PHYSICAL REVIEW B VOLUME 40, NUMBER 10 1 OCTOBER 1989

Velocity-dependent dissipation from free vortices and bound vortex pairs below the Kosterlitz-Thouless transition

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(Received 19 December 1988)

We have developed a model for the onset of nonlinear dissipation in thin superfluid helium films below the static transition temperature (T_{KT}) by extending the linear response theory of Ambegaokar, Halperin, Nelson, and Siggia to include flow velocity. Contributions from both vortex pairs and free vortices are included, the relative contributions of which are controlled by two weakly coupled adjustable parameters: the vortex diffusivity D and a free vortex creation time τ_f. The model does not predict a critical velocity in the usual sense except at $T=0$; however, we define a characteristic velocity for the onset of nonlinear dissipation to compare with experiments. At $T=0$ we find a critical velocity given by the Feynman criterion, where the frequency-dependent diffusion length coincides with the zero in the vortex pair energy. Applications of this model to experiments with ac and dc flows are discussed.

I. INTRODUCTION

A. Static transition

The superfluid transition in thin planar ^{4}He films is believed to be a realization of a two-dimensional Kosterlitz-Thouless (KT)[1] transition, indicated by the loss of topological long-range order above a coverage-dependent temperature T_{KT}. The transition is marked by the appearance of topological excitations (free vortices) which destroy the phase correlation in the local order parameter. Below T_{KT}, vortices of opposite vorticity are associated in pairs which preserve phase correlations.

The existence of the KT transition can be understood by a simple free-energy argument. The creation energy of an isolated vortex in a stationary two-dimensional (2D) superfluid grows as $\ln(R/a)$, where R is the size of the system and a is the vortex core diameter (~ 1 Å). Since R can be macroscopic in size, an isolated vortex is unstable at low temperatures and is rarely created by thermal fluctuations. On the other hand, the creation energy of a vortex pair grows as $\ln(r/a)$, where r is the separation distance between the two vortices, and is independent of the system size. We expect then that the 2D superfluid is populated with vortex pairs with various separations according to a Boltzmann distribution. Since the entropy of a vortex pair also varies as $\ln(r/a)$, the free energy, $F=E-TS$, will fall negative above some critical temperature T_{KT}, indicating the instability of vortex pairs and the favoring of isolated (free) vortices. These free vortices destroy the phase correlation needed for superfluidity, and thus, T_{KT} defines a transition temperature at which the pair with the largest separations unbind and superfluidity is destroyed on this scale. Above the transition, pairs of smaller separation can exist for a time limited by their diffusivity, and superfluidity will be observed if the system is probed on the time scale of these pairs.

A calculation of the superfluid density in the KT theory starts with a *bare* superfluid density σ_0, which already takes into account the effects of the elementary excitations (phonons, rotons, and ripplons). The effect of the vortex pairs on σ_0 is calculated by introducing a length-dependent dielectric constant $\epsilon(r)$ in the creation energy of a vortex pair of separation r to account for the screening by pairs of smaller separation. In this way the smaller length scales are integrated out, and the partition function represents successively coarser-grained systems. For a static superfluid this scheme is continued out to infinite length scales, and the superfluid density is given by

$$\sigma_s = \lim_{r\to\infty} \frac{\sigma_0}{\epsilon(r)} . \tag{1}$$

A depression of the superfluid density near the transition is due to an abundance of vortex pairs which screen the interaction of larger pairs thereby lowering their creation energy. This lower creation energy results in more vortex pairs being produced by thermal fluctuations. The net effect is an increase in $\epsilon(\infty)$ and a corresponding decrease in σ_s. However, the dielectric constant remains finite, as the transition is approached from below and diverges above the transition, producing a square-root cusp and a discontinuous jump in σ_s given by

$$\frac{\sigma_s(T_{\mathrm{KT}}^-)}{k_B T_{\mathrm{KT}}} = \frac{2}{\pi}\left[\frac{m}{\hbar}\right]^2 , \tag{2}$$

where m is the mass of a helium atom. This is the famous universal result, independent of substrate composition, film thickness, or impurities.[2] Numerous experiments have shown the validity of Eq. (2) over a wide range of temperatures and coverages.[3]

B. Dissipation in two-dimensional superfluids

The Kosterlitz-Thouless theory in its original form assumes that, except for the vortex flow fields themselves, the superfluid is at rest. However, for the experiments that use third sound to probe the film characteristics, the superfluid velocity is not zero. Ambegaokar, Halperin, Nelson, and Siggia[4] (AHNS) and Huberman, Myerson, and Doniach[5] (HMD) have extended the static KT theory to include nonzero superfluid velocities. The theory of HMD is restricted to dc flow, whereas the more complete theory of AHNS applies in the frequency domain and thus subsumes the HMD theory.

In the dynamic theory, when the response of vortex pairs to the oscillating flow is taken into account, the finite diffusivity limits the size of those pairs that remain in dynamic equilibrium with the flow. Small vortex pairs can quickly respond and orient themselves perpendicularly to the flow much like an electric dipole in an applied field. These small pairs move along with the superfluid and do not dissipate energy. On the other hand, very large pairs do not have enough time to respond during one oscillation because of their large inertia and finite diffusivity. The maximum dissipation will come from those pairs with separation $r_c(\omega)$, which are maximally (90°) out of phase with the flow, where[4]

$$r_c(\omega) \equiv \sqrt{14D/\omega} \; . \tag{3}$$

The dielectric constant is now a complex function whose real part accounts for the screening of the pair binding energy by pairs with separation less than r_c, and whose imaginary part is a measure of the energy dissipation.

The details of the AHNS theory are carried out assuming that there is a linear response of the vortex pair distribution to an applied superflow. This assumption is valid for low velocities such that $r_c \ll \hbar/mV_s$, and corresponds to $\sim$1 cm/sec at 1000 Hz. Under these conditions, the model is independent of velocity and cannot explain the observed nonlinear behavior shown in Fig. 1.

HMD and AHNS independently used nucleation theory[6] to predict the velocity-dependent dissipation mechanism. In the presence of the fluid flow, the creation energy of a vortex pair, as seen in the lab frame, has a high barrier centered at $r_b \equiv \hbar/mV_s$. The activation of a vortex pair over this barrier results in the breakup of the pair and an accompanying dissipation. A steady state is achieved when the activation rate over the barrier balances the annihilation rate of vortices with other vortices or with the boundaries.

We find nucleation theory to be inadequate for several reasons: (1) our closed geometry has no boundaries, (2) at low temperatures ($\sim$0.1 K), the barrier height can be $\sim$30 K, which precludes thermal activation from ever happening during an experiment, and (3) even under the condition given in (2) we still observe a strong velocity dependence to the dissipation.

Other attempts to explain dissipation in 2D helium films include soliton or shock wave formation[7] and vortex pinning mechanisms.[8] For very thin films in a closed geometry, shock waves cannot form and our line-shape analysis indicates that we indeed do not have shock

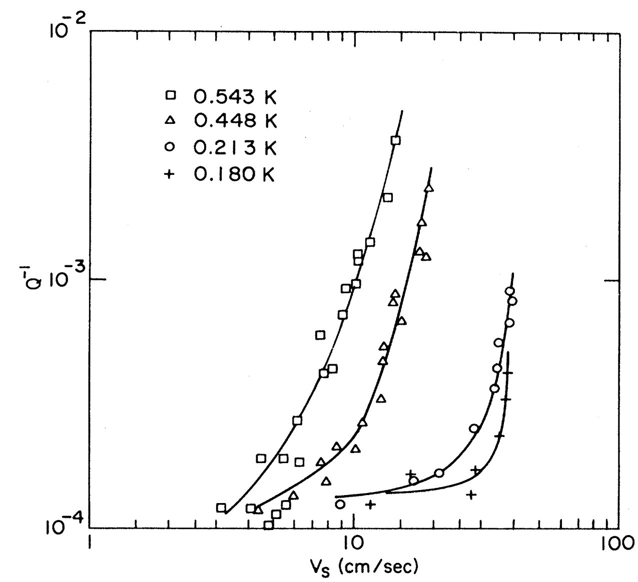

FIG. 1. Dissipation Q^{-1} vs V_s for a thin helium film on neon. At low velocities the resonance Q saturates due to processes other than vortex diffusion. Note the increasing sharpness of the onset of nonlinear dissipation as the temperature is lowered. The solid lines are present to aid the eye. (Taken from Ref. 9.)

waves in our third-sound resonators. Although we cannot conclusively rule out pinning as a possible dissipation mechanism, we wish to explore in further detail the intrinsic properties of the film and invoke pinning only as a last resort.

In this paper we present a model for the onset of nonlinear dissipation in thin superfluid helium films below the static transition temperature T_{KT}. Our model is an extension of the linear response theory of Ambegaokar, Halperin, Nelson, and Siggia to include flow velocity in the recursion relations. Section II describes the development of the model and the various contributions to the dissipation. Emphasis is placed on ac flow since we find the description of this case more illuminating than dc flow. Section III discusses the use of the model to analyze experimental data and the results of fits to the data of Ref. 14. Comparison is also made with the dc flow measurements of Ref. 13.

II. EXTENSION OF THE LINEAR THEORY

A. Recursion relations

Our approach to this problem is to explicitly include the flow velocity in the KT recursion relations and, together with a set of assumptions, carry out the renormalization on a computer in the spirit of AHNS. Our assumptions are (1) vortex pinning is not important for these thin films and low temperatures, (2) vortex pairs are long lived compared to the period of a third-sound oscillation in our resonators, (3) dynamic response is dealt with in the same manner as AHNS, (4) the vortex pair density is low enough that mean-field theory works, (5) thermal activation of vortex pairs over the barrier is not an important dissipation mechanism under our conditions, and (6) contributions from bound as well as free vortices must be taken into account even for temperatures below T_{KT}. The vortex diffusivity D and free vortex creation time τ_{free} appear as weakly coupled adjustable parameters.

The thermodynamics of 2D vortices is effected by the presence of a macroscopic superfluid velocity through the vortex pair energy. In the absence of the vortex pinning, and if the vortex pairs are sufficiently long lived, the energy of a vortex pair in a uniform flow can be obtained by a Galilean transformation to the reference frame of the moving superfluid. The details of this derivation are given by Gillis, Volz, and Mochel[9] and will not be repeated here.

The interaction energy between two oppositely "charged" vortices is modified by a macroscopic flow to

$$U(\mathbf{r})=2\pi\sigma_0\left[\frac{\hbar}{m}\right]^2\int_a^r\left[\frac{dr'}{r'\epsilon(r')}\right]+2\pi\sigma_0\left[\frac{\hbar}{m}\right](\hat{\mathbf{z}}\times\mathbf{V}_s)\cdot\mathbf{r}\ , \tag{4}$$

where $\mathbf{r}\equiv\mathbf{r}_+-\mathbf{r}_-$ is the separation vector of the vortices in the pair, σ_0 is the bare superfluid areal mass density, and $\epsilon(r)$ is a dielectric constant which accounts for the screening effect of smaller pairs. The vortex pair energy is now dependent on the orientation of the pair with respect to the superfluid flow. We can see that there will be asymmetry in the angular distribution of thermally excited vortex pairs. In fact, the energy can actually fall to zero in the presence of a superfluid velocity, which means that the pairs can unbind. It is the unbinding that can occur at any $T<T_{KT}$, due to the macroscopic superfluid flow, that leads to one form of nonlinear dissipation.

Because of the loss of topological order that occurs with the presence of unpaired vortices, the macroscopic flow can actually induce a Kosterlitz-Thouless transition at any temperature, and allow us to use the recursion relations far below the static transition temperature. Although a macroscopic flow induces a pair breaking transition, superfluid behavior can be observed if the fluid is probed on sufficiently small length scales. Driving the film with an ac flow such as with third-sound resonances or torsional oscillator techniques, we can probe the film on a length scale where vortices are bound and superfluid behavior is still observed.

The details of the derivation of the recursive relation were presented in Ref. 9 and will not be repeated in detail here. The important differences between our derivation and that of others is the inclusion of $\mathbf{V}_s$ in the vortex pair energy, and an average over the angular dependence. The angular average is a statement of our assumption that the lifetime of a typical vortex pair is much larger than the period of a third-sound oscillation in our cells, and during our period the vortex pair will be at various angles with the flow.

The recursion relations become

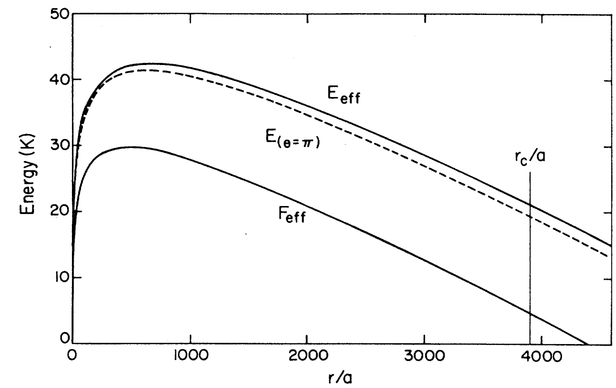

FIG. 2. E_{eff}, $E(\sigma=\pi)$, and F_{eff} are plotted vs r/a at the characteristic velocity of 24.5 cm/sec when $T/T_{KT}=0.5$, $f=2.28$ kHz, and $D=0.01\hbar/m$. The vertical line is the cutoff length r_c/a.

$$[y(l)]^2=(y_0)^2\exp\left[4l-2\pi\int_0^l dl'\,K(l')\right]\times I_0\left[2\pi K_0\left[\frac{V_s}{V_a}\right]e^l\right], \tag{5a}$$

$$[K(l)]^{-1}=(K_0)^{-1}+4\pi^3\int_0^l dl'[y(l')]^2\ , \tag{5b}$$

$$K(l)=\frac{K_0}{\epsilon(l)}\ ,$$

$$K_0=\left[\frac{\hbar}{m}\right]\frac{\sigma_0}{k_BT}\ , \tag{5c}$$

and

$$V_a\equiv\frac{\hbar}{ma}\ .$$

These relations are identical to the original ones except for the Bessel function in Eq. (5a). Note that Eq. (5a) is simply the probability that the system is in a state containing a vortex pair on a length scale l, i.e., $\langle e^{-F(l)/kT}\rangle_\theta$. Figure 2 shows an example of $E(r,V_s)$ (averaged over θ) given by $-kT\ln[a^4\langle\Gamma(r)\rangle_\theta]$, and the effective free energy $-kT\ln[y(l)]^2$ for the case $\epsilon(r)=1$. It is the value of the free energy at the cutoff length scale $l_c(\omega)$ that determines the Q of the macroscopic system.

B. Vortex pair contribution

The dynamic response, calculated by AHNS (Ref. 4) via a Fokker-Planck[10] equation, shows that the pairs on a length scale $l<l_c(\omega)$ can remain in dynamic equilibrium with an oscillating flow, in which case the drag force and the Magnus force are exactly balanced, and the rate of change of the separation vector is given by

$$\frac{d\mathbf{r}}{dt}=\left[-\frac{2D}{k_BT}\right]\nabla U+\boldsymbol{\eta}(t)\ , \tag{6}$$

where U is the interaction energy given by Eq. (4) and $\eta(t)$ is a Gaussian noise term caused by thermal fluctuations. The effect of this motion will be to orient the pairs perpendicular to the flow, as well as change the separation distance. In this sense the angular average may not be necessary since the pairs will always quickly orient themselves to minimize the energy. However, the pairs on a length scale $l \geq l_c$ are unable to maintain this equilibrium, and it is for these pairs that the angular average is most important.

In the original calculation of the vortex pair dynamic response function, AHNS find that their result can be reasonably described by

$$g(r,\omega)=\left[\frac{14Dr^{-2}}{14Dr^{-2}-i\omega}\right]=\frac{1}{1-i\omega\tau_b}, \tag{7}$$

where $g(r,\omega)=1$ corresponds to local equilibrium for pairs with separation r. The real part of $g(r,\omega)$ represents the in-phase response of the pairs to the oscillating flow and illustrates the low pass behavior of the vortex pair system: for $\omega\tau_b \ll 1$ the pairs maintain equilibrium during the oscillations, and for $\omega\tau_b \gg 1$ the pairs are in a nonequilibrium regime and do not respond. The imaginary part of $g(r,\omega)$ shows the out-of-phase response, which is peaked at $r=r_c=(14D/\omega)^{1/2}$ (i.e., $\omega\tau_b=1$), and is an indicator of the dissipation in the system. This function is then approximated by

$$\mathrm{Re}[g(r,\omega)]=\theta(14Dr^{-2}-\omega), \tag{8a}$$

$$\mathrm{Im}[g(r,\omega)]=\tfrac{1}{4}\pi\delta(r-\sqrt{14D}/\omega), \tag{8b}$$

which retains the essential features of Eq. (7). The physical significance of (8a) is that the vortex pairs larger than r_c do not participate in the dynamic response and, hence, do not contribute to the dynamic dielectric constant. The complex dielectric constant is given by

$$\begin{aligned}\epsilon^*(\omega)&=1+\int_a^\infty g(r,\omega)\frac{d\epsilon}{dr}(dr)+\epsilon_{\mathrm{free}}\\&=\epsilon(r_c)+i\frac{\pi}{4}r_c\left[\frac{d\epsilon}{dr}\right]_{r_c}+\epsilon_{\mathrm{free}},\end{aligned} \tag{9}$$

where $r_c(\omega)$ is given by Eq. (3) and ϵ_{free} is a purely imaginary contribution from free vortices, which we will discuss later. The relevant renormalization parameters for 2D superfluid, probed with frequency ω, are found at the length scale l_c:

$$\mathrm{Re}[\epsilon^*(\omega)]=\epsilon(l_c), \tag{10a}$$

$$\begin{aligned}\mathrm{Im}[\epsilon^*(\omega)]&=\frac{\pi}{r}\left[\frac{d\epsilon}{dl}\right]_{l_c}+\mathrm{Im}(\epsilon_{\mathrm{free}})\\&=\pi^4K_0[y(l_c)]^2+\mathrm{Im}(\epsilon_{\mathrm{free}}),\end{aligned} \tag{10b}$$

$$\sigma_s(\omega)=\frac{\sigma_0}{\epsilon(l_c(\omega))}, \tag{10c}$$

$$Q^{-1}=\frac{\mathrm{Im}[\epsilon^*(\omega)]}{\mathrm{Re}[\epsilon^*(\omega)]}, \tag{10d}$$

$$l_c(\omega)=\ln\left[\frac{r_c}{a}\right]=\tfrac{1}{2}\ln\left[\frac{14D}{\omega a^2}\right]. \tag{10e}$$

C. The free vortex contribution

At this point, we must emphasize the conditions under which the dynamic response function Eq. (7) is calculated. AHNS assume the superfluid velocity is low enough that $r_c \ll r_b$, where $r_0=\hbar/mV_s$. Under these conditions the effect of V_s on the vortex pair energy and the vortex pair distribution are neglected. Their conclusions are then that the transition temperature T_{KT} from the static theory remains the pair breaking transition in the dynamic theory, and free vortices are assumed to occur only above the transition or under conditions of very large flow velocities (in which case they rely on nucleation theory to determine the dissipation). Therefore, the contribution from free vortices in Eq. (10) is added only above T_{KT} in their theory.

We believe that for our experiments the vortex pairs up to the cutoff length scale l_c can remain in thermal equilibrium with the fluid (so that their population is described by a Boltzmann distribution) even under the condition $r_c > r_b$, and the oscillating flow field creates an adiabatic change to the vortex pair energy given by Eq. (4) for these pairs. Furthermore, we find that the presence of the superfluid flow can cause the formation of free vortices even below T_{KT} whose contribution cannot be ignored.

Recall from the static theory that the transition occurs when renormalization suppresses the free energy to zero at infinite separation. A zero in the free energy at a length scale l_f indicate the breakup of vortex pairs on that scale, and as the temperature is raised l_f moves to smaller and smaller scales. When the superfluid velocity is induced in the free energy we find that the presence of any nonzero V_s causes the occurrence of such a zero at *all* temperatures and necessitates the inclusion of the free vortex contribution even below T_{KT}.

The free vortex contribution to the dielectric constant is calculated in the Debye-Huckel[11] approximation in which the free vortices diffuse in a macroscopic flow field given by the average of the superfluid velocity over the film. The calculation (which will not be presented here) is easily performed in analogy with a 2D plasma, where we find a conductivity for the 2D charges given by[4]

$$\sigma=n_f q_0^2\left[\frac{D}{k_BT}\right]=n_f\pi\sigma_s\left[\frac{\hbar}{m}\right]^2\frac{D}{k_BT}, \tag{11}$$

where n_f is the free vortex density and q_0 is the charge. The contribution to the complex dielectric constant is found to be

$$\epsilon_{\mathrm{free}}=3\pi i\sigma/\omega=i4\pi^2 n_f\left[\frac{\hbar}{m}\right]^2\frac{D}{\omega k_BT}. \tag{12}$$

To determine n_f notice that its largest possible value is $1/\xi^2$, where ξ is the separation for the zero in the free energy. One might expect that, since the free energy is zero, the system should fill itself with free vortices up to this maximum density. However, if there is a process which limits the production of free vortices, then n_f can be much smaller than $1/\xi^2$. This is, in fact, what we find when we compare the dissipation in our third-sound resonators with the model. We therefore introduce a free vortex creation time τ_{free}, which we use as a fitting parameter to bring the data and the model into agreement. The motivation for this type of parameter is that if the characteristic time to create free vortices is longer than the period of a third-sound oscillation, then n_f will be limited to less than its maximum value. On the other hand, if the creation time is much shorter than the third-sound period than n_f will presumably reach its maximum value during the third-sound oscillation. With this in mind, we assume that the free vortex density is given by

$$n_f = \frac{1/\xi^2}{[1+(\omega\tau_{\text{free}})^2]^{1/2}} . \tag{13}$$

We have remarked that vortices are considered free when the free energy goes to zero. Perhaps a more rigorous definition comes from the two-point correlation function for the vortex charge density. In this case we find that for large separations the correlations die away exponentially, indicating the presence of isolated vortices on this scale. The characteristic length for the decay of these correlations can be taken as the average distance between free vortices. We argue that the vanishing of the free energy is responsible for the breakup of vortex pairs and the loss in correlations. In any event, we have a working definition of the free vortex separation, which may differ by a factor of order 1 from the correlation function definition.

The Kosterlitz-Thouless theory describes a transition from power law to exponential decay with distance in the two-point correlation function caused by the appearance of isolated (or free) vortices for a 2D superfluid. We conclude from our analysis of the free vortex contribution to the dissipation that the presence of a superfluid velocity itself actually induces a Kosterlitz-Thouless-like transition in the film, but the diffusion limited processes allow us to see superfluid behavior when we probe the fluid on small enough length scales.

AHNS propose a model based on nucleation theory to explain the velocity-dependent dissipation due to the formation of free vortices. They expand Eq. (3.11) about the saddle point $r_b = h/mV_s\epsilon$, and calculate from the activation model the rate at which pairs separate. The result is a power-law velocity dependence in the dissipation near the transition. They assume that dissipation results from vortices crossing over the potential barrier that is created by the flow, and this assumption leads to characteristic velocities which are close to V_a given in Eq. (5c) (~ 1500 cm/sec) and much larger than we see experimentally. However, we calculate the rate of vortex pair activation over the barrier for typical velocities from our experiments (10–100 cm/sec) and find the rate to be so small that this process will never happen during an experiment. This result raises issues about the population of vortex pairs at large length scales (beyond the barrier at n_b) and how these pairs were created in the first place. These questions are not new ones, and we will not attempt to address them here except to say that these pairs were probably created by the joining of isolated vortices upon cooling the helium film through the transition temperature.

D. Characteristic versus critical velocities

In bulk helium superfluidity will be observed as long as the flow velocity is smaller than some critical value. Larger velocities favor the creation of excitations which dissipate the kinetic energy and destroy superfluidity. This original idea presented by Landau assumes that these excitations are the elementary excitations of the fluid (phonon, rotons, etc.) and predicts a critical velocity of ~ 60 m/sec. This theory assumes that there are no other excitations which can be created with smaller velocities. In practice the critical velocities that are observed in bulk He are 1–1000 cm/sec indicating that other excitations do indeed exist which have a dispersion with $(E/p)_{\min}$ lower than that for rotons.

Feynman[12] has shown that, since a boundary in a fluid acts as an image plane for vortices (because $\mathbf{V}_s \cdot \hat{\mathbf{n}} = 0$ on the boundary), vortices can be created at the boundary when the flow velocity is sufficiently large that the energy to create a vortex and its image is zero in the reference frame of the wall. In a capillary the excitation is a vortex ring with radius R equal to the radius of the tube. The energy of this ring is dependent on R, and we see that there are instances where a unique critical velocity cannot be defined in terms of intrinsic fluid parameters only. Nevertheless, it seems clear that for a given system of superfluid and its boundaries, well-defined critical velocities do exist in bulk superfluids.

In thin superfluid films, persistent current experiments show that superfluid flow is unstable against decay even when the velocity is very small. We will return to this subject of persistent currents in the section of dc flow, but for now it suffices to say that critical velocities (as seen in bulk helium) have not been observed in thin helium films.

In the ac theory presented here, the concept of critical velocities seems to have no meaning except at $T=0$, as we shall see. The increase in the dissipation is a continuous function of the superfluid velocity, up to the point where the resonance Q has dropped to 1, at which point the resonance is unobservable anyway. To characterize the onset of nonlinear dissipation we define a *characteristic* velocity V_{sc} as that velocity which drives the resonance Q down to some specific value, Q_c from its $V_s=0$ limit Q_0. This somewhat arbitrary, yet well-defined criterion allows us to compare our theory to experiments in a precise way. The characteristic velocity as we have defined it is merely useful as a shortcut for analyzing the third-sound data. Alternatively, we could use a nonlinear least-squares analysis to fit every point of the data, but such a time consuming method would scarcely yield any additional information. Our use of characteristic velocities is a convenient way of characterizing a film for easy comparison. In the $T=0$ limit our characteristic velocity has the added advantage of going smoothly over to a critical velocity as will be discussed below.

The characteristic velocity for a given coverage of superfluid helium is dependent on the vortex diffusivity, the frequency of the film oscillations, the temperature, and Q_c. These dependences are understood by noting the relative position of $r_c(\omega)$ and ξ, where the zero in the free energy occurs. Decreasing r_c means probing the system on smaller length scales where the vortex density is lower (higher free energy) and, thus, so is the dissipation. Increasing the velocity will bring ξ closer to r_c thereby increasing the vortex density and bring the dissipation back to the level Q_c^{-1}.

For $T=0$ we find a Q-independent characteristic velocity below which there is no dissipation and above which the Q is driven down below 1. An expression for this *critical* velocity is similar to a Feynman criterion when the cutoff length coincides with the zero in the free energy of a vortex pair

$$V_{sc}(T=0)=\left[\frac{\hbar}{mr_c}\right]\ln\left[\frac{r_c}{a}\right]+\frac{E_c}{2\pi\sigma_s r_c(\hbar/m)}, \tag{14}$$

and represents the limiting value of $V_{sc}(Q,\omega,T)$ as $T\to 0$. This zero-temperature critical velocity is dependent on the vortex diffusivity and the oscillation frequency through the cutoff length r_c.

E. Application to dc flow and persistent current decay

The model which we have presented thus far describes the effect of ac flow on the superfluid density, and the dissipation that is produced from the movement of vortices. For a given set of film parameters, a film which is oscillating at a fixed frequency ω is being probed on a single length scale $l_c(\omega)$ as far as the bound pairs are concerned. Resonance experiments, such as the ones performed in this lab, are ideal for achieving this. The frequency dependence of the dissipation, etc., can be studied by observing the harmonics.

Some of the first experiments on thin films observed the decay of persistent currents. For example, Eckholm and Hallock[13] established a dc flow in an annular ring and measured the flow velocity with time. For their thinnest films (approximately six layers), they observed a dramatic initial drop in the flow velocity followed by a gradual decrease with time (similar to a decaying exponential). For thicker films the velocity would remain fairly constant for a short time and then slowly decay over long periods of time.

We believe that the model presented here to describe ac flow can also be used to describe the effects of dc flow to the extent that the assumptions given at the beginning of this paper are valid. The response time of a vortex pair in the film is limited by its diffusivity and size, as before, which means that small pairs will respond soonest followed in time by consecutively larger pairs. In short, the cutoff length scale l_c will move to longer length scales as time proceeds. This behavior is accomplished by replacing ω with $1/t$ in Eqs. (10e) (12), and (13). Equation (10e) can now be written as

$$l_c(t)=\ln\left[\frac{r_c(t)}{a}\right]=\tfrac{1}{2}\ln\left[\frac{14Dt}{a^2}\right]=\tfrac{1}{2}\ln\left[\frac{t}{t_0}\right], \tag{15}$$

where $t_0=a^2/14D\approx 10^{-14}$ sec. The renormalization for dc flow begins at an initial time $t=t_0$, which is the time for a vortex pair to diffuse a distance equal to the core diameter a, and velocity $V_s(t_0)$. At a subsequent time t, the renormalization equations are integrated out to the length scale $l_c(t)$ and the dissipation is evaluated. From the dissipation, a new velocity $V_s(t)$ is determined and the procedure starts again for a new time t'. Since the film's initial conditions (namely, the flow velocity and the vortex pair distribution) are changing with time, the effective screening is time dependent, and for this reason the renormalization integrals are recalculated, starting at the smallest length scales, for each time t. In this way we obtain a series of snap shots which together represent the time evolution of the persistent current.

III. RESULTS

A. Characteristic velocities and dissipation in ac flow

Solutions to the model are obtained by numerical integration of the recursion relation given by Eq. (5). The resulting parameters are the dielectric constant, the superfluid density, and Q_v^{-1} given by Eq. (10). The input parameters are the transition temperature T_{KT}, the strength of the square-root cusp b (fixed at 5.5 for this study, see Ref. 3), the oscillator frequency $f_{\mathrm{osc}}=f_{\mathrm{res}}/2$ on resonance, the temperature T, the superfluid velocity V_s, the vortex diffusivity D, and the free vortex creation time τ_{free}. The vortex core diameter a is held fixed at 1 Å. The procedure is (1) neglecting the free vortex contribution, choose a temperature and enter the measured characteristic velocity $V_{sc}(Q_c,\omega,T)$ for V_s. Then, adjust the diffusivity D so that Q_v, calculated by the program using Eq. (10d), is reduced to the appropriate value Q_c. Note that the program does not account for the background Q_0, which is presumed to be limited by effects not related to vortex dynamics or the Kosterlitz-Thouless theory, (2) add the free vortex contribution and adjust τ_{free} so that the total dissipation

$$Q^{-1}=Q_0^{-1}+Q_b^{-1}+Q_f^{-1} \tag{16}$$

best fits the measured curve of Q^{-1} versus V_s. Q_b^{-1} and Q_f^{-1} are the bound and free vortex contributions to the dissipation, respectively. Both D and τ_{free} are found to be only weakly coupled parameters and are assumed to be independent of V_s, (3) at the high temperatures, where the free vortex contribution is largest, Q_f^{-1} may be comparable in size to Q_b^{-1} and steps (1) and (2) should be repeated while including the free vortex contribution to ensure self-consistency.

Figure 1 shows a plot of Q^{-1} versus V_s, measured in our lab by Volz[14] on neon. Note that at the lowest temperatures, the free vortex contribution is small, indicating that the thermal energy is not readily available for the production of free vortices. This is reflected in the long creation times τ_{free} needed to fit the data. At the highest temperatures, the free vortex contribution is as large as or greater than the bound pair contribution because the thermal energy is available at these temperatures to produce a significant free vortex density. The creation times are, appropriately, much shorter at these higher temperatures. At a given temperature the free vortex contribution has an approximate power-law dependence V_s^α with α between 4 and 6 in our temperature range. A plot of $\tau_{\rm free}^{-1}$ versus T is shown in Fig. 3, where we see that the temperature dependence is $\sim T^{5.2}$ for a neon substrate and $\sim T^{1.3}$ for an argon substrate.

The bound vortex contribution always appears as a very strong function of V_s going like $(V_s/V_{sc})^\beta$ with β between 6 and 10 in our temperature range, and it appears that β diverges as $T \rightarrow 0$ indicating $V_{sc}(T=0)$ as the *critical* velocity. Figure 4 shows the temperature dependence of the characteristic velocity for various films on argon. Volz's data is limited to well below $T_{\rm KT}$ because the flow velocity becomes difficult to measure in the pressure of a vapor phase and the dissipation becomes too large as the transition is approached.

Volz finds the diffusivity to have both a temperature and film thickness dependence consistent with measurements on similar films by other labs. Specifically, Fig. 5 shows an increasing diffusivity with temperature changing by as much as 2 orders of magnitude between 0.2 and 0.6 K. With increasing film thickness, the diffusivity drops by as much as three orders of magnitude between 0.3 and 2.75 superfluid layers. There is also evidence of a substrate dependence for the diffusivity in the thinnest heliums. Another interesting feature is that the diffusivity appears to be finite at $T=0$, that is unless its temperature dependence drastically changes below 0.1 K. We will return to this point later.

Recently, Adams and Glaberson[15] have measured the vortex diffusivity of helium films using a torsional oscilla-

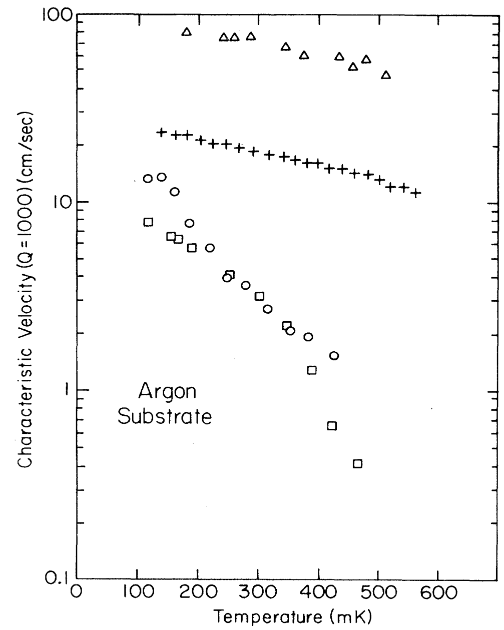

FIG. 4. Characteristic velocities for $Q_c=1000$ for helium on argon. $\triangle=1.95$ superfluid layers, $T_{\rm KT}=1.9$ K; $+=1.65$ superfluid layers, $T_{\rm KT}=1.5$ K; $\bigcirc=0.33$ superfluid layers, $T_{\rm KT}=0.70$ K; $\square=0.27$ superfluid layers, $T_{\rm KT}=0.55$ K. (Taken from Ref. 14.)

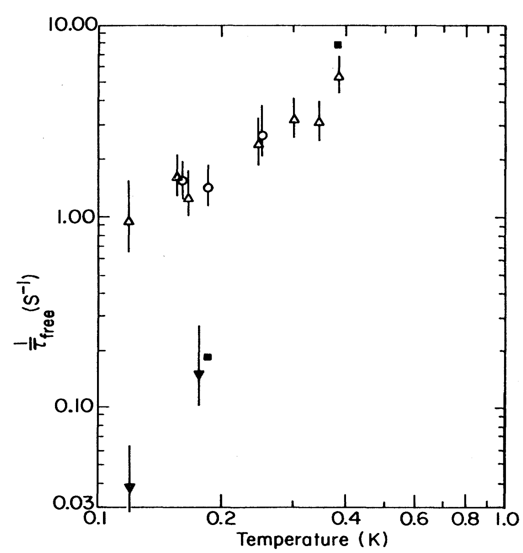

FIG. 3. $\tau_{\rm free}^{-1}$ vs T for helium on argon (open symbols) and neon (solid symbols). $\bigcirc=0.33$ superfluid layers, $T_{\rm KT}=0.7$ K; $\triangle=0.27$ superfluid layers, $T_{\rm KT}=0.55$ K; $\blacktriangledown=0.95$ superfluid layers, $T_{\rm KT}=1.8$ K; $\blacksquare=1.6$ superfluid layers, $T_{\rm KT}=1.3$ K. (Taken from Ref. 14.)

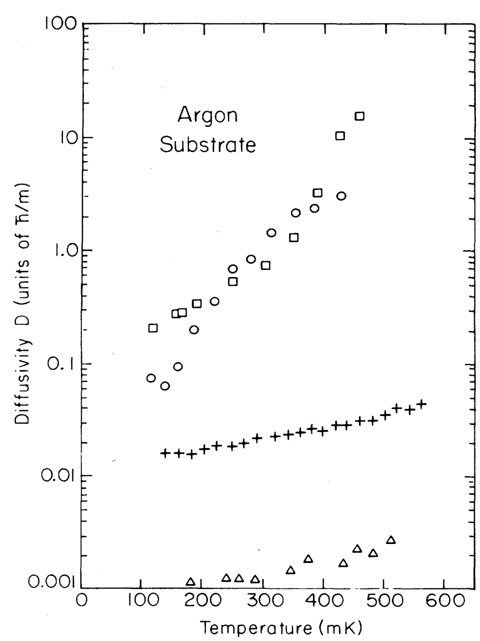

FIG. 5. Vortex diffusivity for helium on argon vs T (same symbols as in Fig. 4). Note an extrapolation to $T=0$ for the three thinnest films gives about the same value of $0.01\hbar/m$. (Taken from Ref. 14.)

tor similar to that of Bishop and Reppy.[3] They compare the dissipation when the cell is at rest to that when the cell is rotating at frequency Ω, the difference being due to an excess number of free vortices $n_\Omega = m\Omega/\pi\hbar$ introduced by the rotation. From this comparison they can extract the vortex diffusivity as a function of film thickness and temperature. Since they know the number of free vortices contributing to their dissipation, they do not need to use the parameter τ_{free}. The torsional oscillator method is extremely sensitive and allows them to measure dissipation at temperatures within a few mK of the dynamic transition temperature T_c. They find that between T_{KT} and T_c (a distance of only about 10–20 mK, usually) the diffusivity increases sharply and apparently diverges as $T \to T_c$. For their two thinnest films $d = 1.8$ and 3.2 superfluid layers, they find diffusivities away from T_{KT} of about $0.1-0.5\hbar/\text{m}$, which is consistent with Volz.

B. Discussion of diffusivities

The data of Volz together with that of Adams *et al.* provides the most convincing evidence thus far of the correctness of the model presented here. The utility of the model will be in determining the diffusivity using third-sound resonance in the low-temperature limit away from T_{KT}. The diffusivity appears to depend on temperature as well as film thickness. A dependence on temperature is expected due to the disappearing thermal excitations, but the exponential dependence of D seems too strong to be caused by this effect. Furthermore, D appears to have a nonzero value at $T = 0$, if we extrapolate below 0.1 K. This indicates that new processes (other than scattering of thermal excitations) must be involved, which lead to suppressed diffusivities at low T and nonzero diffusivities at $T = 0$. Two possible processes are quantum diffusion and interaction with a disordered or rough substrate.

AHNS (Ref. 4) derive the vortex diffusivity from a Langevin equation[16] [Eq. (6)] which describes how the dissipative effects (from the drag F_D on a vortex core) must be balanced by thermal fluctuations in order for our description to be consistent with thermal equilibrium. In this way, they define a diffusivity from a velocity-velocity correlation function

$$\langle \eta^\alpha(t)\eta^\beta(t') \rangle = 4D\delta_{\alpha\beta}\delta(t-t') \ . \tag{17}$$

From the above considerations and the requirement that the average motion be determined by setting $\mathbf{F}_M + \mathbf{F}_D = 0$, the diffusivity D and the convection coefficient C are related to the phenomenological drag coefficients B and B' by

$$D = k_B T \frac{B}{[2\pi(\hbar/m)\sigma_0 - B']^2 + B^2} \ , \tag{18a}$$

$$C = 1 - \frac{[2\pi(\hbar/m)\sigma_0 - B']2\pi\sigma_0(\hbar/m)}{[2\pi(\hbar/m)\sigma_0 - B']^2 + B^2} \ . \tag{18b}$$

If the drag on vortex core is due only to the interaction with the thermal excitations, then as $T \to 0$ we expect B and $B' \to 0$ which forces $D, C \to 0$. This limit describes a convective motion where the vortices move at the local superfluid velocity. The diffusive motion of a vortex, when it occurs, is *perpendicular* to the local superfluid velocity and is thus a diffusion away from convective motion. This description is evident from Eq. (6) in which the diffusive motion is written in terms of the separation vector of a vortex pair and not the position vector. In the opposite limit, where B and B' are large, again $D \to 0$ but $C \to 1$, and the vortices are pinned to the substrate. Figure 6 shows contours of constant C and $[D\sigma_0(\hbar/m)]/k_B T$ in terms of B and B'. Since Volz measured only D and not C we cannot uniquely determine B and B'. However the shaded region in Fig. 6 indicates the values of B and B' that are consistent with Volz's data and the model presented here. Furthermore these values are not very different from their bulk counterparts.[17] Our choice of the region near $C = 0$ is based on the assumption that vortex pinning is neglected. The pinning limit is described by $C \simeq 1$ and $D \simeq 0$ and requires values of B that are 100 times larger than our chosen range to describe Volz's thickest coverage on Ar.

Assuming the B and B' have similar origins as their bulk counterparts, B arises at least in part by the scattering of elementary excitations (phonons and rotons) from the vortex care, whereas B' arises from an asymmetry in the roton scattering.[17] Volz's data indicate that drag in the thick film is due mainly to phonon scattering, whereas the rotons become increasingly important as the film gets thinner. This result is understandable since there is evidence that the roton gap gets smaller as the film is thinned below three layers.[18] We envision the phenomenological drag coefficient B to have at least two contributions: B_e due to the scattering of thermal excitations, and B_{sub} due to interactions with the substrate. We write each coefficient as M_{eff}/τ_i, where M_{eff} is some

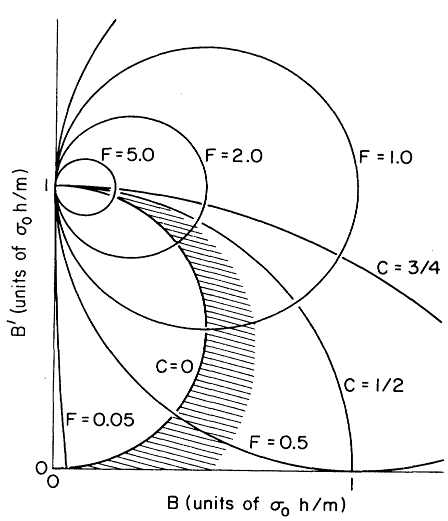

FIG. 6. Contours of constant $F = (D\sigma_0\hbar/\text{m})/k_B T$ and C are plotted here as functions of the drag coefficients B and B'. The shaded region represents the values of B and B' from our analysis of data from Ref. 14.

effective mass and τ_i is an interaction scattering time. For a given film thickness, the temperature dependence is determined, in part, by $1/\tau_c$ the scattering rate from thermal excitations, which increases with temperature and also by a possible (yet unknown) temperature dependence in $1/\tau_{\rm sub}$.

The strong film thickness dependence of D probably reflects the relative importance of the substrate interaction and the scattering of thermal excitations. We expect that in the thicker films, D is dominated by intrinsic effects (thermal excitations), whereas substrate effects become important in thin films. If this is true, then we may find that D becomes independent of film thickness in the limit of very thin films and very low temperatures.

Comparing the diffusivities found on argon and neon, Volz finds that, of his two films on neon, the thickest film falls in line with the argon films. This finding supports a dependence of D on superfluid film thickness and not on the strength of the van der Waals force. The thinner film on neon, however, shows an anomalously high and almost T-independent diffusivity of $\sim 3\hbar/m$.

Theoretical work needs to be done concerning the propagation of third sound on disordered or rough substrates taking into account the vortex dynamics. The scattering of vortices off substrate inhomogeneities may quite possibly contribute to the vortex diffusivity even at $T=0$.

C. Discussion of free vortex creation times

The similarity of the temperature dependences of the free vortex creation time $\tau_{\rm free}$ on neon and the thermal response time of the substrate $\tau_{\rm sub}=C_{\rm sub}/K_B$ may reflect the interaction (between the substrate and the superfluid) which is involved in the creation of vortices. ($C_{\rm sub}$ is the heat capacity of the substrate, which is probably $\sim T$, and K_B is the thermal boundary conductance between the substrate and the helium film $\sim T^{6.3\pm1}$.) McMillan[19] has proposed a model to calculate K_B which couples substrate phonons with the 2D phonons in the helium film. He finds that, neglecting two-phonon processes because they cannot conserve momentum, the three-phonon contribution goes as $(k_BT/\hbar S_{\rm sub})$,[8] where $S_{\rm sub}$ is the substrate sound velocity. The discrepancy between the theory and experiment can be due to several possible sources which cannot be described here. Note that, since McMillan's theory underestimates the conductivity, an additional channel may need to be included. Perhaps the interaction between substrate phonons and vortices may be this needed channel.

The long times obtained for $\tau_{\rm free}$ at low temperatures mean that it may be possible to observe the buildup of free vortices in the film. In principle, the film would have to be allowed to come to rest by waiting at least $\Delta t=10Q/\omega$, and then driving the film at the resonant frequency while watching the thermometer signal. The idea is that the initial free vortex density will be zero and the dissipation will be due to the background only (assuming the velocity is not too high). As the resonance builds up, it does so with a time constant $\tau_0=Q_0/\omega$. If τ_0 is very different from $\tau_{\rm free}$ then two distinguishable cases emerge: (1) $\tau_0>\tau_{\rm free}$. In this case we would see a change in the rate of build up as the free vortex density establishes itself. (2) $\tau_0<\tau_{\rm free}$. In this case it may actually be possible for the resonance to build up before n_f is established, and for the resonance amplitude to overshoot the steady-state value. To actually observe this may be very difficult because it requires that Q_0^{-1} be smaller than Q_f^{-1} by enough to make the amplitude difference observable, while at the same time keeping $\tau_0<\tau_{\rm free}$. For example, suppose that we require τ_0 to be at least five times smaller than $\tau_{\rm free}$ to ensure good resolution, and we require Q_f^{-1} to be at least ten times larger than Q_0^{-1}. These two conditions can be restated concisely as

$$\frac{1}{5\omega\tau_{\rm free}}<Q_0^{-1}<\tfrac{1}{10}Q_f^{-1}\ . \qquad (19)$$

Below 0.3 K the free vortex contribution is too small to make any significant change in the resonance line shape, and above 0.4 K the creation time is becoming too small (typically, $Q_0\sim10^4$). Increasing the frequency helps to reduce the lower bound, but the upper bound is reduced by more because Q_f^{-1} goes as ω^{-2}. Most of the time the two requirements are mutually exclusive.

D. dc flow: The decay of persistent currents

We will discuss some measurements of persistent current decay presented several years ago by Eckholm and Hallock[13] and explain to what extent our model can describe their results. The experiment consists of establishing a dc fluid flow in an annular ring and then using doppler shifted third sound to measure the flow velocity as a function of time. Their results are shown in Fig. 7. These measurements were done at the same temperature 1.45 K for various films thicknesses between 6.5 and 9.6 layers. For the thinnest films, $-dV_s/d(\log t)$ is largest immediately after the flow is established and then diminishes at longer times. For the thickest films,

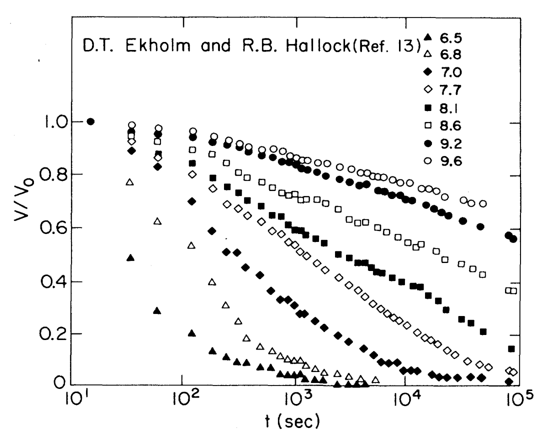

FIG. 7. Persistent current decay for various film thicknesses, $T=1.45$ K and $V_0=25$ cm/sec from Ref. 13. Note the change in behavior of the decay in the thinnest films.

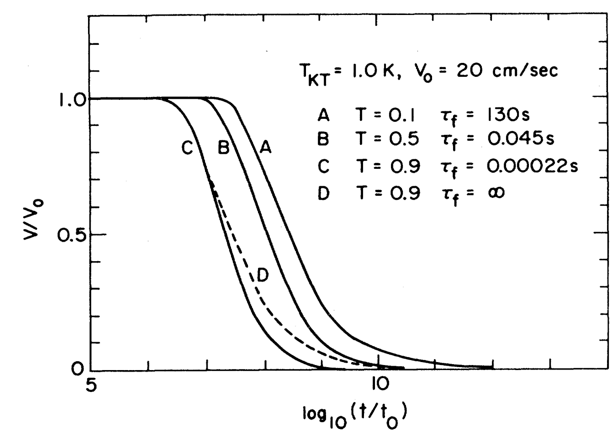

FIG. 8. Computer generated persistent current decay for various temperatures. The dotted line is the decay omitting the free vortex contribution for $T=0.9$ K.

$-dV_s/d(\log t)$ is near zero at first and then gradually increases to a value that is constant for more than two decades in time.

This experiment is quite complicated, not only because it is done in the time domain, but because of the high temperature and large film thicknesses. We originally hoped that, by extending our model to include these effects, (i.e., Landau excitations) we would be able to describe Eckholm and Hallock's data over the entire range of film thicknesses. Unfortunately, we found this not to be the case. We find that at best our model can describe the behavior of the thinnest film of 6.5 layers.

Figure 8 shows the result of our computer model using the method described. We see that the flow velocity does not change until some characteristic time, after which the velocity drops in a fashion similar to Eckholm and Hallock's thinnest film. The physics of the behavior is governed by the time-dependent length scale $l(t)=\frac{1}{2}\ln(t/t_0)$ on which the film is being probed. At short times, the film is being probed on length scales where there are very few pairs, and the dissipation is negligible. At some characteristic time (determined by the flow velocity V_{s0} and the temperature) the film is being probed on a length scale that contains enough pairs to produce some dissipation and a subsequent decrease in V_s. From then on, the probe length scale and the zero in the free energy chase each other, toward larger length scales, producing the rather uniform decay rates. Finally, the zero in the free energy wins out causing a reduction in the dissipation and the decay rate.

The parameters D and τ_{free} are obtained from the ac results for a film with a similar T_{KT}. The dashed line in Fig. 8 shows the effect of leaving out the free vortex contribution.

Yu[20] has analyzed Eckholm and Hallock's data for the thicker films and concludes that the wide range of behavior is due to the competition of three effects. For the thickest films, the decay is dominated by nucleation of free vortices at the boundaries of the annulus, and the intermediate films are governed by pinning effects. Only in the thinnest film is the decay dominated by intrinsic processes. This explanation is consistent with our findings, since our model neglects the extrinsic processes which dominate the thick films.

IV. CONCLUSION

Our model of nonlinear dissipation in thin He films not only allows us to interpret our third-sound resonance experiments and extract such phenomenological parameters as the vortex diffusivity and the free vortex creation time, but also provides us with some unique insight into nature of the superfluid transition in the presence of superfluid flow. The most important of these insights are the presence of isolated vortices below the static transition temperature T_{KT} and the prediction of a critical velocity only at $T=0$. For the experimentalist, we stress that ac experiments are far easier to interpret than dc (persistent currents) because the diffusion limited vortex motion ensures that ac flows will be probing the system on only one length scale.

Future work is needed on a microscopic theory of the drag coefficients and their relationship to the elementary excitations of the film as well as the effect of substrate inhomogeneities. These theories are necessary for the interpretation of the experimental measurements of the diffusivity and its dependence on temperature and film thickness.

ACKNOWLEDGMENTS

We gratefully acknowledge many helpful conversations with A. J. Leggett and M. Wortis. This research received the support of the National Science Foundation under Grant No. NSF-DMR84-13790 as well as the support of the Materials Research Laboratory under Grant No. NSF-DMR86-12860 and the Department of Physics at the University of Illinois.

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