{Re}\,m\sim\exp(-\pi a/2|\theta|)\ , \qquad (22)$$

up to an algebraic (in $|\theta|$) prefactor.

It is interesting that (22) also holds for the present model below the percolation threshold (i.e., for $c<1$) at infinitesimal temperature. This is because the cluster size distribution has precisely the form assumed above, with[6] $a=c-1-\ln c$. This also agrees with the result obtained by Harris[9] for the diluted Bethe lattice, to which the present model is equivalent[6] for $c<1$.

A heuristic argument that the form (21) holds generally for short-range interactions can be constructed following the approach used by Bray[3] in a related context. In its simplest form, the argument assumes that the dominant contribution to $\rho(\theta)$ for small θ comes from large compact regions of fully occupied sites or bonds (for site or bond dilution, respectively). Since, for $T_C<T<T_G$, $\rho(0)$ is nonzero for the pure system, and proportional to the spontaneous magnetization per site m, a fully occupied compact region of linear dimension L will contribute a closest (to the real axis) YL zero at $\theta\sim 1/mL^d$ (for spatial dimension d). Even if the compact region is not an isolated cluster, the extensivity of $\rho(\theta)$ suggests that the position of the zero will be only weakly perturbed by the coupling to the rest of the system. Now the probability per site to belong to a compact cluster of size L is of order $\exp(-cL^d)$, where c depends on the site/bond occupation probability p. Hence the large compact clusters yield a contribution to $\rho(\theta)$ of the form (21), with $A(T)\sim c/m$. In particular, A is predicted to diverge as $(T_G-T)^{-\beta}$ for $T\to T_{G-}$, where β is the critical exponent of the pure system. Contributions from zeros other than the closest are exponentially suppressed for $\theta\to 0$, just as the contributions to (16) from $s>1$ are suppressed.

The above argument yields, in fact, a lower bound for $\rho(\theta)$, since only a subset of possible contributions is included, namely those due to regions of pure system. A refinement of the argument, including more contributions, can be given following Bray.[3] The idea is to include regions characterized by a local site/bond concentration $p'>p$, such that the regions are within the ordered phase of the bulk at the given temperature, i.e., $T<T_C(p')$. The closest zero for such a region is then $\theta\sim 1/m(p')L^d$ [where $m(p')$ is the magnetization per site in the bulk for concentration p'], while the probability weight for such a region is of order $\exp[-f(p')L^d]$, where $f(p')$ is given by Eq. (2) of Bray.[3] In particular, $f(p')$ vanishes quadratically for $p'\to p$: $f(p')\sim(p'-p)^2$. For fixed p', Eq. (21) is recovered with $A\sim f(p')/m(p')$. The optimal choice of p' is that which minimizes A.

This approach is particularly fruitful for p close to $p_C(T)$, the boundary between Griffiths and ferromagnetic phases. Then we anticipate that the optimal p' will also be close to $p_c(T)$. Using $m(p')\sim[p'-p_C(T)]^{\beta_r}$, where β_r is the critical exponent of the *dilute* system, yields

$$A\sim\min_{p'}(p'-p)^2/[p'-p_C(T)]^{\beta_r}$$
$$\sim[p_C(T)-p]^{2-\beta_r}\sim[T-T_C(p)]^{2-\beta_r}\ .$$

Thus A vanishes as $T\to T_C$, consistent with a nonvanishing $\rho(0)$ for $T<T_C$.

In conclusion, exact results for $\rho(\theta)$ obtained previously have been limited to infinitesimal temperature below the percolation threshold. The result (19) is, as far as we are aware, the first exact result for $\rho(\theta)$ above the percolation threshold for any model, and holds for general temperatures in the Griffiths phase. While heuristic arguments suggest that the detailed form of the essential singularity in $\rho(\theta)$ may differ from (19) in short-range models [probably by the absence of the logarithm, as in (21)], the qualitative features, in particular the temperature dependence, are correctly captured by (19).

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*On leave of absence from Department of Physics, Hunan Normal University, Changsha, Hunan, People's Republic of China.

[1]R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969).
[2]C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952); **87**, 410 (1952).
[3]M. Randeria, J. P. Sethna, and R. G. Palmer, Phys. Rev. Lett. **54**, 1321 (1985); A. J. Bray, *ibid.* **59**, 586 (1987).
[4]D. Dhar, in *Stochastic Processes: Formalism and Applications,* edited by G. S. Agarwal and S. Dattagupta (Springer, Berlin, 1983); A. J. Bray, Phys. Rev. Lett. **60**, 720 (1988); D. Dhar, M. Randeria, and J. P. Sethna, Europhys. Lett. **5**, 485 (1988); A. J. Bray and G. J. Rodgers, J. Phys. C **21**, L243 (1988); Phys. Rev. B **38**, 9252 (1988); S. Jain, J. Phys. C **21**, L1045 (1988); A. J. Bray, J. Phys. A **22**, L81 (1989); S. Colborne and A. J. Bray, *ibid.* **22**, 2505 (1989).
[5]L. Viana and A. J. Bray, J. Phys. C **18**, 3037 (1985).
[6]A. J. Bray and G. J. Rodgers, Phys. Rev. B **38**, 11 461 (1988).
[7]I. Kanter and H. Sompolinsky, Phys. Rev. Lett. **58**, 164 (1987).
[8]G. J. Rodgers and A. J. Bray, Phys. Rev. B **37**, 3557 (1988).
[9]A. B. Harris, Phys. Rev. B **12**, 203 (1975).
[10]M. Schwartz, Phys. Rev. B **18**, 2364 (1978).
[11]H. Kunz and B. Souillard, J. Stat. Phys. **19**, 77 (1978).