PHYSICAL REVIEW B VOLUME 40, NUMBER 10 1 OCTOBER 1989

Gap energy and long-range order in the boson-fermion model of superconductivity

R. Friedberg and T. D. Lee
Columbia University, New York, New York 10027
(Received 23 May 1989)

From the observations of very small coherent lengths for all high-temperature superconductors, we conclude that the pair state is reasonably well localized in the coordinate space and therefore can be represented phenomenologically by a local boson field ϕ. The underlying mechanism for superconductivity is assumed to be through the "s-channel" reaction $2e \rightarrow \phi \rightarrow 2e$. This leads to a mixed boson-fermion model. We examine the long-range order, gap energy, and Meissner effect in such a theory.

I. INTRODUCTION

A. Small coherence length

The observation[1,2] of a small "coherence length" $\xi(\approx 10$ Å) in the newly discovered high-temperature superconductors[3,4] indicates that the pairing between electrons, or holes, in these materials is reasonably localized in the coordinate space. Hence, the pair state can be well approximated by a phenomenological local boson field $\phi(\mathbf{r})$, whose mass M is $\approx 2m_e$ and whose elementary charge unit is $2e$, where m_e and e are the mass and charge of an electron. It follows then that the transition

$$2e \rightarrow \phi \rightarrow 2e \tag{1.1}$$

must occur, in which e denotes either an electron or a hole; furthermore, the localization of ϕ implies that phenomena at distances larger than the physical extension of ϕ [which is $O(\xi)$] are insensitive to the interior of ϕ. Since ξ is of the same order as the scale of a lattice unit cell, it becomes possible to develop a phenomenological theory of superconductivity based *only* on the local character of ϕ. The purpose of this and a previous paper[5] is to demonstrate that this is indeed the case.

Of course, physics at large does depend on several overall properties: the spin of ϕ, the stability of an individual ϕ quantum, the isotropicity and homogeneity (or their absence) of the space containing ϕ, and so on. The situation is analogous to that in particle physics: the smallness of the radii of pions, ρ-mesons, kaons, ... makes it possible for us to handle much of the dynamics without any reference to their internal structure, such as quark-antiquark pairs or bag models. Hence, the origin of their formation becomes a problem separate from the description of their mechanics. An important ingredient in this type of phenomenological approach is the selection of the basic interaction Hamiltonian that describes the underlying dominant process. In the usual low-temperature superconductors, ξ varies from about 10^4 to a few hundred Å. The corresponding pairing state ϕ is too extended and ill defined in the coordinate space; therefore (1.1) does not play an important role. Instead, the BCS theory of superconductivity[6] is based on the emission and absorption of phonons,

$$2e \rightarrow 2e + \text{phonon} \rightarrow 2e \ . \tag{1.2}$$

In the language of particle physics, (1.1) is an s-channel process, while (1.2) is t channel. (See Fig. 1 for nomenclature.) The BCS theory may be called the t-channel theory. As we shall see, the s-channel reaction (1.1) leads to a new theory (which, however, shares many features in common with the BCS theory) of superconductivity, whose validity rests only on the localization of ϕ, and is independent of the detailed microscopic origin of the pairing mechanism; in addition, its long-range order can be represented by the macroscopic occupation number of the zero-momentum bosons, as in the Bose-Einstein condensation. Together, these two (s-channel and t-channel) formulations provide a rich body of theoretical means, which may prove useful in analyzing the large variety of superconductivity and superfluidity phenomena that exist in nature.

The use of a boson field for the superfluidity of liquid He II has had a long history. However, there are some major differences in the following application to (high-temperature) superconductors.

(1) The ϕ quantum is charged, carrying $2e$, while the helium atom is neutral.

(2) We assume each individual ϕ quantum to be unstable, with 2ν as its excitation energy. (As we shall see, this assumption makes it possible for the s-channel theory to exhibit many BCS-like characteristics, yet without the isotope effect.)

In the rest frame of a single ϕ quantum, the decay

$$\phi \rightarrow 2e \tag{1.3}$$

occurs, in which each e carries an energy

$$\frac{k^2}{2m} = \nu \ .$$

Consequently, in a large system, there are macroscopic numbers of both bosons (the ϕ quanta) and fermions (electrons or holes), distributed according to the principles of statistical mechanics.

At temperature $T < T_c$, there is always a macroscopic

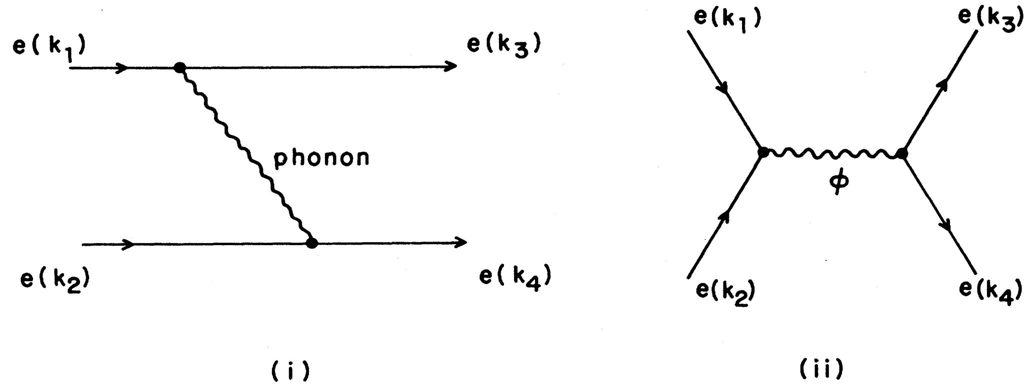

FIG. 1. In $e(k_1)+e(k_2)\rightarrow e(k_3)+e(k_4)$, in terms of the two initial and two final four-momenta there are only two independent kinematic scalars (excluding $k_1^2,\ldots,k_4^2$). It is customary to label $s=(k_1+k_2)^2=(k_3+k_4)^2$ and $t=(k_1-k_3)^2=(k_4-k_2)^2$. (i) is a t-channel reaction and (ii) is s channel.

distribution of zero-momentum bosons coexisting with a Fermi distribution of electrons (or holes). Take the simple example of *zero* temperature. Let ε_F be the Fermi energy. When $\varepsilon_F=\nu$, the decay $\phi\rightarrow 2e$ cannot take place because of the exclusion principle; therefore the bosons are present. Even when $\varepsilon_F<\nu$, there is still a macroscopic number of (virtual) zero-momentum bosons in the form of a static coherent field amplitude whose source is the fermion pairs. This then leads to the following essential features of the present boson-fermion model.

As we shall see,[5] below the critical temperature T_c the long-range order in the boson-fermion model can always be described by the zero-momentum bosonic amplitude B of the ϕ field, as in the Bose-Einstein condensation (and therefore similar to liquid He II). Because of the transition (1.1), the zero momentum of the boson in the condensate forces the two e's to have equal and opposite momenta, forming a Cooper pair. Therefore the same long-range order also applies to the Cooper pairs of the fermions. Furthermore, the gap energy Δ of the fermion system is related to B by

$$\Delta^2=|gB|^2 ,$$

where g is the coupling for $\phi\rightarrow 2e$.

B. T_c versus (carrier density/mass)

Recently, Uemura *et al.*[7] discovered that in all (high-temperature) cupric superconductors there is a universality law:

$$T_c\propto\rho/m^* , \tag{1.4}$$

where ρ is the number density of superconducting charge carriers and m^* their effective mass; the proportionality constant is the same for all materials, about

$$40\ \mathrm{K}\ \ \text{to}\ \ 4\times10^{20}\ \mathrm{cm}^{-3}/m_e , \tag{1.5}$$

assuming each carrier bears a charge e. Uemura *et al.* point out that in a two-dimensional system the density of a Fermi distribution is proportional to the Fermi energy ε_F. In these cupric superconductors, the charge carriers concentrate on the two-dimensional CuO_2 plane; their tunneling between these planes gives rise to the three-dimensional character. The average separation c between CuO_2 planes is approximately constant for different materials:

$$c\cong 6\ \text{Å} .$$

Introducing a two-dimensional density

$$\sigma\equiv\rho c ,$$

one may express (1.4) as

$$T_c\propto\sigma/m^* . \tag{1.4$'$}$$

Because in a two-dimensional Fermi distribution of mass m^*,

$$\varepsilon_F=\pi\sigma/m^* ,$$

the experimental results (1.4) and (1.5) can also be stated as

$$\varepsilon_F/k_BT_c\cong 16$$

for all cupric superconductors where k_B is the Boltzmann constant. (The above ratio is independent of the effective mass m^*.) However, the BCS theory relates T_c not to ε_F but to a Debye frequency $\omega_D\ll\varepsilon_F$, which is quite different. Furthermore, it seems difficult to account for the large constant ratio 16. (See, however, the mechanism proposed by Emery and Reiter,[8] in which ω_D is replaced by a much higher-energy scale $\sim\varepsilon_F$.) In an s-channel theory, the relevant fermionic energy scale is ε_F; therefore it is natural to have $T_c\propto\varepsilon_F$. [See, e.g., (5.23) below.]

One may explore an alternate possibility. If the measured ρ is interpreted as due to bosons of charge $2e$ on planes with the same spacing c, the proportionality constant (1.5) would be reduced by a factor 4; it becomes

$$40\ \mathrm{K}\ \ \text{to}\ \ 10^{20}\ \mathrm{cm}^{-3}/m_e . \tag{1.5$'$}$$

In a Bose-Einstein transition, the following two lengths should be of comparable size:

$$d \equiv \sigma^{-1/2} \quad \text{and} \quad \lambda_T \equiv \sqrt{2\pi/Mk_BT_c} \ ,$$

where d is the interparticle distance, λ_T the thermal wavelength, and the boson mass M is now the carrier's mass m^*. Set $T_c=40$ K. Using (1.5′) and $c \cong 6$ Å, one finds the corresponding two-dimensional density-to-mass ratio to be

$$\sigma/M = \rho c/m^* \cong 6\times 10^{12}\ \text{cm}^{-2}/m_e \ .$$

Hence, in terms of the boson picture, the experimental results (1.4) and (1.5) may also be stated as

$$\lambda_T^2\sigma \cong 8 \ , \tag{1.6}$$

i.e.,

$$\lambda_T/d \cong 2\sqrt{2}$$

for all cupric superconductors. (Again these numbers are independent of M; i.e., m^*.)

For an ideal two-dimensional boson system, there is no Bose-Einstein condensation; the corresponding values for (1.6) would be logarithmically ∞. However, the cupric superconductors are three-dimensional structures, made of parallel layers of CuO_2 planes. Even without a definite theoretical idea, one may approach the problem heuristically by using the ideal two-dimensional boson formula but introducing an infrared cutoff l^{-1} for the boson momentum k; this gives

$$\lambda_T^2\sigma = \ln(2Ml^2k_BT) \cong 8$$

and (for $M \sim 2m_e$ and $T \sim 40$ K)

$$l \sim 10^3 \ \text{Å} \ ,$$

which is of a reasonable magnitude. (The same formula would imply a variation of about 10% from La, 214, to Th, 2223.)

In practice, there can be several candidates for l: the transitions between CuO_2 planes render the system three dimensional and give rise to superconductivity and Meissner effect (the typical London length is about the same order as the above l). Alternatively, in our model the logarithmical divergence is removed by the presence of fermions, which (as we shall see) can cause a change in the bosonic low-energy excitation spectrum; in addition the phase transition can take place at a chemical potential lower than the boson threshold. We do not yet have a clear theoretical understanding of l. Nevertheless it seems promising to interpret (1.4′) as due to some combined two-dimensional action of fermionic Cooper pairs and bosonic pairing states. All these possibilities give an important additional impetus for the study of the boson-fermion model. [The charge density deduced from the recent muon spin-resonance experiment[7] is lower than that inferred from stoichiometric measurements; the new value gives (1.6), which alters our previous view[5] concerning the relevance of Bose-Einstein condensation.]

C. A prototype s-channel model

In this paper, we discuss the prototype of a boson-fermion model (or an s-channel theory of superconductivity). We assume ϕ to be of spin 0 and that the space containing ϕ is a three-dimensional homogeneous and isotropic continuum (except in Sec. VII when we discuss the two-dimensional model). As noted before, for realistic applications, a more appropriate approximation of the latter would be the product of a two-dimensional x,y continuum (simulating the CuO_2 plane) and a discrete lattice of spacing c along the z direction. The two-dimensional layer character of CuO_2 planes helps in the localization of the pair state in the z direction, making the ϕ quantum disc shaped. The space that ϕ moves in becomes a three-dimensional continuum when $c \to 0$, but two dimensional when $c \to \infty$. This interesting case, plus the generalization to higher spin, are planned to be discussed in a separate publication.

Here we consider an idealized system consisting of the local scalar field ϕ and the electron (or hole) field ψ_σ where $\sigma = \uparrow$ or $\downarrow$ denotes the spin. The Hamiltonian is ($\hbar=1$)

$$H = H_0 + H_1 \tag{1.7}$$

in which the free Hamiltonian is

$$H_0 = \int \left[\phi^\dagger \left[2\nu_0 - \frac{1}{2M}\nabla^2\right]\phi + \psi_\sigma^\dagger \left[-\frac{1}{2m}\nabla^2\right]\psi_\sigma\right] d^3r \tag{1.8}$$

with the repeated spin index σ summed over and $\dagger$ denoting the Hermitian conjugate. The interaction H_1 can be either a local Hamiltonian,

$$H_1 = g\int (\phi^\dagger \psi_\uparrow \psi_\downarrow + \text{H.c.})d^3r \ , \tag{1.9}$$

or a nonlocal one,

$$H_1 = g\int d^3r \int d^3l \left[\phi^\dagger(\mathbf{r})\psi_\uparrow\left[\mathbf{r}+\frac{\mathbf{l}}{2}\right] \times \psi_\downarrow\left[\mathbf{r}-\frac{\mathbf{l}}{2}\right] + \text{H.c.}\right]\hat{u}(\mathbf{l}) \tag{1.9′}$$

with the coupling constant g and the form factor $\hat{u}(\mathbf{l})$ both real, and $\hat{u}(\mathbf{l})$ satisfying

$$\int \hat{u}(\mathbf{l})d^3l = 1 \ .$$

Both ϕ and ψ_σ are the usual quantized field operators whose equal-time commutator and anticommutator are

$$[\phi(\mathbf{r}), \phi^\dagger(\mathbf{r}')] = \delta^3(\mathbf{r}-\mathbf{r}') \tag{1.10a}$$

and

$$\{\psi_\sigma(\mathbf{r}), \psi_{\sigma'}^\dagger(\mathbf{r}')\} = \delta_{\sigma\sigma'}\delta^3(\mathbf{r}-\mathbf{r}') \ . \tag{1.10b}$$

The total particle number operator is defined to be

$$N = \int (2\phi^\dagger\phi + \psi_\sigma^\dagger\psi_\sigma)d^3r \tag{1.11}$$

which commutes with H and is therefore conserved.

Expand the field operators in Fourier components inside a volume Ω with periodic boundary conditions:

$$\psi_\sigma(\mathbf{r}) = \sum_k \Omega^{-1/2} a_{\mathbf{k},\sigma} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.12a}$$

and

$$\phi(\mathbf{r})=\sum_k \Omega^{-1/2} b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1.12b}$$

with $\{a_{\mathbf{k},\sigma}, a^\dagger_{\mathbf{k}',\sigma'}\}=\delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma\sigma'}$, $[b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}]=\delta_{\mathbf{k},\mathbf{k}'}$, etc. Equation (1.9′) can then be written as

$$H_1=\frac{g}{\sqrt{\Omega}}\sum_{p,k}(b^\dagger_{\mathbf{p}} a_{\mathbf{p}/2+\mathbf{k},\uparrow} a_{\mathbf{p}/2-\mathbf{k},\downarrow} u_{\mathbf{k}} + \text{H.c.}) \ , \tag{1.13}$$

where

$$u_{\mathbf{k}}=\int \hat{u}(l) e^{i\mathbf{k}\cdot l} d^3 l \ . \tag{1.14}$$

In (1.7), $2\nu_0$ is the "bare" excitation energy of ϕ. Because of the interaction, the "physical" (i.e., renormalized) excitation energy 2ν in reaction (1.3) is given by

$$2\nu=2\nu_0+\frac{g^2}{2\Omega}\sum_k \text{P}\frac{|u_k|^2}{\nu-\omega_k} \ , \tag{1.15}$$

where P denotes the principal value and

$$\omega_k=\frac{k^2}{2m} \ . \tag{1.16}$$

The decay width Γ is given by

$$\Gamma=(g^2/\pi)m^{3/2}\sqrt{\nu/2}|u_k|^2 \tag{1.17}$$

at $k=(2m\nu)^{1/2}$. In the following, we assume for simplicity

$$u_k=1 \quad \text{for } k\le\Lambda \ . \tag{1.18}$$

Because the theory (1.7)–(1.9) is renormalizable, most of the physical applications are insensitive to the ultraviolet cutoff Λ; i.e., we may take $\Lambda=\infty$.

In Sec. II we examine the definition of the long-range-order parameter B. The thermodynamic functions are calculated in Sec. IV by using the Bogoliubov-Valatin transformation[9,10] introduced in Sec. III. The details of temperature and density variations of the gap energy Δ and the chemical potential μ are given in Sec. V. One sees how the typical BCS-type formulas can be analytically connected to the standard Bose-Einstein expressions. In this way, these two approaches become further unified in the boson-fermion model.

The Meissner effect[11] is examined in Sec. VI. For completeness, we give a self-contained analysis of the (well-known) spontaneous symmetry-breaking mechanism. The difference between the particle number $N=0$ sector (as in the usual standard electroweak theory in elementary particle physics) and the N macroscopic and $\neq 0$ sector (important for superconductivity) is emphasized. For example, in a relativistic theory without the electromagnetic coupling (i.e., $e=0$), the Nambu-Goldstone boson[12,13] travels with the velocity of light c in the former case, but with the sound velocity $\ll c$ in the latter. Yet, all these can be brought into a single formulation. As we shall see, the s-channel theory has an intrinsically simpler structure than the t-channel theory; this makes it possible to take a deductive approach, thereby rendering the model attractive on the pedagogical level.

In Sec. VII we discuss the case when the space dimension $D=2$. It is well known[14] that the long-range-order parameter B of the Bose condensate disappears in this case. We give arguments which suggest that the gap energy $\Delta=g(B^*B)^{1/2}$ may remain. If so, there is still a phase transition in $D=2$, and it should at least exhibit *quasisuperconductivity*.

II. LONG-RANGE ORDER

To derive the long-range-order parameter, it is convenient to add to the Hamiltonian H of (1.7) an infinitesimal term that breaks the N conservation. Define

$$H_j\equiv H+\int(j^*\phi+j\phi^\dagger)d^3r \ , \tag{2.1}$$

where j is a constant (i.e., $\mathbf{r}$-independent) infinitesimal. The corresponding grand partition function is

$$Q_j=\text{tr}e^{-\beta(H_j-\mu N)} \tag{2.2}$$

with μ being the chemical potential (same as the Gibbs thermodynamic function per particle), and $\beta=(k_B T)^{-1}$. The ensemble average of any operator O is

$$\langle O\rangle\equiv Q_j^{-1}\text{tr}(Oe^{-\beta(H_j-\mu N)}) \ . \tag{2.3}$$

Regard $\ln Q_j$ as a function of j, j^*, and μ (besides T and Ω). Because the partial derivatives of $H_j-\mu N$ with respect to j^*, j, and μ are $\Omega^{1/2}b_0$, $\Omega^{1/2}b_0^\dagger$, and $-N$, we have (with b_0, $b_0^\dagger$ denoting $b_{\mathbf{k}}$ and $b^\dagger_{\mathbf{k}}$ at $\mathbf{k}=0$)

$$\begin{aligned}\langle b_0\rangle&=-k_B T\Omega^{-1/2}\frac{\partial \ln Q_j}{\partial j^*} \ ,\\ \langle b_0^\dagger\rangle&=-k_B T\Omega^{-1/2}\frac{\partial \ln Q_j}{\partial j} \ ,\end{aligned} \tag{2.4}$$

and

$$\langle N\rangle=k_B T\frac{\partial \ln Q_j}{\partial\mu} \ . \tag{2.5}$$

The long-range-order parameter B is given by the double limit

$$B\equiv\lim_{j\to 0}\lim_{\Omega\to\infty}\Omega^{-1/2}\langle b_0\rangle \ . \tag{2.6}$$

As we shall see, below the critical temperature T_c, $B\neq 0$; furthermore,

$$B^*B=\lim_{j\to 0}\lim_{\Omega\to\infty}\Omega^{-1}\langle b_0^\dagger b_0\rangle \ . \tag{2.7}$$

The order of double limit in (2.6) is important, since

$$\lim_{\Omega\to\infty}\lim_{j\to 0}\langle b_0\rangle=0 \ , \tag{2.8}$$

even though in the same double limit

$$\lim_{\Omega\to\infty}\lim_{j\to 0}\Omega^{-1}\langle b_0 b_0\rangle=B^*B \ , \tag{2.7$'$}$$

identical to (2.7).

Introduce p_j as the Legendre transform of $\Omega^{-1}k_B T\ln Q_j$ in the limit of infinite volume:

$$p_j \equiv \lim_{\Omega\to\infty} \Omega^{-1} \left[k_B T \ln Q_j + \left\langle \int (j^*\phi + j\phi^\dagger) d^3r \right\rangle \right] \tag{2.9}$$

which, through (2.4), can be regarded as a function of

$$\begin{aligned} B_j &\equiv \lim_{\Omega\to\infty} \Omega^{-1/2} \langle b_0 \rangle , \\ B_j^* &\equiv \lim_{\Omega\to\infty} \Omega^{-1/2} \langle b_0^\dagger \rangle , \end{aligned} \tag{2.10}$$

and μ. We find

$$\frac{\partial p_j}{\partial B_j} = j^* \tag{2.11}$$

and

$$\frac{\partial p_j}{\partial B_j^*} = j \ . \tag{2.12}$$

In the limit $j \to 0$ we have $B_j \to B$, $B_j^* \to B^*$, and p_j becomes the (physical) pressure of the system

$$p \equiv \lim_{j\to 0} p_j \ . \tag{2.13}$$

To evaluate the ground-state energy $\mathcal{E}_{\text{gd}}$ of H_j, or the partition function Q_j, we may regard H_1 of (1.9) as the perturbation and

$$H_0 + \int (j^*\phi + j\phi^\dagger) d^3r \tag{2.14}$$

as the zeroth-order Hamiltonian. Either $\mathcal{E}_{\text{gd}}$ or $\ln Q_j$ can be expressed as a sum of one-loop, two-loop, . . . , diagrams of the perturbation series. The summation of all one-loop diagrams can be done explicitly. In the next section, we shall calculate the same sum by a simpler method through the use of a canonical transformation U which is a product of a Bogoliubov-Valatin transformation[9,10] times a translation in the ϕ space.

A. Remark

Since the theory is invariant under a constant phase transformation

$$\phi \to \phi e^{2i\alpha}$$

and (2.15)

$$\psi_\sigma \to \psi_\sigma e^{i\alpha} ,$$

the absolute phase angle of the long-range parameter B is not an observable. The introduction of an infinitesimal symmetry-breaking j which determines the phase of B [given by (4.11) in Sec. IV] is only a mathematical device. All physical quantities depend on B^*B. However, the relative phase between $\phi(x)$ at two different spacelike separated points x and x' is an observable. It is the long-range coherence in this relative phase that gives rise to the superfluidity of liquid He II, in the example when the ϕ field represents a macroscopic system of helium atoms. When ϕ carries an electric charge, as is the case here, with the inclusion of the electromagnetic field the theory is also invariant under an arbitrary x-dependent $\alpha(x)$ transformation. Superfluidity is then connected with the long-range coherence of the gauge-invariant relative phase

$$\phi^\dagger(x') \exp\left[2ie \int_x^{x'} A_\mu dx_\mu \right] \phi(x) , \tag{2.16}$$

where A_μ is the electromagnetic four-potential, $2e$ is the charge carried by ϕ, and x and x' have spacelike separation.

III. A CANONICAL TRANSFORMATION

Let

$$\begin{aligned} \tilde{a}_{\mathbf{k},\uparrow} &= a_{\mathbf{k},\uparrow} \cos\theta_k - e^{i\gamma} a^\dagger_{-\mathbf{k},\downarrow} \sin\theta_k , \\ \tilde{a}_{-\mathbf{k},\downarrow} &= e^{i\gamma} a^\dagger_{\mathbf{k},\uparrow} \sin\theta_k + a_{-\mathbf{k},\downarrow} \cos\theta_k , \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \tilde{b}_0 &= b_0 - \Omega^{1/2} B , \\ \tilde{b}_{\mathbf{k}} &= b_{\mathbf{k}} \quad (\mathbf{k} \neq 0) , \end{aligned} \tag{3.2}$$

where

$$B = |B| e^{i\gamma} , \tag{3.3}$$

$$\begin{aligned} &\sin 2\theta_k = g|B|/E_k , \quad \cos 2\theta_k = (\omega_k - \mu)/E_k , \\ &E_k = [(\omega_k - \mu)^2 + g^2 |B|^2]^{1/2} . \end{aligned} \tag{3.4}$$

Evidently $\{\tilde{a}_{\mathbf{k},\sigma}, \tilde{a}^\dagger_{\mathbf{k}',\sigma'}\} = \delta_{\mathbf{k},\mathbf{k}'} \delta_{\sigma\sigma'}$, $[\tilde{b}_{\mathbf{k}}, \tilde{b}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$; the transformation (3.1) and (3.2) is therefore canonical.

Then from (1.7)–(1.11) and (2.1) we have

$$\mathcal{H} \equiv H_j - \mu N = \sum_k \left[\left(\frac{k^2}{2M} + 2(\nu_0 - \mu) \right) b^\dagger_{\mathbf{k}} b_{\mathbf{k}} + (\omega_k - \mu) a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k},\sigma} \right] + H_1 + \Omega \varepsilon_j ,$$

which can be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \Omega \varepsilon_j + \mathcal{H}_{\text{int}} , \tag{3.5}$$

where, in terms of the transformed operators $\tilde{a}_{\mathbf{k},\sigma}$ and $\tilde{b}_{\mathbf{k}}$,

$$\mathcal{H}_0 = \sum_k (\omega_k - \mu - E_k) + 2(\nu_0 - \mu) \Omega B^* B + \sum_k \left[\left(\frac{k^2}{2M} + 2(\nu_0 - \mu) \right) \tilde{b}^\dagger_{\mathbf{k}} \tilde{b}_{\mathbf{k}} + E_k \tilde{a}^\dagger_{\mathbf{k},\sigma} \tilde{a}_{\mathbf{k},\sigma} \right] , \tag{3.6}$$

$$\mathcal{H}_1 = 2(\nu_0 - \mu) \Omega^{1/2} (B^* \tilde{b}_0 + B \tilde{b}^\dagger_0) , \tag{3.7}$$

$$\begin{aligned} \varepsilon_j &= \Omega^{-1/2} (j^* b_0 + j b_0^\dagger) \\ &= j^* (B + \tilde{b}_0 / \Omega^{1/2}) + j (B^* + \tilde{b}^\dagger_0 / \Omega^{1/2}) , \end{aligned} \tag{3.8}$$

and $\mathcal{H}_{\text{int}}$ is cubic in $\tilde{b}_{\mathbf{k}}$, $\tilde{a}_{\mathbf{k},\sigma}$, and their Hermitian conjugate (but independent of j). The partition function (2.2) can be evaluated by using

$$Q_j = \text{tr} e^{-\beta \mathcal{H}} . \tag{3.9}$$

In (3.2), B is just a constant parameter. Anticipating (2.6) and (2.10), we require (since $j\to 0$ in the end)

$$\Omega^{1/2}B=Q_j^{-1}\mathrm{tr}(b_0 e^{-\beta\mathcal{H}}) , \tag{3.10}$$

which, because of (3.2), is equivalent to

$$Q_j\langle\tilde{b}_0\rangle=\mathrm{tr}(\tilde{b}_0 e^{-\beta\mathcal{H}})=0 . \tag{3.11}$$

We note that $\tilde{a}_{\mathbf{k},\sigma},\tilde{b}_{\mathbf{k}}$ can also be obtained from the unitary transformation defined by $U=U_1U_2$ where

$$U_1=\exp(\Omega^{1/2}Bb_0^\dagger-\Omega^{1/2}B^*b_0) , \tag{3.12}$$

$$U_2=\exp\sum_k\theta_k(e^{-i\gamma}a_{\mathbf{k},\uparrow}a_{-\mathbf{k},\downarrow}-e^{i\gamma}a^\dagger_{-\mathbf{k},\downarrow}a^\dagger_{\mathbf{k},\uparrow}) \tag{3.13}$$

as

$$\tilde{a}_{\mathbf{k},\sigma}=U_2a_{\mathbf{k},\sigma}U_2^\dagger,\quad \tilde{b}_{\mathbf{k}}=U_1b_{\mathbf{k}}U_1^\dagger . \tag{3.14}$$

The ground state of $\mathcal{H}_0$ is just $U|0\rangle$ where $a_{\mathbf{k},\sigma}|0\rangle=b_{\mathbf{k}}|0\rangle=0$ for all $\mathbf{k},\sigma$. (Therefore $\tilde{a}_{\mathbf{k},\sigma}U|0\rangle=\tilde{b}_{\mathbf{k}}U|0\rangle=0$.) The state $U|0\rangle$ is analogous to the trial wave function in BCS theory. Our theory has the mathematical advantage that the corrections to $U|0\rangle$ can be evaluated perturbatively in a systematic way.

We may either regard H_1 of (1.9) as the perturbation or, alternatively, regard the corresponding

$$\mathcal{H}_1+\mathcal{H}_{\mathrm{int}} \tag{3.15}$$

as the perturbation. The former contains one-loop as well as two-loop, three-loop, ..., diagrams; the latter has *no* one-loop diagram when (3.11) is satisfied. The application of the canonical transformation (3.1) and (3.2) is then equivalent to summing over all one-loop diagrams. Note that $\mathcal{H}_1$ is linear in $\tilde{b}_0$ and $\tilde{b}_0^\dagger$; because of (3.11) it is reasonable to include $\mathcal{H}_1$ as part of the perturbation (3.15). Condition (3.11) is equivalent to (2.10), which led to (2.11) and (2.12). As we shall show, it also determines B. [See (4.3) for $|B|$ and (4.11) for the phase of B.]

IV. THERMODYNAMIC FUNCTIONS

The zeroth-order Q_j is $\mathrm{tr}e^{-\beta(\mathcal{H}_0+\Omega\epsilon_j)}$. By using (3.6), (3.8), (2.9), and (2.13) we find that the zeroth-order expression of the pressure p as a function of T, μ, and B is (setting $j\to 0$ in the end)

$$\begin{aligned}p=&-2(\nu_0-\mu)|B|^2+\Omega^{-1}\sum_k(E_k+\mu-\omega_k)\\&+2(\beta\Omega)^{-1}\sum_k\ln(1+e^{-\beta E_k})\\&-(\beta\Omega)^{-1}\sum_k\ln\{1-\exp\beta[2\mu-2\nu-(k^2/2M)]\} ,\end{aligned} \tag{4.1}$$

where, as before, $\omega_k=k^2/2m$ and

$$E_k=[(\omega_k-\mu)^2+g^2|B|^2]^{1/2} . \tag{4.2}$$

On account of (2.11)–(2.13), $(\partial p/\partial|B|)_{\mu,T}=0$, which gives

$$\nu_0-\mu-\Omega^{-1}\frac{g^2}{4}\sum_k\frac{1}{E_k}\tanh\tfrac{1}{2}\beta E_k=0 .$$

[This is obtained to leading order by using (4.1) which omits $\mathcal{H}_{\mathrm{int}}$ entirely. The same equation can be obtained directly from (3.11) but only by including $\mathcal{H}_{\mathrm{int}}$ to first order.] By using (1.15), we may express the above formula in terms of the physical excitation energy 2ν of the ϕ quantum:

$$\nu-\mu=\Omega^{-1}\frac{g^2}{4}\sum_k\left[\frac{1}{E_k}\tanh\tfrac{1}{2}\beta E_k+\mathrm{P}\frac{1}{\nu-\omega_k}\right] , \tag{4.3}$$

where P denotes the principal value. The right-hand side is convergent in the ultraviolet region. Hence, we may take the ultraviolet cutoff Λ of (1.18) to be ∞. The particle density $\rho\equiv\langle N\rangle/\Omega$ is given by $(\partial p/\partial\mu)_{T,B}$, which gives

$$\begin{aligned}\rho=&2|B|^2+2\Omega^{-1}\sum_k(e^{\beta[2\nu+(k^2/2M)-2\mu]}-1)^{-1}\\&+\Omega^{-1}\sum_k[E_k(1+e^{-\beta E_k})]^{-1}\\&\times[E_k+\mu-\omega_k+(E_k-\mu+\omega_k)e^{-\beta E_k}] .\end{aligned} \tag{4.4}$$

From (4.3) and (4.4), μ and $|B|^2$ can be determined as functions of ρ and T. [Equation (4.3) is similar to the gap equation in the BCS theory, and Eq. (4.4) is the generalization of the density equation in the Bose-Einstein condensation.]

Let $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{S}$ be the energy, Helmholtz free energy, and entropy of the system. We have

$$\Omega^{-1}\mathcal{F}(T,\rho)=\rho\mu-p . \tag{4.5}$$

At a fixed $\langle N\rangle$, $\delta\mathcal{F}=-\mathcal{S}\delta T-p\delta\Omega$. Since $\delta p=\rho\delta\mu+(\partial p/\partial T)_\mu\delta T$, we have

$$\Omega^{-1}\mathcal{S}=-\left[\frac{\partial}{\partial T}(\Omega^{-1}\mathcal{F})\right]_\rho=\left[\frac{\partial p}{\partial T}\right]_{\mu,B} \tag{4.6}$$

which can be readily calculated by using (4.1). The thermodynamic energy $\mathcal{E}=\mathcal{F}+T\mathcal{S}$ is then given by

$$\begin{aligned}\Omega^{-1}\mathcal{E}=&2\nu|B|^2+\Omega^{-1}\sum_k(\omega_k-\mu-E_k)\\&+\Omega^{-1}\sum_k\left[m_k\left[2\nu+\frac{k^2}{2M}\right]+2n_kE_k\right] ,\end{aligned} \tag{4.7}$$

where m_k and n_k are the ensemble average of $\tilde{b}_{\mathbf{k}}^\dagger\tilde{b}_{\mathbf{k}}$ and $\tilde{a}^\dagger_{\mathbf{k},\sigma}\tilde{a}_{\mathbf{k},\sigma}$:

$$m_k=(e^{\beta[2\nu+(k^2/2M)-2\mu]}-1)^{-1} \tag{4.8}$$

and

$$n_k=(e^{\beta E_k}+1)^{-1} . \tag{4.9}$$

From (4.2) we see that the fermion excitation has a gap energy, which is related to the long-range order B of the boson field:

$$\Delta=|gB| \ . \tag{4.10}$$

The phase of B depends on the phase of the symmetry-breaking infinitesimal j. In accordance with (2.10), (2.11), and (3.3), it can be shown that

$$\gamma=\Phi(B)=\pi+\Phi(j) \ , \tag{4.11}$$

where Φ is the phase.

From (4.1), it follows that

$$\left[\frac{\partial^2 p}{\partial(|B|^2)^2}\right]_{T,\mu}=\frac{\beta^2 g^4}{16\Omega}\sum_k \frac{1}{E_k}\frac{d}{dx}\left[\frac{\tanh x}{x}\right]<0 \ , \tag{4.12}$$

where $x=\frac{1}{2}\beta E_k$. Hence, the magnitude of the long-range-order parameter B is determined by the *maximum* of $p(T,\mu,B)$ at fixed T and μ.

We shall now prove that $|B|$ can also be determined by the *minimum* of the Helmholtz free energy, but at fixed T and ρ. First, use (4.4) alone to solve for $\mu=\mu(T,\rho,B)$, and substitute it into (4.5) to obtain $\Omega^{-1}\mathcal{F}(T,\rho,B)$. When this is done, let us minimize $\mathcal{F}$ with respect to B at fixed T,ρ. (In the following, write $|B|$ as B, for convenience.)

Indeed, at fixed T we have

$$\begin{aligned}\Omega^{-1}\left[\frac{\partial\mathcal{F}}{\partial B}\right]_\rho &=\rho\left[\frac{\partial\mu}{\partial B}\right]_\rho-\left[\frac{\partial p}{\partial B}\right]_\rho\\ &=\rho\left[\frac{\partial\mu}{\partial B}\right]_\rho-\left[\frac{\partial p}{\partial B}\right]_\mu-\left[\frac{\partial p}{\partial \mu}\right]_B\left[\frac{\partial\mu}{\partial B}\right]_\rho\\ &=-\left[\frac{\partial p}{\partial B}\right]_\mu\end{aligned} \tag{4.13}$$

since $\rho=(\partial p/\partial\mu)_B$; therefore

$$\begin{aligned}\Omega^{-1}\left[\frac{\partial^2\mathcal{F}}{\partial B^2}\right]_\rho &=-\left[\frac{\partial}{\partial B}\right]_\rho\left[\frac{\partial p}{\partial B}\right]_\mu\\ &=-\left[\frac{\partial^2 p}{\partial B^2}\right]_\mu-\left[\frac{\partial\mu}{\partial B}\right]_\rho\left[\frac{\partial}{\partial\mu}\right]_B\left[\frac{\partial p}{\partial B}\right]_\mu\\ &=-\left[\frac{\partial^2 p}{\partial B^2}\right]_\mu+\left[\frac{\partial\mu}{\partial\rho}\right]_B\left[\frac{\partial\rho}{\partial B}\right]_\mu\left[\frac{\partial^2 p}{\partial\mu\partial B}\right]\\ &=-\left[\frac{\partial^2 p}{\partial B^2}\right]_\mu+\left[\frac{\partial\rho}{\partial\mu}\right]_B^{-1}\left[\frac{\partial\rho}{\partial B}\right]_\mu^2 .\end{aligned} \tag{4.14}$$

On account of

$$\left[\frac{\partial\rho}{\partial\mu}\right]_B=[\partial^2 p/\partial\mu^2]_B=(\beta\Omega)^{-1}(\langle N^2\rangle-\langle N\rangle^2)>0 \ ,$$
$$\left[\frac{\partial^2\mathcal{F}}{\partial B^2}\right]_\rho>0 \tag{4.15}$$

since $(\partial^2 p/\partial B^2)_\mu<0$ as in (4.12). Hence, the magnitude of the long-range order B can also be determined by the *minimum* of $\Omega^{-1}\mathcal{F}(T,\rho,B)$ at fixed T and ρ, in agreement with the general principles of thermodynamics.

V. GAP ENERGY AND CHEMICAL POTENTIAL

The gap energy $\Delta=|gB|$ and the chemical potential μ are functions of the temperature T and the particle density ρ determined by (4.3) and (4.4). In this section, we discuss these two functions $\Delta(T,\rho)$ and $\mu(T,\rho)$. All formulas here pertain to three dimensions.

A. $T=0$

At zero temperature, denote

$$\Delta_0\equiv\Delta(0,\rho) \ \text{ and } \ \mu_0\equiv\mu(0,\rho) \ ; \tag{5.1}$$

μ_0 is the same as the Fermi energy. Neglecting $(\Delta_0/\mu_0)^2$, we find (4.3) and (4.4) to be (proved in the Appendix)

$$\nu-\mu_0=\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}(\mu_0)^{1/2}\left[-2+\ln\frac{8\mu_0}{\Delta_0}\right] \tag{5.2}$$

and

$$\rho=2|B_0|^2+(3\pi^2)^{-1}(2m\mu_0)^{3/2} \tag{5.3}$$

with

$$\Delta_0=|gB_0| \ . \tag{5.4}$$

It is convenient to introduce the dimensionless coupling constant

$$\hat{g}^2\equiv\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}\frac{1}{\sqrt{\nu}} \ . \tag{5.5}$$

From (5.2), it follows that

$$\Delta_0=8\mu_0\exp\left[-2-\frac{\nu-\mu_0}{\hat{g}^2(\nu\mu_0)^{1/2}}\right] \ . \tag{5.6}$$

Since (5.2) is derived under the assumption $(\Delta_0/\mu_0)^2\ll 1$, the exponent in the above expression should be negative and not small; hence,

$$\nu>\mu_0 \ . \tag{5.7}$$

Define

$$\rho_\nu\equiv(3\pi^2)^{-1}(2m\nu)^{3/2} \ , \tag{5.8}$$

the fermionic density when the Fermi energy equals ν, with the excitation energy of the ϕ quantum being equal to 2ν. (After the completion of this work, we learned that in the particular case of $\rho<\rho_\nu$ but not near ρ_ν, which will be discussed below, our model is related to the one studied recently by Newns, Rasolt, and Pattnaik.[15])

(*i*) $\rho<\rho_\nu$ *and* $(\rho_\nu-\rho)/\rho_\nu=O(1)$. From (5.3), $\rho>(3\pi^2)^{-1}(2m\mu_0)^{3/2}$ which means in this case $(\nu-\mu_0)/\nu=O(1)$. In the weak-coupling limit $\hat{g}^2\ll 1$, both $(\Delta_0/\mu_0)^2$ and $|B_0|^2/\rho$ are, in accordance with (5.6), exponentially small. Hence (5.3) gives

$$\mu_0\cong(3\pi^2\rho)^{3/2}/2m \ . \tag{5.9}$$

Upon substitution into (5.6), this determines Δ_0 as a function of ρ. These formulas are similar to those in the BCS theory.

(*ii*) $\rho>\rho_\nu$. As the zeroth approximation, set $\mu_0\cong\nu$;

therefore, from (5.3)

$$|B_0|^2=\Delta_0^2/g^2\cong\tfrac{1}{2}(\rho-\rho_\nu)\ . \tag{5.10}$$

We may use the above formula and (5.2) to derive the first-order correction:

$$\mu_0=\nu\left[1-\hat{g}^2\left[\ln\frac{8\nu}{g|B_0|}-2\right]\right] \tag{5.11}$$

which gives, in the weak-coupling limit,

$$(\nu-\mu_0)/\nu=O(\hat{g}^2\ln\hat{g})\ . \tag{5.12}$$

Note that in case (i), the long-range order B_0 of the Bose-condensate amplitude appears only as a result of the s-channel "virtual" transition (1.1); in the weak-coupling limit, the boson density $|B_0|^2$ is much smaller than the fermion density. The system exhibits a BCS-like characteristic. In case (ii) when ρ is $>\rho_\nu$, the Fermi energy μ_0 approaches ν; the system now deviates from the typical behavior of the BCS theory. Its Bose-condensate amplitude B_0 builds up steadily with increasing density, similar to that in the usual Bose-Einstein condensation. In both cases, the Bose-condensate amplitude determines the gap energy of the fermion system.

(*iii*) ρ *near* ρ_ν. To examine more closely the transition from $\rho<\rho_\nu$ to $\rho>\rho_\nu$, we introduce the following dimensionless quantities:

$$x\equiv(\mu_0/\nu)^{3/4},\quad y\equiv(2|B_0|^2/\rho_\nu)^{1/2}\ , \tag{5.13a}$$

and

$$\hat{\rho}\equiv\rho/\rho_\nu\ . \tag{5.13b}$$

Equations (5.2) and (5.3) become

$$y=f(x)\equiv(4\sqrt{3}/e^2\hat{g})x^{4/3}\exp[-(x^{-2/3}-x^{2/3})/\hat{g}^2] \tag{5.14}$$

and

$$\hat{\rho}=x^2+y^2\ . \tag{5.15}$$

Figure 2 gives $f(x)$ versus x (plotted for the example of $\hat{g}=0.3$). The intersection of $f(x)$, solid curve, and the circle (5.15), dashed curve, determines x and y. For $\hat{g}=0.3$, when x is <0.8, the curve $f(x)$ is near zero which gives $x\cong\hat{\rho}^{1/2}$ and $y\cong f(\hat{\rho}^{1/2})$, in accordance with case (i). When x is near 1, $f(x)$ rises almost vertically parallel to the y axis; hence, $y\cong(\hat{\rho}-1)^{1/2}$ in agreement with (5.10).

B. Critical temperature T_c

At T_c, both the long-range order B and the gap energy Δ are zero. Denote

$$\mu_c=\mu(T_c,\rho),\quad \beta_c=(k_BT_c)^{-1}\ , \tag{5.16}$$

and the fugacity of the bosons at the critical temperature

$$z_c=\exp[2\beta_c(\mu_c-\nu)]\ . \tag{5.17}$$

Neglecting $(k_BT_c/\mu_c)^2$ and $\exp(-\mu_c\beta_c)$, we find (4.3) and (4.4) to be (proved in the Appendix)

$$\nu-\mu_c=\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}(\mu_c)^{1/2}\left[-2+\gamma+\ln\frac{8\mu_c\beta_c}{\pi}\right] \tag{5.18}$$

and

$$\rho=\frac{(2m\mu_c)^{3/2}}{3\pi^2}+2\left[\frac{Mk_BT_c}{2\pi}\right]^{3/2}g_{3/2}(z_c)\ , \tag{5.19}$$

where

$$g_{3/2}(z)=\sum_{l=1}^{\infty}\frac{z^l}{l^{3/2}} \tag{5.20}$$

and Euler's constant

$$\gamma=0.5772\ . \tag{5.21}$$

For $z<1$, $g_{3/2}(z)$ is analytic in z. At $z=1$, $g_{3/2}(z)=\sum_1^\infty l^{-3/2}=2.612$, but its derivative is ∞. Hence, in (5.19) $z_c\le 1$, and therefore

$$\nu\ge\mu_c\ . \tag{5.22}$$

From (5.5) and (5.18), we have

$$k_BT_c=(8\mu_c/\pi)\exp\left[-2+\gamma-\frac{\nu-\mu_c}{\hat{g}^2(\nu\mu_c)^{1/2}}\right]\ . \tag{5.23}$$

(*i*) $\rho<\rho_\nu$ *and* $(\rho_\nu-\rho)/\rho_\nu=O(1)$. In this case, $(\nu-\mu_c)/\nu=O(1)$. In the weak-coupling limit $\hat{g}^2\ll1$, we have k_BT_c/μ_c exponentially small, and therefore z_c and $g_{3/2}(z_c)$ are (exponentially)^2 small. Hence, in (5.19), the boson density (second term on the right-hand side) may be neglected and

$$\mu_c\cong(3\pi^2\rho)^{2/3}/2m\ . \tag{5.24}$$

Combining it with (5.6), (5.9), and (5.23), we derive

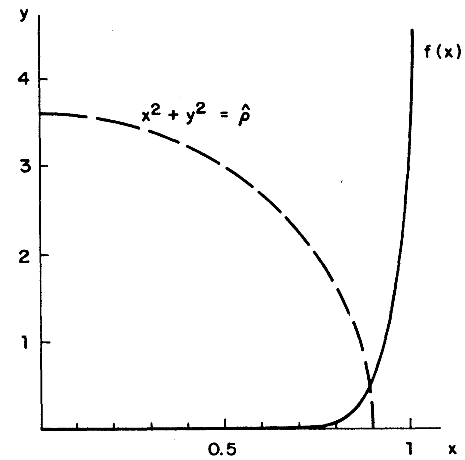

FIG. 2. The solid curve is $y=f(x)$, defined by (5.14) and plotted for $\hat{g}=0.3$. The dashed curve is the circle $x^2+y^2=\hat{\rho}$ (here, for $\hat{\rho}=0.9$). The intersection between these two curves determines x and y, and therefore, through (5.13), also the gap energy $\Delta_0(\rho)$ and the chemical potential $\mu_0(\rho)$ at $T=0$.

$$\frac{\Delta_0}{k_B T_c} = \pi e^{-\gamma} = 1.7639 , \tag{5.25}$$

the same relation as in the BCS theory. The chemical potential μ_c (at the critical temperature T_c) increases with density ρ, and μ_c approaches ν as $\rho \to \rho_\nu$.

(*ii*) $\rho > \rho_\nu$. On account of (5.22), as the zeroth approximation, set $\mu_c \cong \nu$; hence, $z_c \cong 1$. From (5.19), the critical temperature is given by

$$2 \left[\frac{M k_B T_c}{2\pi} \right]^{3/2} = (\rho - \rho_\nu)/2.612 . \tag{5.26}$$

The combination of (5.10) and (5.26) gives, instead of (5.25),

$$\Delta_0^2 = 2.612 g^2 \left[\frac{M k_B T_c}{2\pi} \right]^{3/2} ; \tag{5.27}$$

i.e.,

$$\Delta_0^2 = 2.612 \hat{g}^2 \sqrt{\pi\nu} (k_B T_c M/m)^{3/2} . \tag{5.27'}$$

In this case, as ρ increases (but still assuming $k_B T_c \ll \mu$), the fermion density remains approximately a constant $\cong \rho_\nu$, while the boson density increases according to $\frac{1}{2}(\rho - \rho_\nu)$; the same quantity is approximately $|B_0|^2$ at $T=0$, but becomes $2.612\ (M k_B T_c/2\pi)^{3/2}$ at the critical temperature, and that leads to the above relation.

We note that when ρ is larger than but near ρ_ν, it is possible to neglect $(k_B T_c/\mu_c)^2$. However, as ρ increases further, terms proportional to $(k_B T_c/\mu_c)^2$ must be included. As shown in the Appendix, (5.18) and (5.19) are replaced by

$$\nu - \mu_c = \left[\frac{g}{\pi} \right]^2 \left[\frac{m}{2} \right]^{3/2} (\mu_c)^{1/2} \left[-2 + \gamma + \ln \frac{8\mu_c}{\pi k_B T_c} + \frac{\pi^2}{96} \left[\frac{k_B T_c}{\mu_c} \right]^2 + \frac{5!!}{8!!} \frac{7\pi^4}{120} \left[\frac{k_B T_c}{\mu_c} \right]^4 + \cdots \right] \tag{5.28}$$

and

$$\rho = 2 \left[\frac{M k_B T_c}{2\pi} \right]^{3/2} g_{3/2}(z_c) + \frac{(2m\mu_c)^{3/2}}{3\pi^2} \left[1 + \frac{3\pi^2}{48} \left[\frac{k_B T_c}{\mu_c} \right]^2 + \frac{1}{8!!} \frac{21\pi^4}{10} \left[\frac{k_B T_c}{\mu_c} \right]^4 + \cdots \right] . \tag{5.29}$$

For $\hat{g}^2 \ll 1$, (5.28) indicates that $\mu_c = \nu[1 - O(\hat{g}^2)]$. Setting $\mu_c \cong \nu$ in (5.29), we derive the zeroth approximation of T_c, now including the $(k_B T_c/\mu_c)^2$ corrections; substituting the result into (5.28), we derive the first correction to $\nu - \mu_c$. In this way, T_c and μ_c can be derived as a power series in $\hat{g}^2$.

As ρ keeps on increasing, so does $k_B T_c$, and eventually $k_B T_c$ becomes larger than ν; when that happens, the fermions are no longer degenerate and, in the approximation $\mu_c \cong \nu$, (5.29) is replaced by

$$\rho = 2(2.612) \left[\frac{M k_B T_c}{2\pi} \right]^{3/2} + 2 \left[\frac{m k_B T_c}{2\pi} \right]^{3/2} g_{3/2}(\zeta) , \tag{5.30}$$

where

$$\zeta = -e^{\nu/k_B T_c} \tag{5.31}$$

and $g_{3/2}$ is defined by (5.20).

For $\rho > \rho_\nu$ and $(\rho - \rho_\nu)/\rho_\nu = O(1)$ we have $k_B T_c/\nu = O(1)$. But on account of (5.10), $\Delta_0^2/\nu^2 = O(\hat{g}^2)$; hence in the weak-coupling limit

$$\Delta_0^2 \ll (k_B T)^2 ,$$

which is quite different from the low-density formula (5.25). When the density is so large that the distance between bosons becomes comparable to the intrinsic size of the boson, then our local-field approximation of the pairing state breaks down.

(iii) ρ near ρ_ν. To see more clearly the transition from $\rho < \rho_\nu$ to $\rho > \rho_\nu$, we introduce, similar to (5.14) and (5.15), the following dimensionless quantities:

$$x_c \equiv (\mu_c/\nu)^{3/4} \quad \text{and} \quad y_c \equiv \tfrac{1}{2}(\sqrt{3}\pi e^{-\gamma})(\nu \beta_c \hat{g})^{-1} . \tag{5.32}$$

Equations (5.18) and (5.19) become

$$y_c = f(x_c) \tag{5.33}$$

and

$$\hat{\rho} = x_c^2 + \frac{3^{1/4}}{\sqrt{2}\pi} (M e^{\gamma} \hat{g} y_c/m)^{3/2} g_{3/2}(z_c) , \tag{5.34}$$

where $\hat{\rho}$, $\hat{g}$, and $f(x)$ are given by (5.14) and (5.15), and $z_c = e^{2\beta_c(\mu_c - \nu)}$. From (5.22) it follows that

$$x_c \le 1 . \tag{5.35}$$

The graphic determination of x_c and y_c follows closely the steps described in case (iii) of Sec. V A. Change the coordinates x,y in Fig. 2 to x_c and y_c; the solid curve $f(x)$ becomes $f(x_c)$. Replace the dashed circle by (5.34), which has a similar overall shape. The intersection of the solid and dashed curves now gives x_c and y_c. The steep rise of $f(x_c)$ near $x_c = 1$ "explains" the rapid change from case (i) to case (ii).

C. Variation of Δ versus T

Equation (4.3) can be written as

$$\nu-\mu=\tfrac{1}{2}\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2} f(\Delta,\beta)\ , \tag{5.36}$$

where

$$f(\Delta,\beta)=\int_0^\infty(\omega_k)^{1/2}d\omega_k\left[\frac{1}{E_k}\tanh\tfrac{1}{2}\beta E_k+P\frac{1}{\nu-\omega_k}\right]. \tag{5.37}$$

Neglecting $(k_BT/\mu)^2$, $e^{-\mu/k_BT}$, and $(\Delta/\mu)^2$, we find

$$f(\Delta,\beta)=2\sqrt{\mu}\left[-2+\ln\frac{2\mu\beta}{\pi}+F(\beta\Delta)\right] \tag{5.38}$$

and $F(\beta\Delta)$ is given by

$$F=\lim_{L\to\infty}\left\{\sum_{l=0}^{L-1}\left[(l+\tfrac{1}{2})^2+\left[\frac{\beta\Delta}{2\pi}\right]^2\right]^{1/2}-\ln L\right\}. \tag{5.39}$$

Hence, for $\beta\Delta/2\pi<\frac{1}{2}$

$$F=\gamma+2\ln 2+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{2n!!}(2^{2n+1}-1)\times\zeta(2n+1)\left[\frac{\beta\Delta}{2\pi}\right]^{2n}, \tag{5.40}$$

where $\zeta(3)=1.202$, $\zeta(5)=1.0369,\ldots$ are the values of the ζ function, and as before, $\gamma=0.5772$ is Euler's number. An alternative expansion, which is useful for large $\beta\Delta/2\pi$, is

$$F=\ln(4\pi/\beta\Delta)+2\sum_{m=1}^{\infty}(-1)^mK_0(m\beta\Delta)\ , \tag{5.41}$$

where $K_0(z)$ is related to the Bessel and Neumann functions by $K_0(z)=\frac{1}{2}\pi i[J_0(iz)+iN_0(iz)]$, and $K_0(z)\to(\frac{1}{2}\pi/z)^{1/2}e^{-z}$ as $z\to\infty$. The proof of (5.41) is given in the Appendix. Table I lists F versus $\beta\Delta$.

Neglecting $(k_BT/\mu)^2$, $e^{-\mu/k_BT}$, and $(\Delta/\mu)^2$, we find that (4.4) can be written as

$$\rho=2|B|^2+2\left[\frac{Mk_BT}{2\pi}\right]^{3/2}g_{3/2}(z)+\frac{(2m\mu)^{3/2}}{3\pi^2}\ , \tag{5.42}$$

where $g_{3/2}(z)$ is defined by (5.20) and

$$z=\exp[2(\mu-\nu)/k_BT]\ . \tag{5.43}$$

(i) $\rho<\rho_\nu$ and $(\rho_\nu-\rho)/\rho_\nu=O(1)$. A good zeroth approximation in this case is, as in (5.9) and (5.24),

$$\mu\simeq(3\pi^2\rho)^{2/3}/2m\ . \tag{5.44}$$

Substituting it into (5.36)–(5.39), we determine $\Delta(T,\rho)$. In (5.39), the function F depends only on the product $\beta\Delta$. Hence, Δ at any temperature can be related to Δ at a different temperature. Since at the critical temperature T_c, $\Delta=0$, we may use (5.40) to determine Δ at any temperature T not too much lower than T_c $(\beta\Delta<\pi)$:

TABLE I. The function $F(\beta\Delta)$ vs $\beta\Delta$. See (5.39) for the definition of $F(\beta\Delta)$.

$\beta\Delta$	$F(\beta\Delta)$
0.2	1.9593
0.4	1.9467
0.6	1.9261
0.8	1.8983
1.0	1.8641
1.5	1.7567
2.0	1.6302
2.5	1.4970
3.0	1.3653
4.0	1.1227
5.0	0.9142
6.0	0.7368
8.0	0.4513
10.0	0.2284
12.0	0.0461
15.0	−0.1770
20.0	−0.4647
30.0	−0.8702
50.0	−1.3810
100.0	−2.0741

$$F(\beta\Delta)=(5.40)=\gamma+\ln(4T/T_c)\ ; \tag{5.45}$$

likewise, since at zero temperature $\Delta=\Delta_0$, it is more convenient to use (5.41) to determine Δ at very low temperature $(\beta\Delta>1)$:

$$F(\beta\Delta)=(5.41)=\ln(4\pi k_BT/\Delta_0)\ . \tag{5.46}$$

Thus, as $T\to T_c-$, from (5.45)

$$\frac{\Delta}{k_BT_c}=2\pi\left[\frac{2}{7\zeta(3)}\right]^{1/2}\left[1-\frac{T}{T_c}\right]^{1/2} =3.0633\left[1-\frac{T}{T_c}\right]^{1/2}, \tag{5.47}$$

and as $T\to 0+$, from (5.46)

$$\frac{\Delta}{\Delta_0}=1-\left[\frac{2\pi k_BT}{\Delta_0}\right]^{1/2}e^{-\Delta_0/k_BT}\ . \tag{5.48}$$

(ii) $\rho>\rho_\nu$. As the zeroth approximation, we may again set

$$\mu\cong\nu\ . \tag{5.49}$$

In accordance with (5.43), $z\cong 1$, and consequently (5.42) gives

$$\Delta^2=g^2|B|^2\cong g^2\left[\tfrac{1}{2}(\rho-\rho_\nu)-2.612\left[\frac{Mk_BT}{2\pi}\right]^{3/2}\right]. \tag{5.50}$$

Combining this expression with (5.10) and (5.27), we find

$$\Delta^2\cong\Delta_0^2\left[1-\left[\frac{T}{T_c}\right]^{3/2}\right], \tag{5.51}$$

which is quite different from case (i).

VI. MEISSNER EFFECT

A. Spontaneous symmetry breaking

The Meissner effect[11] is closely connected with the spontaneous symmetry breaking of the electromagnetic gauge invariance inside the superconductor. For mathematical convenience, we shall use a relativistic local field Φ to represent the bosons; hence, Φ and $\Phi^\dagger$ commute at equal time, in contrast to (1.10a). Let A_μ be the electromagnetic four-vector and ψ_σ the same nonrelativistic field for the electron, as before, with $\sigma=\uparrow$ and $\downarrow$. The Lagrangian density (in units $\hbar=c=1$) can be written as a sum of terms:

$$\mathcal{L}_A+\mathcal{L}_\phi+\mathcal{L}_e+\mathcal{L}_1 , \tag{6.1}$$

where

$$\mathcal{L}_A=-\frac{1}{4}\left[\frac{\partial A_\nu}{\partial x_\mu}-\frac{\partial A_\mu}{\partial x_\nu}\right]^2 ,$$

$$\mathcal{L}_\phi=-\left[\left[\frac{\partial}{\partial x_\mu}+2ieA_\mu\right]\Phi^\dagger\right]\times\left[\frac{\partial}{\partial x_\mu}-2ieA_\mu\right]\Phi-M^2\Phi^\dagger\Phi , \tag{6.2}$$

$$\mathcal{L}_e=\psi_\sigma^\dagger\left[i\frac{\partial}{\partial t}-eA_0-m\right]\psi_\sigma-\frac{1}{2m}[(\nabla+ie\,\mathbf{A})\psi_\sigma^\dagger](\nabla-ie\,\mathbf{A})\psi_\sigma ,$$

and

$$\mathcal{L}_1=g(2M)^{1/2}(\Phi^\dagger\psi_\uparrow\psi_\downarrow+\Phi\psi_\downarrow^\dagger\psi_\uparrow^\dagger) .$$

The repeated spin index σ is summed over $\uparrow$ and $\downarrow$, the Greek indices μ and ν are summed over from 1 to 4, with $x_4=it$ and $A_4=iA_0$. The electric charge e is negative if ψ_σ represents the electron field. For simplicity, we set the "bare" excitation energy of the boson to be

$$2\nu_0=M-2m . \tag{6.3}$$

The theory is invariant under the gauge transformation

$$\Phi\to\Phi e^{2i\alpha} ,$$

$$\psi_\sigma\to\psi_\sigma e^{i\alpha} ,$$

and (6.4)

$$A_\mu\to A_\mu+\frac{1}{e}\frac{\partial\alpha}{\partial x_\mu} ,$$

where α is an arbitrary real function of x_μ. The electric charge density in units of e is given by

$$\rho=-2i[(\dot\Phi^\dagger-2ieA_0\Phi^\dagger)\Phi-\Phi^\dagger(\dot\Phi+2ieA_0\Phi)]+\psi_\sigma^\dagger\psi_\sigma , \tag{6.5}$$

where the dot denotes the time derivative.

Write

$$\Phi(x)=\left[C+\frac{1}{\sqrt{2}}R(x)\right]\exp[2i\theta(x)] \tag{6.6}$$

with C, $R(x)$, $\theta(x)$ all real and C a constant. Note that for any complex function $\Phi(x)$ and real constant C, we can use the above equation to define $R(x)$ and $\theta(x)$. Because of (6.4), introduce

$$\Psi_\sigma(x)\equiv\psi_\sigma(x)\exp[-i\theta(x)] \tag{6.7}$$

and

$$V_\mu(x)\equiv A_\mu(x)-\frac{1}{e}\frac{\partial\theta}{\partial x_\mu} \tag{6.8}$$

with $V_4=iV_0$. As we shall see, the constant C will turn out to be related to the long-range-order parameter B by

$$C=(2M)^{-1/2}|B| . \tag{6.9}$$

In terms of these transformed variables, $\mathcal{L}_A$, $\mathcal{L}_\phi$, $\mathcal{L}_e$, and $\mathcal{L}_1$ can be written as

$$\mathcal{L}_A=-\frac{1}{4}\left[\frac{\partial V_\nu}{\partial x_\mu}-\frac{\partial V_\mu}{\partial x_\nu}\right]^2 ,$$

$$\mathcal{L}_\phi=-\frac{1}{2}\left[\frac{\partial R}{\partial x_\mu}\right]^2-[4e^2(\mathbf{V}^2-V_0^2)+M^2]\left[C+\frac{1}{\sqrt{2}}R\right]^2 , \tag{6.10}$$

$$\mathcal{L}_e=\Psi_\sigma^\dagger\left[i\frac{\partial}{\partial t}-eV_0-m\right]\Psi_\sigma-\frac{1}{2m}(\nabla+ie\mathbf{V})\Psi_\sigma^\dagger\cdot(\nabla-ie\mathbf{V})\Psi_\sigma ,$$

and

$$\mathcal{L}_1=g(2M)^{1/2}\left[C+\frac{1}{\sqrt{2}}R\right](\Psi_\uparrow\Psi_\downarrow+\Psi_\downarrow^\dagger\Psi_\uparrow^\dagger) .$$

The density (6.5) becomes

$$\rho=-8eV_0\left[C+\frac{1}{\sqrt{2}}R\right]^2+\Psi_\sigma^\dagger\Psi_\sigma ; \tag{6.11}$$

its integral is the total charge of the system (in units of e)

$$N=\int\rho\,d^3r . \tag{6.12}$$

Because in our problem the quanta of both Ψ_σ and Φ carry charges of the same sign, the above charge density ρ is also of the same sign everywhere. There should be, in addition, a background *constant* charge distribution $\rho_{\rm ext}$ of opposite sign due to external sources, which can be introduced through the gauge-invariant Lagrangian density

$$\mathcal{L}_{\rm ext}\equiv eV_0\rho_{\rm ext} . \tag{6.13}$$

The total Lagrangian density $\mathcal{L}$ is

$$\mathcal{L}=\mathcal{L}_A+\mathcal{L}_\phi+\mathcal{L}_e+\mathcal{L}_1+\mathcal{L}_{\rm ext} . \tag{6.14}$$

So far C is just a constant parameter. In the next two sections, we shall show that $C\neq 0$ for both the ground-state energy and the partition function (at temperature $T<T_c$). Hence, in (6.8) and (6.10), the massless A_μ field joins the "Nambu-Goldstone" field[12,13] $\partial\theta/\partial x_\mu$ to form a massive vector field V_μ of mass squared

$$m_V^2=8e^2C^2 \; , \tag{6.15}$$

given by the constant part of the coefficient of $\frac{1}{2}\mathbf{V}^2$ in (6.10), as in the Higgs mechanism.[16] (Further discussions will be given below.)

The conjugate momenta of $\mathbf{V}$, R, and Ψ_σ can be obtained by differentiating the Lagrangian density $\mathcal{L}$:

$$-\mathbf{E}\equiv\partial\mathcal{L}\Big/\partial\left[\frac{\partial\mathbf{V}}{\partial t}\right]=\frac{\partial\mathbf{V}}{\partial t}+\nabla V_0 \; , \tag{6.16}$$

$$\Pi\equiv\partial\mathcal{L}/\partial\dot{R}=\dot{R} \; , \tag{6.17}$$

and

$$i\Psi_\sigma^\dagger=\partial\mathcal{L}/\partial\dot{\Psi}_\sigma \; . \tag{6.18}$$

Because $\dot{V}_0$ is absent in $\mathcal{L}$, V_0 does not have a conjugate momentum; instead, we use the equation $\delta\int\mathcal{L}\,d^3r/\delta V_0=0$,

$$V_0=\left[8e^2\left[C+\frac{1}{\sqrt{2}}R\right]^2\right]^{-1}(-\nabla\cdot\mathbf{E}+e\Psi_\sigma^\dagger\Psi_\sigma-e\rho_{\rm ext}) \; , \tag{6.19}$$

to regard V_0 as a function of R, $\mathbf{E}$, and Φ_σ and $\rho_{\rm ext}$ (which is an external constant parameter). In terms of ρ given by (6.11), the above equation is simply Gauss's law:

$$\nabla\cdot\mathbf{E}=e(\rho-\rho_{\rm ext}) \; . \tag{6.20}$$

In the following section, when we pass from the Lagrangian to the Hamiltonian, we shall freely partially integrate; this is possible provided

$$\int\nabla\cdot\mathbf{E}\,d^3r=0 \; , \tag{6.21}$$

which implies

$$\rho_{\rm ext}=\Omega^{-1}\int\rho\,d^3r=\Omega^{-1}N \; , \tag{6.22}$$

with Ω equal to the volume of the system, as before. Substituting (6.11) into (6.22) and expanding $\int\rho\,d^3r$ as the average of a power series in V_0, R, and Ψ_σ, we have

$$\rho_{\rm ext}=-8eC^2\overline{V}_0-8\sqrt{2}eC\overline{RV}_0+\cdots \; , \tag{6.23}$$

where $\overline{f}\equiv\Omega^{-1}\int f\,d^3r$ for any f, so that

$$\overline{V}_0=\Omega^{-1}\int V_0d^3r \; , \tag{6.24}$$

$\overline{RV}_0=\Omega^{-1}\int RV_0d^3r$, etc.

B. Hamiltonian and quantization

The Hamiltonian H is given by

$$H=\int\left[-\mathbf{E}\cdot\frac{\partial\mathbf{V}}{\partial t}+\Pi\dot{R}+i\Psi_\sigma^\dagger\dot{\Psi}_\sigma-\mathcal{L}\right]d^3r \; . \tag{6.25}$$

By using (6.10) and (6.13)–(6.19), we find

$$\begin{aligned}H=\int d^3r\Bigg\{&\tfrac{1}{2}\mathbf{E}^2+\tfrac{1}{2}(\nabla\times\mathbf{V})^2+\left[C+\frac{1}{\sqrt{2}}R\right]^2(4e^2\mathbf{V}^2+M^2)\\&+\frac{1}{2}\left[8e^2\left[C+\frac{1}{\sqrt{2}}R\right]^2\right]^{-1}[(\nabla\cdot\mathbf{E})^2-e^2(\Psi_\sigma^\dagger\Psi_\sigma-\rho_{\rm ext})^2]+\tfrac{1}{2}\Pi^2+\tfrac{1}{2}(\nabla R)^2+\Psi_\sigma^\dagger(eV_0+m)\Psi_\sigma\\&+\frac{1}{2m}(\nabla+ie\mathbf{V})\Psi_\sigma^\dagger\cdot(\nabla-ie\mathbf{V})\Psi_\sigma+g(2M)^{1/2}\left[C+\frac{1}{\sqrt{2}}R\right](\Psi_\uparrow\Psi_\downarrow+\Psi_\downarrow^\dagger\Psi_\uparrow^\dagger)-eV_0\rho_{\rm ext}\Bigg\} \; .\end{aligned} \tag{6.26}$$

Keeping $\mathbf{E}$, $\mathbf{V}$, Π, R, Ψ_σ, and $\Psi_\sigma^\dagger$ (or course, also the constants C, e, g, and M) fixed, but taking $\rho_{\rm ext}$ as a variable and regarding V_0 as a dependent function on $\rho_{\rm ext}$ through (6.19), we derive

$$\partial H/\partial\rho_{\rm ext}=-e\Omega\overline{V}_0 \; . \tag{6.27}$$

Introduce its Legendre transform

$$\mathcal{H}\equiv H+e\Omega\rho_{\rm ext}\overline{V}_0 \tag{6.28}$$

and treat $\mathcal{H}$ as a function of $\overline{V}_0$ and the field variables $\mathbf{E}$, $\mathbf{V}$, Π, R, Ψ_σ, and $\Psi_\sigma^\dagger$ (with $\rho_{\rm ext}$ now as a dependent variable; it follows then that

$$\partial\mathcal{H}/\partial(e\overline{V}_0)=\Omega\rho_{\rm ext}=N \; , \tag{6.29}$$

on account of (6.12) and (6.22). Recalling that in our previous nonrelativistic case $\partial(H-\mu N)/\partial(-\mu)=N$, we identify $-e\overline{V}_0$ as the Gibbs energy per particle, but including the rest mass m; i.e.,

$$-e\overline{V}_0=m+\mu \; , \tag{6.30}$$

where μ is the chemical potential. Correspondingly, (6.28) becomes

$$\mathcal{H}=H-(m+\mu)N \; . \tag{6.31}$$

Define

$$v_0\equiv V_0-\overline{V}_0 \; , \tag{6.32}$$

where, by definition,

$$\overline{v}_0=\Omega^{-1}\int v_0d^3r=0 \; . \tag{6.33}$$

By using (6.30), (6.32), and (6.33) we can rewrite (6.23) as

$$\rho_{\text{ext}}=\sigma_{\text{ext}}+\Omega^{-1}\int \Psi_\sigma^\dagger\Psi_\sigma d^3r \ , \tag{6.34}$$

where

$$\sigma_{\text{ext}}\equiv 8(m+\mu)(C^2+\sqrt{2}C\overline{R}+\tfrac{1}{2}\overline{R^2})-8\sqrt{2}eC\overline{Rv_0}+\cdots \tag{6.35}$$

is the average of $-8eV_0[C+(1/\sqrt{2})R]^2$. Combining (6.19) with (6.30)–(6.33), we obtain the following expansion of v_0 as a power series of the field variables:

$$v_0=-m_V^{-2}(\nabla\cdot\mathbf{E})+4m_V^{-1}(m+\mu)(R-\overline{R})+\cdots \tag{6.36}$$

in which $m_V=2\sqrt{2}eC$ and $\cdots$ contains quadratic and higher powers of R, $\mathbf{E}$, Ψ_σ with their averages subtracted. Likewise, it follows from (6.34) that the following expression [occurring in H of (6.26)] becomes

$$\begin{aligned}\int \left[C+\frac{1}{\sqrt{2}}R\right]^{-2}&(\Psi_\sigma^\dagger\Psi_\sigma-\rho_{\text{ext}})^2d^3r\\ &=\int\left[\left[C+\frac{1}{\sqrt{2}}R\right]^{-2}\sigma_{\text{ext}}^2+\cdots\right]d^3r\\ &=64\Omega\left[(m+\mu)^2(C^2+\sqrt{2}C\overline{R}+2\overline{R}^{\,2}-\tfrac{3}{2}\overline{R^2})\right.\\ &\qquad\left.+\frac{m+\mu}{m_V}\overline{R\nabla\cdot\mathbf{E}}\right]+\cdots \ ,\end{aligned}\tag{6.37}$$

where $\cdots$ contains only *cubic* and higher powers of these field variables.

In the same way, $\mathcal{H}$, defined by (6.28), can be written as

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_0'+\mathcal{H}_1+\mathcal{H}_{\text{int}} \tag{6.38}$$

where $\mathcal{H}_0$ depends quadratically on Ψ_σ and $\Psi_\sigma^\dagger$, $\mathcal{H}_0'$ is a quadratic function on other field variables, and $\mathcal{H}_1$ is linear in R:

$$\begin{aligned}\mathcal{H}_0=\int d^3r\,\Big[&[M^2-4(m+\mu)^2]C^2+\frac{1}{2m}\nabla\Psi_\sigma^\dagger\cdot\nabla\Psi_\sigma\\ &-\mu\Psi_\sigma^\dagger\Psi_\sigma+g(2M)^{1/2}C(\Psi_\uparrow\Psi_\downarrow+\Psi_\downarrow^\dagger\Psi_\uparrow^\dagger)\Big] \ ,\end{aligned}\tag{6.39}$$

$$\begin{aligned}\mathcal{H}_0'=\tfrac{1}{2}\int d^3r\{&\mathbf{E}^2+m_V^{-2}(\nabla\cdot\mathbf{E})^2+(\nabla\times\mathbf{V})^2+m_V^2\mathbf{V}^2+\Pi^2\\ &+(\nabla R)^2+[M^2+12(m+\mu)^2]R^2\\ &-8m_V^{-1}(m+\mu)R\nabla\cdot\mathbf{E}\} \ ,\end{aligned}\tag{6.40}$$

$$\mathcal{H}_1=\int d^3r\{[M^2-4(m+\mu)^2]\sqrt{2}C-8(m+\mu)^2\overline{R}\}R \ , \tag{6.41}$$

where $\overline{R}=\int R\,d^3r$; the interaction Hamiltonian $\mathcal{H}_{\text{int}}$ consists only of cubic and quartic terms of field operators $\mathbf{E}$, $\mathbf{V}$, Π, R, Ψ_σ, and $\Psi_\sigma^\dagger$. In (6.40), the product term $R\nabla\cdot\mathbf{E}$ is somewhat unusual; it comes from the σ_{ext}^2 term in (6.37), which in turn arises from the $\overline{Rv_0}$ term in (6.35), with v_0 given by (6.36); for its physical significance, see remarks 2 and 3 at the end of Sec. VI C.

For quantization, we require $\mathbf{V}$, R, Ψ_σ, and their conjugate momenta to satisfy the following commutation and anticommutation relations:

$$\begin{aligned}&[V_i(\mathbf{r},t),E_j(\mathbf{r}',t)]=-i\delta_{ij}\delta^3(\mathbf{r}-\mathbf{r}') \ ,\\ &[\Pi(\mathbf{r},t)R(\mathbf{r}',t)]=-i\delta^3(\mathbf{r}-\mathbf{r}') \ ,\\ &\{\Psi_\sigma(\mathbf{r},t),\Psi_{\sigma'}^\dagger(\mathbf{r}',t)\}=\delta_{\sigma\sigma'}\delta^3(\mathbf{r}-\mathbf{r}') \ ,\\ &\{\Psi_\sigma(\mathbf{r},t),\Psi_{\sigma'}(\mathbf{r}',t)\}=0\end{aligned}\tag{6.42}$$

with all other equal-time commutators between them zero.

As remarked earlier, when we make the Legendre transformation (6.28) from H of (6.26) to $\mathcal{H}$ of (6.38)–(6.41), there is a switch of independent variables: in H we regard one of the independent variables to be ρ_{ext} with $\overline{V}_0$ dependent on ρ_{ext}, whereas in $\mathcal{H}$ the independent variable is $\overline{V}_0$ (or equivalently, the chemical potential $\mu=-e\overline{V}_0-m$) with ρ_{ext} as the dependent variable given by (6.34) and (6.35). In the quantized theory, the right sides of (6.34) and (6.35) are operators; so then is $\rho_{\text{ext}}=\Omega^{-1}N$. This is particularly useful for the grand canonical ensemble, since at a given μ it is now possible for us to include states of different eigenvalues of N.

C. Determination of C

The grand partition function is

$$\mathcal{Q}=\text{tr}e^{-\beta\mathcal{H}} \tag{6.43}$$

where, as before, $\beta=(k_BT)^{-1}$. Similar to (3.11), in order to determine C we require

$$\text{tr}(Re^{-\beta\mathcal{H}})=0 \ . \tag{6.44}$$

In evaluating $\mathcal{Q}$, we regard

$$\mathcal{H}_1+\mathcal{H}_{\text{int}} \tag{6.45}$$

as the perturbation. As noted before, $\mathcal{H}_1$ is linear in R; because of (6.44), it can be included as part of the perturbation, as in (3.15). The zeroth-order partition function is

$$\text{tr}e^{-\beta(\mathcal{H}_0+\mathcal{H}_0')} \ . \tag{6.46}$$

To determine C to the *zeroth* order in e^2, we need only consider

$$q\equiv\text{tr}e^{-\beta\mathcal{H}_0} \ . \tag{6.47}$$

Write

$$\mathcal{H}_0=h+\Omega[M^2-4(m+\mu)^2]C^2 \ , \tag{6.48}$$

where, on account of (6.39),

$$\begin{aligned}h=\sum_k&\left[\frac{k^2}{2m}-\mu\right]a_{\mathbf{k},\sigma}^\dagger a_{\mathbf{k},\sigma}\\ &+g(2M)^{1/2}C\sum_k(a_{\mathbf{k},\uparrow}a_{-\mathbf{k},\downarrow}+a_{-\mathbf{k},\downarrow}^\dagger a_{\mathbf{k},\uparrow}^\dagger) \ ,\end{aligned}$$

and $a_{\mathbf{k},\sigma}$, $a_{\mathbf{k},\sigma}^\dagger$ are the Fourier components of Ψ_σ and $\Psi_\sigma^\dagger$:

$$\Psi_\sigma(\mathbf{r})=\sum_k\Omega^{-1/2}a_{\mathbf{k},\sigma}e^{i\mathbf{k}\cdot\mathbf{r}}$$

with $\{a_{\mathbf{k},\sigma}, a^{\dagger}_{\mathbf{k}',\sigma'}\} = \delta_{\mathbf{kk}'}\delta_{\sigma\sigma'}$, as in (1.12). The matrix h can be diagonalized by the Bogoliubov-Valatin transformation U_2 of (3.13) with $\gamma=0$ and $C=(2M)^{-1/2}|B|$ in agreement with (6.9). We have

$$h=\sum_k [(\omega_k-\mu-E_k)+E_k \tilde{a}^{\dagger}_{\mathbf{k},\sigma}\tilde{a}_{\mathbf{k},\sigma}] , \tag{6.49}$$

where, as in (3.4)

$$E_k=[(\omega_k-\mu)^2+g^2|B|^2]^{1/2} \tag{6.50}$$

and $\omega_k=k^2/2m$. Define p_0 to be the partial pressure:

$$p_0 \equiv k_B T \Omega^{-1} \ln q . \tag{6.51}$$

Using (6.3), (6.9), and neglecting $(\nu_0/m)^2$ and $(\mu/m)^2$, in the nonrelativistic limit the constant term $\Omega[M^2-4(m+\mu)^2]C^2$ in (6.48) is

$$2\Omega(\nu_0-\mu)|B|^2 .$$

Hence,

$$p_0=-2(\nu_0-\mu)|B|^2+\Omega^{-1}\sum_k (E_k+\mu-\omega_k) + 2(\beta\Omega)^{-1}\sum_k \ln(1+e^{-\beta E_k}) . \tag{6.52}$$

By following the same steps leading from (3.11), $\mathrm{tr}(\tilde{b}_0 e^{-\beta\mathcal{H}})=0$, to $(\partial p/\partial|B|)_{\mu,T}=0$, which gives the gap equation (4.3), we can start from the corresponding condition (6.46), and derive $(\partial p_0/\partial|B|)_{\mu,T}=0$ for the present problem. In this way, as expected, we derive the same gap equation (4.3):

$$\nu-\mu=\Omega^{-1}\frac{g^2}{4}\sum_k \left[\frac{1}{E_k}\tanh\tfrac{1}{2}\beta E_k + P\frac{1}{\nu-\omega_k}\right] . \tag{6.53}$$

Substituting the result into (6.15), we find the vector mass squared to be

$$m_V^2=4(e^2/M)|B|^2 . \tag{6.54}$$

Recall that the gap energy Δ of the fermion system is $g|B|$; we obtain

$$m_V^2=4(e^2/Mg^2)\Delta^2 . \tag{6.55}$$

Since the mass term has an effect similar to the gap term on the spectrum, the above equation relates the two "gap" energies, one for the bosons and the other for the fermions.

Because the V quantum is moving in a medium composed of the electron field Ψ_σ and the "Higgs field" R, from (6.10) we see that the coefficient of $-\frac{1}{2}\mathbf{V}^2$ in $\mathcal{L}_\phi+\mathcal{L}_e$ is

$$8e^2\left[C+\frac{1}{\sqrt{2}}R\right]^2+\frac{e^2}{m}\Psi^\dagger_\sigma\Psi_\sigma=\frac{e^2}{m}\rho , \tag{6.56}$$

where, in accordance with (6.11), $e\rho$ is the charge density in the nonrelativistic limit, since in that limit ev_0/m and $\mu/m \to 0$; therefore $eV_0=e(\bar{V}_0+v_0)$ becomes

$$eV_0 \cong e\bar{V}_0 \cong -m . \tag{6.57}$$

London's length[17] λ_L for the Meissner effect is determined by the ensemble average of (6.56):

$$\lambda_L^{-2}=e^2\langle\rho\rangle/m , \tag{6.58}$$

where

$$\langle\rho\rangle=\mathcal{Q}^{-1}\mathrm{tr}(\rho e^{-\beta\mathcal{H}}) . \tag{6.59}$$

D. Remarks

(1) From (6.40), one sees that the coefficient of $\frac{1}{2}R^2$ in $\mathcal{H}_0'$ is

$$\mathcal{M}_0^2=M^2+12(m+\mu)^2 . \tag{6.60}$$

If we include the second-order effect of the trilinear coupling $g(2M)^{1/2}(R/\sqrt{2})(\psi_\uparrow\psi_\downarrow+\psi^\dagger_\downarrow\psi^\dagger_\uparrow)$ term in $\mathcal{H}_{\rm int}$, then $\mathcal{M}_0^2$ is replaced by its renormalized value

$$\mathcal{M}^2=16(m+\mu)^2(1+u^2) , \tag{6.61}$$

where

$$u^2=g^4|B|^2\sum_k (8mE_k^3\Omega)^{-1} \tag{6.62}$$

with Ω being the volume and E_k given by (6.50). As we shall see, u is the sound velocity (when $e=0$). From (6.3), $M^2=4(m+\nu)^2$; here, ν_0 is also replaced by its renormalized value ν. In the nonrelativistic limit ν/m, μ/m, and u^2 are $\ll 1$; therefore

$$\mathcal{M}_0^2 \cong \mathcal{M}^2 \cong (2M)^2 \tag{6.63}$$

which is the threshold mass squared in the creation of a pair of Φ and anti-Φ quanta. For nonrelativistic applications, the R-mode excitation can be neglected. (It is the product of our artifice of using a relativistic field operator Φ for the bosons.)

(2) If we were in the sector with total particle number $N=0$, then $\rho_{\rm ext}=0$; in that case, as can be seen from (6.26), there is no direct coupling between R and $\nabla\cdot\mathbf{E}$ in H. [In (6.26) the $R\nabla\cdot\mathbf{E}$ coupling appears in the last term $-eV_0\rho_{\rm ext}$ through V_0, which is determined by (6.19).] At a given momentum $\mathbf{k}$, the vector meson $\mathbf{V}$ has two transverse modes and one longitudinal mode, all of the same energy $(m_V^2+k^2)^{1/2}$.

(3) In our present case, $N\neq 0$ and is macroscopic; this necessitates $\rho_{\rm ext}\neq 0$ and leads to an $R\nabla\cdot\mathbf{E}$ product term in $\mathcal{H}_0'$, given by (6.40). The normal modes of $\mathcal{H}_0'$ for a given momentum $\mathbf{k}$ consist of two transverse vector mesons of energy $(m_V^2+k^2)^{1/2}$, one longitudinal vector meson of energy ω_-, and one R quantum of energy ω_+ where

$$\omega_\pm^2=k^2+\tfrac{1}{2}(\mathcal{M}^2+m_V^2) \pm[16k^2(m+\mu)^2+\tfrac{1}{4}(\mathcal{M}^2-m_V^2)^2]^{1/2} . \tag{6.64}$$

In deriving this expression, we have replaced in $\mathcal{H}_0'$ the coefficient $\mathcal{M}_0^2$ of $\frac{1}{2}R^2$ by its renormalized value $\mathcal{M}^2$. For m_V/m and $\mu/m \ll 1$ and small k^2.

$$\omega_+^2 \cong \mathcal{M}^2+2k^2 \cong (2M)^2+2k^2 \tag{6.65}$$

confirming (6.63). Correspondingly,

$$\omega_-^2 = m_V^2 + k^2u^2 + (k^4/4M^2) \ . \tag{6.66}$$

When $e=0$, $m_V=0$, and $\omega_-^2 \to k^2u^2$ as $k\to 0$; therefore u is the "sound" velocity for the propagation of the phase angle θ in Φ when $e\to 0$.

When $e\to 0$, the longitudinal vector quantum becomes the Goldstone boson; its velocity in the section $N=0$ is the light velocity $c=1$, but becomes $u \ll 1$ in the sector N macroscopic and $\neq 0$.

(4) Historically, the inspiration of the Higgs mechanism came from the Landau-Ginsburg equation[18] for superconductivity. Our theory differs from the Landau-Ginsburg equation in being a mechanical system. We start from a (temperature-independent) Lagrangian and the thermodynamics is derived from the partition function, whereas the Landau-Ginsburg equation is, by construction, a thermodynamical model with its Lagrangian temperature dependent.

VII. GAP ENERGY IN TWO DIMENSIONS

So far we have considered only the three-dimensional continuum case. In this section we turn to the two-dimensional problem. It is well known that when the space dimension $D=2$, the long-range-order parameter B, defined by (2.6), vanishes. The proof due to Hohenberg[14] can be readily generalized to the present case of a mixed fermion-boson model.

Write the boson field ϕ as

$$\phi(\mathbf{r}) = |\phi(\mathbf{r})| e^{i\theta(\mathbf{r})} \ . \tag{7.1}$$

The long-range-order parameter B, defined by (2.6), denotes the coherence of $\theta(\mathbf{r})$ over a macroscopic distance. When $D=2$, $\theta(\mathbf{r})$ has to have sizable fluctuations over a length

$$l \approx \lambda_T e^{\sigma\lambda_T^2/2} \ , \tag{7.2}$$

where $\lambda_T = (Mk_BT/2\pi)^{-1/2}$ is the thermal wavelength and σ the two-dimensional particle density; consequently $B=0$. However, the gap energy Δ of the fermion system depends only on the constancy of boson density $|\phi|^2$ over a macroscopic distance, which can be realized in $D=2$ as well as in $D=3$, as we shall see.

Consider the analog problem of liquid helium. In $D=3$, liquid He II has both a constant density $|\phi|^2$ and a long-range coherent phase angle θ. Liquid He I has only the same constant liquid density $|\phi|^2$. The vanishing of the long-range coherent phase parameter is the origin of the Λ transition from He II to He I. The phase transition from He I to helium gas is connected to a change in $|\phi|^2$. In the hypothetical case of $D=2$, He II would cease to exist because of (7.2), but the phase transition between He I and helium gas would remain. Of course, He I is not a superfluid.

For practical applications, set $M\approx 5m_e$ and $T=T_c \approx 10^2$ K, then

$$\lambda_T \approx 30 \text{ Å}$$

Since on a typical CuO_2 plane of any of the high-T_c superconductors, σ is $\sim(10 \text{ Å})^{-2}$, the length l would be $\approx 2\times 10^3$ Å. (In reality, these superconductors are all three dimensional. Here, we consider the hypothetical case of a two-dimensional material.) Imagine a division of an infinite two-dimensional system into regions of size $<l$, but much larger than ξ (the coherence length). Within each region, at sufficiently low temperature the parameter B exists. The phase of B wanders from region to region, but its magnitude $|B|$ remains the same. Since the gap energy Δ depends only on $|B|$, we have the same Δ for the entire system. Hence, for such an infinite two-dimensional system there can be a genuine phase transition in the gap energy Δ; at $T<T_c$ we have $\Delta\neq 0$. Because the fermion system is closely tied to the boson system, we are not able to establish that there would be genuine superconductivity in a strictly two-dimensional system, even though the fermions may have a gap energy. However, it seems reasonable to expect at least *quasisuperconductivity* in $D=2$; i.e., superconductivity (including Meissner effect) over a finite distance $\approx l$. In the following, we shall compute the critical temperature.

As will be shown in the Appendix, for $D=2$, instead of (5.2) and (5.3), the gap energy Δ_0 and the chemical potential μ_0 at $T=0$ are given by (neglecting Δ_0^2/μ_0^2)

$$\nu - \mu_0 = \frac{g^2 m}{4\pi} \ln \frac{2(\mu_0\nu)^{1/2}}{\Delta_0} \tag{7.3}$$

and

$$\sigma = 2(\Delta_0/g)^2 + (m\mu_0/\pi) \ , \tag{7.4}$$

where σ is the two-dimensional particle density. At $T=T_c$, the gap energy is zero; instead of (5.18) and (5.19), T_c and the chemical potential μ_c are given by

$$\nu - \mu_c = \frac{g^2 m}{4\pi} \left[\gamma + \ln \frac{2(\mu_c\nu)^{1/2}}{\pi k_B T_c} \right] \tag{7.5}$$

and

$$\sigma = -\frac{Mk_BT_c}{\pi} \ln(1 - e^{2(\mu_c - \nu)/k_BT_c}) + \frac{mk_BT_c}{\pi} \ln(1 + e^{\mu_c/k_BT_c}) \ . \tag{7.6}$$

We neglect $(k_BT_c/\mu_c)^2$ in (7.5), but (7.6) holds to all orders in k_BT_c/μ_c. In (7.6), the first term on the right-hand side is two times the boson density which diverges at $\mu_c=\nu$; hence

$$\mu_c < \nu \ . \tag{7.7}$$

As in (5.8), define

$$\sigma_\nu \equiv m\nu/\pi \ . \tag{7.8}$$

the two-dimensional density when the Fermi energy equals ν.

(i) $\sigma < \sigma_\nu$ and $(\sigma_\nu - \sigma)/\sigma_\nu = O(1)$. In the weak-coupling limit, both Δ_0/μ_c and k_BT_c/μ_c are exponentially small. Hence, from (7.4) and (7.6), we derive as an approximation

$$\mu_0=\mu_c=\pi\sigma/m \tag{7.9}$$

and therefore, from (7.3) and (7.5)

$$\frac{\Delta_0}{k_B T_c}=\pi e^{-\gamma}=1.7639 , \tag{7.10}$$

the same relation as in (5.25).

(*ii*) $\sigma>\sigma_\nu$. As the zeroth approximation we may set $\mu_0=\nu$ in (7.4) and $\mu_c=\nu$ in the last term on the right-hand side of (7.6):

$$\Delta_0^2=\frac{g^2}{2}(\sigma-\sigma_\nu) \tag{7.11}$$

and

$$\sigma=\frac{Mk_BT_c}{\pi}\ln(1-e^{2(\mu_c-\nu)/k_BT_c})+\frac{mk_BT_c}{\pi}\ln(1+e^{\nu/k_BT_e}) . \tag{7.12}$$

By using (7.5) and (7.12) we can solve for μ_c and T_c. Substituting (7.11) into (7.3), we derive the first-order correction in $\nu-\mu_0$.

In $D=2$, the boson density diverges at $\mu_c=\nu$. Hence, as noted before in (7.7), $\mu_c<\nu$. Except for this important difference, the overall dependence of Δ_0 on T_c is rather similar to the three-dimensional case.

ACKNOWLEDGMENTS

We thank Y. J. Uemura and M. K. Wu for informative discussions. This research was supported in part by the U.S. Department of Energy.

APPENDIX

1. $D=2$

In two dimensions, the gap equation (4.3) becomes

$$\nu-\mu=(8\pi)^{-1}mg^2f_2(\Delta,\beta) , \tag{A1}$$

where

$$f_2(\Delta,\beta)=\int_0^\infty d\omega\left[\frac{1}{E}\tanh\tfrac{1}{2}E\beta+P\frac{1}{\nu-\omega}\right] \tag{A2}$$

and

$$E=[(\omega-\mu)^2+\Delta^2]^{1/2} . \tag{A3}$$

Theorem. Neglecting $O(e^{-\mu/k_BT})$ and $O(\Delta^2/\mu^2)$,

$$f_2(\Delta,\beta)=2F(\Delta\beta)+\ln[\beta^2\nu\mu/(2\pi)^2] , \tag{A4}$$

where, as in (5.39), $F(\Delta\beta)$ is given by

$$F=\lim_{N\to\infty}\left\{\sum_{n=0}^{N-1}\left[(n+\tfrac{1}{2})^2+\left[\frac{\beta\Delta}{2\pi}\right]^2\right]^{1/2}-\ln N\right\} . \tag{A5}$$

[Note that the same function $F(\Delta\beta)$ appears in the formulas for $D=2$ and $D=3$.]

Proof. Let $z\equiv\beta E$ and $\xi\equiv\beta\Delta>0$. Replace the upper limit in (A2) by $z=Z$, with $Z\to\infty$ in the end. Since $d\omega/E=dz/(z^2-\xi^2)^{1/2}$, (A2) can be written as

$$f_2(\Delta,\beta)=\lim_{Z\to\infty}[I(Y,Z)+\ln(\beta\nu/Z)] , \tag{A6}$$

where

$$I(Y,Z)=\left[\int_\xi^Y dz+\int_\xi^Z dz\right](z^2-\xi^2)^{-1/2}\tanh\tfrac{1}{2}z \tag{A7}$$

and $Y\equiv(\mu^2+\Delta^2)^{1/2}\beta\cong\mu\beta$.

From (A6), we obtain (A4) directly provided that F is defined by

$$2F(\beta\Delta)=\lim_{Y,Z\to\infty}\left[I(Y,Z)+\ln\left[\frac{4\pi^2}{YZ}\right]\right] . \tag{A8}$$

In the complex $z=x+iy$ plane, $(z^2-\xi^2)^{1/2}$ has two branch points: $z=\pm\xi$. Arrange the cuts along the real axis from $x=-\infty$ to $-\xi$ and then from $x=\xi$ to ∞, as illustrated in Fig. 3. Immediately above the cuts $z=x+i0+$, choose $(z^2-\xi^2)^{1/2}$ positive when $x>\xi$. Therefore it is negative when $x<-\xi$, i.e., $(z^2-\xi^2)^{1/2}$ is odd in x above these two cuts. On the real axis, between two cuts $(z^2-\xi^2)^{1/2}=i(\xi^2-x^2)^{1/2}$ which is even in x. Hence, along the real axis the integrand in (A8) is *even* in z immediately above the cuts and *odd* in z between the cuts. Thus we may write

$$I(Y,Z)=\int_{-Y}^{Z}\frac{dz}{(z^2-\xi^2)^{1/2}}\tanh\frac{z}{2} \tag{A9}$$

along the solid path indicated in Fig. 3.

Let C be the closed contour consisting of the solid path and the three dashed paths

C_1: $z=Z$ to $Z+2i\pi N$,

C_2: $z=Z+2i\pi N$ to $-Y+2i\pi N$,

and

C_3: $z=-Y+2i\pi N$ to $-Y$,

where N is a very large integer. Since $\tanh\frac{1}{2}z$ has poles at $z=i\pi(2n+1)$ with n being any integer,

$$\oint_C\frac{dz}{(z^2-\xi^2)^{1/2}}\tanh\frac{z}{2}=4\pi\sum_{n=0}^{N-1}[\pi^2(2n+1)^2+\xi^2]^{-1/2} . \tag{A10}$$

Along C_1, neglecting e^{-Z} we may approximate $\tanh(z/2)\cong1$; likewise, along C_3, neglecting $e^{-Y}\cong e^{-\beta\mu}$ we may approximate $\tanh(z/2)\cong-1$. Keep Y and Z fixed (but large) and choose $2\pi N\gg Z$, or Y. The sum of the integrals along C_1 and C_3 [neglecting also $(\Delta/\mu)^2$; this approximation can be easily improved] is $\ln(4\pi^2N^2/YZ)$, while the integral along $C_2\to0$ as $N\to\infty$.

Thus, subtracting $C_1+C_2+C_3$ from (A10), we find

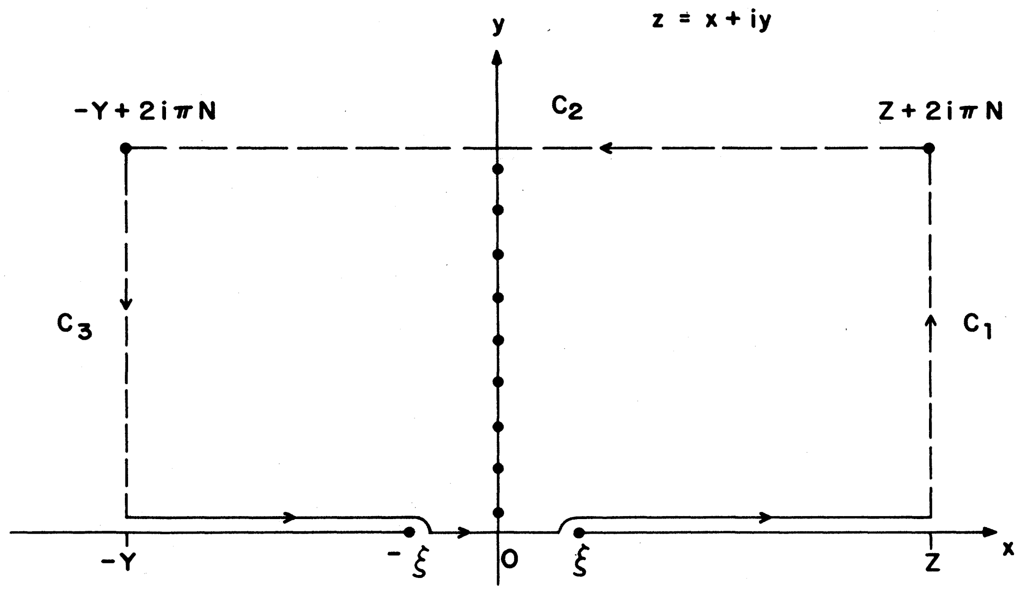

FIG. 3. The contour C for the integration of (A9) in the complex z plane.

$$I(Y,Z)=\lim_{N\to\infty}\left\{2\sum_{n=0}^{N-1}\left[(n+\tfrac{1}{2})^2+\left[\frac{\xi}{2\pi}\right]^2\right]^{-1/2}-\ln\frac{4\pi N^2}{YZ}\right\} \tag{A11}$$

which with (A8) gives us (A5).

Expanding $\tanh(\frac{1}{2}E\beta)$ in the integrand of (A2) as a power series in $e^{-E\beta}$, and comparing with (A4), we find that F is also given by (5.41). Equation (7.3) or (7.5) now follows from (A1) and (A4) by taking the leading term from (5.41) or (5.40), respectively.

2. $D=3$

In three dimensions, (4.3) becomes

$$\nu-\mu=\frac{1}{2}\left[\frac{g}{\pi}\right]^2\left[\frac{m}{2}\right]^{3/2}f_3(\Delta,\beta)\ , \tag{A12}$$

where $f_3(\Delta,\beta)=f(\Delta,\beta)$ of (5.37):

$$f_3(\Delta,\beta)=\int_0^\infty\sqrt{\omega}\,d\omega\left[\frac{1}{E}\tanh\tfrac{1}{2}E\beta+P\frac{1}{\nu-\omega}\right]. \tag{A13}$$

As we shall show, neglecting $(k_BT/\mu)^2$, $e^{-\mu/k_BT}$, and $(\Delta/\mu)^2$, $f_3(\Delta,\beta)$ is given by (5.38) and (5.39).

Choose $\mu\gg\tau\gg\Delta$ with $\tau\beta\gg1$ [but $\Delta\beta=O(1)$]. Separate the integral into three sectors with ω varying from

(i) 0 to $\mu-\tau$,
(ii) $\mu-\tau$ to $\mu+\tau$,
(iii) $\mu+\tau$ to λ,

and $\lambda\to\infty$ in the end. In (i) and (iii), we neglect $e^{-\tau/\beta}$, $(\tau/\mu)^2$, and $(\Delta/\tau)^2$; therefore $\tanh\frac{1}{2}E\beta\cong1$. The integrations of $\sqrt{\omega}(\tanh\frac{1}{2}E\beta)/E$ over (i) and (iii) give

$$(\text{i})+(\text{iii})=2\sqrt{\mu}(-2+\ln4\mu/\tau)+2\sqrt{\lambda}\ ; \tag{A14}$$

as $\lambda\to\infty$ the term $2\sqrt{\lambda}$ just cancels the last term in (A13). For sector (ii), we neglect $(\tau/\mu)^2$ and therefore $\sqrt{\omega}\cong\sqrt{\mu}$; next we regard $\tau/\Delta\gg1$ which makes the integral over (ii) just $\sqrt{\mu}I(W,W)$ with I given by (A7), and $W=(\tau^2\beta^2+\xi^2)^{1/2}\cong\tau\beta$.

Combining the sectors and canceling $2\sqrt{\lambda}$, we obtain

$$f_3(\Delta,\beta)=2\sqrt{\mu}\left[-2+\ln\frac{4\mu\beta}{W}\right]+\sqrt{\mu}I(W,W) \tag{A15}$$

which on taking $W\gg1$ reduces to (5.38) in view of (A8) and $f_3(\Delta,\beta)=f(\Delta,\beta)$. The expressions (5.39)–(5.41) have already been shown equivalent to (A8).

Now (5.2) or (5.18) follows from (A12) with (5.38) by using the leading term of (5.41) or (5.40). The higher corrections in (5.28) and (5.29) are obtained from the standard treatment of a nearly degenerate Fermi gas.

[1]P. Chaudhari *et al.*, Phys. Rev. B **36**, 8903 (1987).
[2]T. K. Worthington, W. J. Gallagher, and T. R. Dinger, Phys. Rev. Lett. **59**, 1160 (1987).
[3]J. C. Bednorz and K. A. Müller, Z. Phys. B **64**, 189 (1986).
[4]M. K. Wu, J. R. Ashburn, C. J. Torng, P. H. Hor, R. L. Meng, L. Gao, Z. J. Huang, Y. Q. Wang, and C. W. Chu, Phys. Rev. Lett. **58**, 908 (1987); Z. X. Zhao, L. Chen, Q. Yang, Y. Huang, G. Chen, R. Tang, G. Liu, C. Cui, L. Chen, L. Wang, S. Guo,

S. Lin, and J. Bi, Kexue Tongbao **6**, 412 (1987).
[5]R. Friedberg and T. D. Lee, Phys. Lett. A (to be published).
[6]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **106**, 162 (1957); **108**, 1175 (1957).
[7]Y. J. Uemura *et al.*, Phys. Rev. Lett. **62**, 2317 (1989).
[8]V. J. Emery and G. Reiter, Phys. Rev. B **38**, 4547 (1988).
[9]N. N. Bogoliubov, Nuovo Cimento **7**, 6 (1958); **7**, 794 (1958).
[10]J. Valatin, Nuovo Cimento **7**, 843 (1958).
[11]W. Meissner and R. Ochsenfeld, Naturwissenschaften **21**, 787 (1933).
[12]Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960).
[13]J. Goldstone, Nuovo Cimento **19**, 154 (1961).
[14]P. Hohenberg, Phys. Rev. **158**, 383 (1967).
[15]D. M. Newns, M. Rasolt, and P. C. Pattnaik, Phys. Rev. B **38**, 6513 (1988).
[16]P. W. Higgs, Phys. Lett. **12**, 132 (1964).
[17]F. London, *Superfluids* (Wiley, New York, 1954).
[18]V. L. Ginsburg and L. D. Landau, J. Exp. Theor. Phys. (USSR) **20**, 1064 (1950).