# Percolation in two-dimensional lattices. II. The extent of universality

Fumiko Yonezawa and Shoichi Sakamoto
*Department of Physics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223, Japan*

Motoo Hori
*Department of Applied Physics, Tokyo Institute of Technology, 2-12-1 Ohokayama, Meguro-ku, Tokyo 152, Japan*
(Received 8 December 1988)

By making use of finite-size scaling and Monte Carlo simulations, we study the so-called "*universality*" concerning critical exponents of percolation in several two-dimensional lattices. In particular, our main purpose is to clarify *how universal is "universality."* For this purpose, we choose the following five lattices—square and *kagomé* (both periodic where the number $z$ of nearest-neighbor sites is single valued, i.e., $z=4$), dice (periodic where $z$ is mixed valued, i.e., $z=3$ and 6, the average $\bar{z}$ being four), Penrose tiling (nonperiodic where $z$ is mixed valued, i.e., $z=3$, 4, 5, 6, and 7, the average $\bar{z}$ being four), and the dual lattice of Penrose (nonperiodic where $z$ is single valued, i.e., $z=4$). For both site and bond percolation of these lattices, we analyze the results of our Monte Carlo simulations and evaluate six critical exponents, all of which are in good agreement with respective values predicted theoretically. Our results indicate that "universality" is really universal irrespective of classes of problems, i.e., whether bond or site; irrespective of kinds of lattices, i.e., whether periodic or nonperiodic; and irrespective of types of coordination, i.e., whether single valued or mixed valued.

## I. INTRODUCTION

Among various problems concerning percolation, two major themes are the determination of percolation thresholds and the investigation of the critical phenomena near a threshold. The previous paper (the first paper of this series of papers, which hereafter is referred to as I)[1] is concerned with the former theme, while the present paper deals with the latter theme.

In I, we proposed a method of deriving a reliable value for the percolation threshold $p_c$ of an infinite lattice from the information of finite lattices. The key point was the appropriate definition of the effective threshold for a finite lattice, which turned out to be a good approximation for the percolation threshold of an infinite system. Using Monte Carlo (MC) simulations and employing our method thus proposed, we obtained the "reliable" thresholds of several two-dimensional (2D) lattices both for bond and site percolation.

The scaling theory has been put forward in order to describe the behaviors of some quantities near a critical point $p_c$ of a phase transition; these behaviors are normally called critical phenomena.[2,3] Percolation[4–8] is one example for which the peculiar properties near $p_c$ are expressed by means of critical exponents introduced by the scaling hypothesis. The most important assertion concerning critical exponents is that they are independent of the details of the lattice structure but depend on the dimension $d$ of the lattice. This aspect is considered to hold for thermal phase transitions in general, as well as for our problem of site and bond percolation. This feature is widely amounted by a somewhat magnified terminology of "*universality*."

Now the main purpose of the present paper is to examine *how universal is "universality."* The reason why we are interested in this theme is that, although periodic structures are not invoked in the scaling theory,[9–12] almost all MC simulations performed so far[8,13–19] have dealt with periodic lattices in which the coordination number $z$ is single valued. Consequently, it is only for those periodic lattices which have single-valued coordination that no violation of "*universality*" has been ascertained. Actually, the status of research on "*universality*" remains as described in Table I, i.e., lattices analyzed previously are only those belonging to the category in the top left column. Then, it directly follows that, in order to ensure the validity of "*universality*" more firmly, it is required to study lattices in other columns as well. This is exactly what we do in this paper.

On the basis of Stauffer's cluster number scaling hypothesis,[8] and MC simulations, we evaluate the following six critical exponents of some 2D lattices periodic or nonperiodic where the coordination number is either single valued or mixed valued, i.e., exponent $\nu$ defining the correlation length, exponent $\tau$ defining the sizes of clusters at $p_c$, the fractal dimension $D$, exponent $\alpha$ defining the total number of clusters, exponent $\beta$ defining the

TABLE I. Has the existence of "*universality*" been ascertained by previous work?

| | Type of lattice | |
|---|---|---|
| $z$ | Periodic lattice | Nonperiodic lattice |
| single valued | seems to have | not studied yet |
| mixed valued | not studied yet | not studied yet |

 

TABLE II. Lattices we study in this paper: the coordination number $z$ of each lattice is either single valued or mixed valued; $z=4$ for the former case while the average coordination number $\bar{z}=4$ for the latter case.

| $z$ | Type of lattice: Periodic lattice | Type of lattice: Nonperiodic lattice |
|---|---|---|
| single valued | square<br>kagomé | dual of Penrose |
| mixed valued | dice<br>($z=3$ and 6) | Penrose<br>($z=3$, 4, 5, 6, and 7) |

probability that a site or bond belongs to a percolating cluster, and exponent $\gamma$ defining the mean cluster size.

Aside from the critical exponents, there exist other quantities in disordered systems which, in general, have been asserted to be dimensional invariants in the sense that they depend only on the dimension of the system. The problem of disorder and dimensionality has long been discussed in various contexts. The appearance of magnetic[20,21] or crystalline[22,23] order and the electron localization[24–27] in disordered systems are some examples which are explained in relation with dimensionality. Concerning the percolation problem, it has been empirically pointed out that the following two quantities are dimensional invariants, that is, the critical volume fraction[7,28] $v_c \equiv f p_c$(site) for site percolation, $f$ being the filling factor, and the critical value $\eta_c$ for the effective coordination number defined by $\eta_c \equiv \bar{z} p_c$(bond).[7] We will show that, according to our results, both $v_c$ and $\eta_c$ are, respectively, almost equivalent for all lattices we study, thus indicating that they are really dimensional invariants in a wide variety of lattices.

Our method is explained in Sec. II and results are given in Sec. III. The critical volume $v_c$ and the critical value $\eta_c$ for the effective coordination number are mentioned in Sec. IV and discussions are found in Sec. V.

## II. SYSTEMS AND METHOD

As we have mentioned in the preceding section, it is required, for our purpose of confirming "*universality*," to study lattices in the columns in Table I which are denoted by "not studied yet." In order to meet this requirement, we choose five lattices listed in Table II according to the criterion that either the coordination number $z$ itself is four in the case of single-valued coordination or the average $\bar{z}$ is four in the case of mixed-valued coordination.[29]

Stauffer's cluster number scaling hypothesis for percolation[8] asserts that quantities listed in Table III show the critical behaviors as described in the table, where $p$ is the concentration of intact elements (bonds or sites), $p_c$ the percolation threshold of an infinite system, $d$ the dimension of the lattice, $n_s(p)$ the number of $s$ clusters at $p$, an $s$ cluster being a cluster composed of $s$ connected elements (bonds or sites); $L$ is the linear dimension of the system defined by $L=N^{1/2}$, $N$ being the system size, $s_{\text{perc}}$ the size of a percolating cluster, and $f(x)$ as an appropriate function of $x$. The strength $P$ of an infinite network is the probability that an element (a bond or site) belongs to a percolating cluster. Most of the quantities as listed in Table III are related to the number $n_s(p)$ of $s$ clusters. Detailed explanations about the critical behaviors of these quantities are given in the attractive book by Stauffer.[8]

In thermodynamic phase transitions such as the paraferro magnetic transition in a magnetic system and the liquid-gas transition in a fluid, exponent $\alpha$ defines the critical behavior of the specific heat; exponent $\beta$ defines the magnetization in a magnetic system and the density difference between liquid and gas in a fluid; and exponent $\gamma$ defines isothermal magnetic susceptibility in a magnetic system and isothermal compressibility in a fluid. Exponent $\nu$, defining the correlation length, has the common meaning in all these systems.

Although the scaling theory in its original form is applied to infinite systems, similar arguments are developed for finite systems, which is termed as the finite-size scaling and is very useful in predicting the behaviors of infinite systems from the results for finite systems. The finite-size scaling is characterized by the fact that the linear dimension $L$ of the system appears in the discussions. In our analyses to follow, we use the idea of the finite-size scaling. The scaling hypothesis gives rise to the relationships among critical exponents, which are generally referred to as the scaling law. When the dimension $d$ of a system appears in the scaling law, the theory is called hyperscaling and it is implicitly assumed that the relations hold only for such $d$ as satisfies $2 \le d \le 6$, because the mean-field theory is exact for $d > 6$. The scaling law for seven critical exponents in Table III is expressed by five independent equations in Table IV. This means that at least two out of seven exponents must be

TABLE III. Critical exponents and critical behavior of quantities (after Stauffer).

| Exponent | Quantity | Critical behavior |
|---|---|---|
| $\alpha$ | total number of clusters | $M_0 \propto \lvert p-p_c\rvert^{(2-\alpha)}$ |
| $\beta$ | strength of infinite ($p>p_c$) network | $P \propto (p-p_c)^{\beta}$ |
| $\gamma$ | mean size of finite clusters | $S \propto \lvert p-p_c\rvert^{-\gamma}$ |
| $\nu$ | correlation length at $p=p_c$ | $\xi \propto \lvert p-p_c\rvert^{-\nu}$ |
| $\tau$ | cluster number at $p=p_c$ | $n_s(p_c) \propto s^{-\tau}$ |
| $D$ | fractal dimension at $p=p_c$ | $s_{\text{perc}} \propto L^{D}$ |
| $\sigma$ | cluster number for $p \neq p_c$ | $n_s(p) \propto s^{-\tau} f((p-p_c)s^{\sigma})$ |

TABLE IV. The scaling law ($d$: dimension of system).

| |
|---|
| $\alpha+2\beta+\gamma=2$ |
| $2-\alpha=\nu d$ |
| $\beta=\nu(d-D)$ |
| $D=1/\nu\sigma$ |
| $\tau=1+d/D$ |

determined by other methods. Note that the same scaling law as listed in Table IV holds for the phase transition in a magnetic and fluid system.

We carry out MC simulations in the same way as stated in I. That is to say, for a given lattice of size $N$, we choose $M$ elements (either bonds or sites) at randon out of $N$ elements and count the number $n_s$ of $s$ clusters for each $s$. When the cluster of the maximum size extends from one side to the other side of the lattice, say from top to bottom or from right to left, we call the cluster "percolating." We denote this size by $s_{\text{perc}}$.

In the above process, it is necessary to identify each $s$ cluster composed of $s$ elements, which is not an easy task to carry out unless $N$ is very small.[30] Since this job of identifying clusters is required at each MC run, the efficiency of the algorithm for the cluster analysis is the key factor in the economy of computer time. Actually, the smart way of dealing with the cluster analysis is of central importance in the computer simulations of phase transitions including percolation and magnetic phase transitions. One of the present authors (Sakamoto[31]) has worked out a very efficient algorithm for the cluster analysis, which enables us to treat systems of considerable sizes and to try many MC runs.

For each $p \equiv M/N$ with each increment $\Delta p \equiv 0.002$, we carry out $n$ MC runs and count the number $m$ of runs in which we find a percolating cluster. Then, the probability $R_N^A(p)$ of finding a percolating cluster is identified by $m/n$. Several explanations have been proposed to verify that $R_N^A(p)$ near the percolation threshold has the form of an error function defined by the mean $p_c^A(N)$ and the standard deviation, $\Delta_N^A$. In I, we have shown that, when a percolating cluster is approximately defined, $p_c^A(N)$ thus determined gives a very good approximation of the percolation threshold in an infinite system even if $N$ is not very large.

Using this method, we have obtained in I the percolation thresholds, each with three significant figures, both for bond and site percolation in several 2D lattices. Since very accurate values for thresholds are required in the scaling analyses, it is a great advantage that we have reliable values for thresholds at the starting point.

## III. CRITICAL EXPONENTS AND SCALING LAW

Now let us try and estimate some critical exponents from the information made available in our simulations and analyses. For our study, we choose six exponents $\alpha$, $\beta$, $\gamma$, $\nu$, $\tau$, and $D$ in Table III. We omit $\sigma$ because it is rather difficult to determine $\sigma$ accurately. In what follows, we demonstrate the numerical results for these six exponents.

### A. Exponent $\nu$

In a system of finite size $N=L^2$, let us pick up $p$ which is slightly larger than the effective threshold $p_c^A(N)$ such that $p=p_c^A(N)+B\Delta_N^A$, where $p_c^A(N)$ and $\Delta_N^A$ are, respectively, the mean and the standard deviation of Gaussian

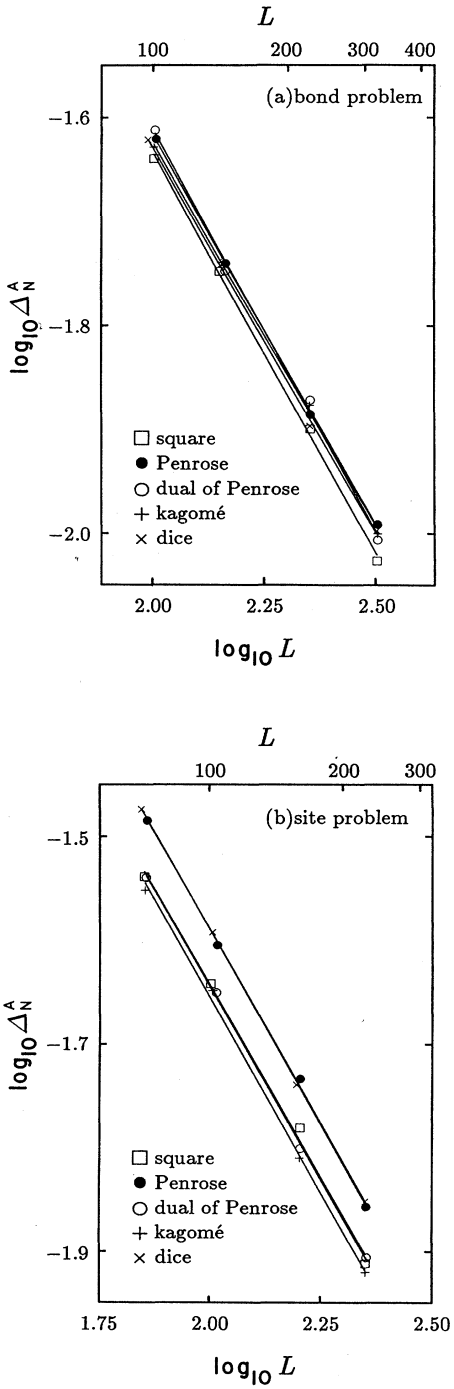


FIG. 1. $\text{Log}_{10}\Delta_N^A$ vs $\log_{10}L$ where $\Delta_N^A$ is the standard deviation which defines $R_N^A(p)$. (a) the bond problems; (b) the site problem.

TABLE V. Values of some critical exponents obtained from our simulations.

| | Lattice | $\nu/\nu_0$ | $D/D_0$ | $\tau/\tau_0$ | $\frac{2-\alpha}{2-\alpha_0}$ | $\beta/\beta_0$ | $\gamma/\gamma_0$ |
|---|---|---|---|---|---|---|---|
| Bond | square | 1.01 | 1.00 | 0.96 | 0.95 | 0.95 | 0.92 |
| | kagomé | 1.02 | 0.98 | 0.96 | 0.95 | 0.76 | 1.03 |
| | dice | 1.01 | 1.01 | 0.97 | 0.94 | 0.94 | 0.95 |
| | Penrose | 1.01 | 0.98 | 0.97 | 0.94 | 0.79 | 1.03 |
| | dual of Penrose | 0.98 | 1.02 | 0.96 | 0.95 | 0.90 | 1.04 |
| Site | square | 1.06 | 0.96 | 0.96 | 1.00 | 1.02 | 0.93 |
| | kagomé | 1.01 | 0.99 | 0.96 | 0.99 | 0.78 | 0.95 |
| | dice | 1.01 | 1.02 | 0.95 | 1.00 | 1.09 | 0.88 |
| | Penrose | 1.01 | 1.00 | 0.94 | 1.00 | 1.09 | 1.05 |
| | dual of Penrose | 1.02 | 1.02 | 0.97 | 0.99 | 0.73 | 1.16 |
| Theoretical prediction | | $\nu_0$ 1.333 $\left[=\frac{4}{3}\right]$ | $D_0$ 1.896 $\left[=\frac{91}{48}\right]$ | $\tau_0$ 2.055 $\left[=\frac{187}{91}\right]$ | $2-\alpha_0$ 2.667 $\left[=2-\left[-\frac{2}{3}\right]\right]$ | $\beta_0$ 0.139 $\left[=\frac{5}{36}\right]$ | $\gamma_0$ 2.389 $\left[=\frac{43}{18}\right]$ |

distribution related with the error function defining $R_N^A(p)$, and $B$ is a small positive constant satisfying $0<B\ll 1$. Noting that $p_c^A(N)$ is almost identical with the percolation threshold $p_c$ of an infinite system and the correlation length $\xi$ at this $p$ is about $L$, we can write the critical behavior $\xi \propto |p-p_c|^{-\nu}$ in the form

$$|p_c^A(N)-p_c| \propto L^{-1/\nu} . \tag{3.1}$$

Then, it is easy to see that

$$\Delta_N^A \propto L^{-1/\nu} . \tag{3.2}$$

In Fig. 1(a) and 1(b) are shown $\log_{10}\Delta_N^A$ versus $\log_{10}L$, respectively, for bond and site percolation in the five lattices in Table II. According to Eq. (3.2), the slope of each line corresponds to $(-1/\nu)$. The values of $\nu$ thus estimated are listed in the first column of Table V in relation with the theoretically predicted value $\nu_0=\frac{4}{3}$. For the bond problem, the deviation of $\nu$ from the theoretically predicted value is 2% at most, while for the site problem, the largest deviation is 6%. When we remember the way in which $\Delta_N^A$ are determined, we can say that the deviation of this magnitude is small enough and we can conclude that the exponent $\nu$ for either bond or site percolation is a dimensional invariant, taking the theoretically predicted value $\nu_0=\frac{4}{3}$.

It is interesting to note that we employed the theoretical value $\nu_0=\frac{4}{3}$ when we estimated in I the error bar for the percolation threshold from the extrapolations of the upper and lower bound of the effective thresholds at finite size $L=N^{1/2}$. In the evaluation of a percolation threshold itself, on the other hand, each $p_c^A(N)$ as determined and enumerated in I is almost independent of $N$ or $L$ when $N\gtrsim 5000$ (see Table I and Figs. 6 and 7 in I), and therefore it is practically unnecessary to assume the value for $\nu$ in the estimation of a percolation threshold. In other words, our method for the estimation of percolation thresholds as proposed in I makes it possible to derive percolation thresholds without a knowledge of the critical exponent $\nu$, which is an immense advantage of our method.

### B. Fractal dimension $D$

In order to evaluate the fractal dimension $D$ as defined in Table III, we have to express $s_{\text{perc}}$ at $p_c$ as a function of the linear dimension $L$ of the system. We pick up four sizes —5000, 10 000, 25 000, and 50 000 for site percolation and twice of each of these sizes for bond percolation. We carry out 500 MC runs to take the average.

The $\log_{10}s_{\text{perc}}$ versus $\log_{10}L$ plot is illustrated in Fig. 2(a) for bond percolation and in Fig. 2(b) for site percolation. The slope of this line gives an estimation of the fractal dimension $D$. The values $D$ thus obtained are presented in the second column of Table V in relation with the theoretically predicted value $D_0=\frac{91}{48}\simeq 1.896$. The deviation from the theoretically predicted value is 2% at most for bond percolation and 4% for site percolation, which undoubtedly shows that fractal dimension $D$ is a dimensional invariant.

### C. Exponent $\tau$

From the definition of $\tau$ in Table III we see that the slope of the $\log_{10}[n_s(p_c)]$ versus $\log_{10}s$ plot gives $(-\tau)$. The relations $\log_{10}[n_s(p_c)]$ versus $\log_{10}s$ are illustrated in Fig. 3 for bond percolation and in Fig. 4 for site percolation. The results presented in this figure have been obtained from the average of 100 MC runs in lattices of size 400 000 for bond percolation and of size 200 000 for site percolation.

Each line is drawn so that the fitting is the best for intermediate values of $s$. For small $s$, the data points are lower than the straight line since the simple power law of the scaling theory is not valid for small $s$. For very large $s$, the data points become higher than the straight line since clusters of larger sizes are cut into several pieces by the existence of the boundaries.

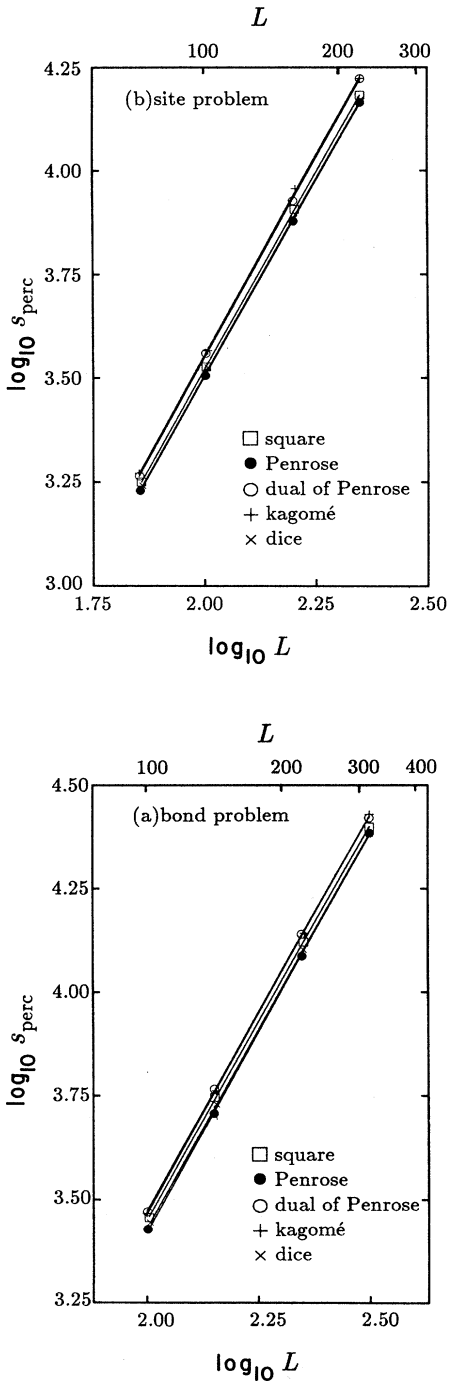


FIG. 2. $\mathrm{Log}_{10}s_{\mathrm{perc}}$ vs $\log_{10}L$ at $p=p_c$. The value of $s_{\mathrm{perc}}$ is determined by taking the average of the results obtained from 500 MC simulations at $p=p_c$. (a) the bond problem; (b) the site problem.

The values $\tau$ calculated from the slopes of the lines in Figs. 3 and 4 are given in the third column of Table V in relation with the theoretically predicted value $\tau_0=\frac{187}{91}\simeq 2.055$. The deviation from the theoretical value is 4% at most for bond percolation and 6% for site percolation. Here again, agreement is remarkable, and $\tau$ is also a dimensional invariant.

### D. Exponents $\alpha$, $\beta$, and $\gamma$

When Eq. (3.1) is inserted, the equations defining $\alpha$, $\beta$, and $\gamma$ are written, for a finite system of size $N=L^2$, as

$$M_0(p)\propto L^{-(2-\alpha)/\nu}, \tag{3.3}$$

$$P(p)\propto L^{-\beta/\nu}, \tag{3.4}$$

$$S(p)\propto L^{\gamma/\nu}, \tag{3.5}$$

where $p=p_c^A(N)+B\Delta_N^A$. When these relations are illustrated in the logarithmic-logarithmic scale, the slopes of the obtained lines are, respectively, $-(2-\alpha)/\nu$, $-\beta/\nu$ and $\gamma/\nu$, from which we can derive $(2-\alpha)$, $\beta$, and $\gamma$ on assuming that $\nu=\nu_0=\frac{4}{3}$.

The results thus determined are presented in the last three columns in relation with the respective predictions $(2-\alpha_0)=\frac{8}{3}\simeq 2.667$, $\beta_0=\frac{5}{36}\simeq 0.139$, and $\gamma_0=\frac{43}{18}\simeq 2.389$. As for $(2-\alpha)$, the deviation from the theoretical value is 6% at most for bond percolation and 1% for site percolation, and therefore we can say that $\alpha$ is a dimensional invariant. As for the probability $P$ that an element (bond or site) belongs to a percolating cluster, the critical behavior of $P$ as described in Table III makes sense only for $p>p_c$ when the system size is infinite. In a finite system, this is not the case. Actually in a finite system, the probability $P$ is proportional to $s_{\mathrm{perc}}$, where $s_{\mathrm{perc}}$ is the size of a percolating cluster which is proportional to $L^D$. Noting that $N=L^2$, we have $P\propto L^{D-2}$. Comparing this result with Eq. (3.4), we obtain $\beta/\nu=2-D$. Since the fractal dimension $D$ is slightly lower than the physical dimension $d$, the difference $(d-D)$ or $(2-D)$ for our 2D systems is small, which implies that $\beta$ is a parameter very difficult to determine accurately from simulations. This is the reason why the deviation of $\beta$ from the theoretical value $\beta_0=\frac{5}{36}\simeq 0.139$ becomes as large as 27% in some lattices. Exponent $\gamma$ suffers a similar difficulty, although the deviation of $\gamma$ from the theoretical value $\gamma_0=\frac{43}{18}\simeq 2.389$ is not so serious as in the case of $\beta$. However, when we recollect that it is generally much more subtle to estimate critical exponents than to evaluate thresholds, the agreement as a whole is reasonable both for $\beta$ and $\gamma$, which proves that $\beta$ and $\gamma$ are also critical exponents.

To summarize, we conclude that the scaling hypothesis is valid and "*universality*" is fulfilled irrespective of the class of the problem, i.e., whether it is bond or site percolation, irrespective of the type of a lattice, i.e., whether it is periodic or not, and irrespective of the kind of coordination therein, i.e., whether it is single valued or mixed valued.

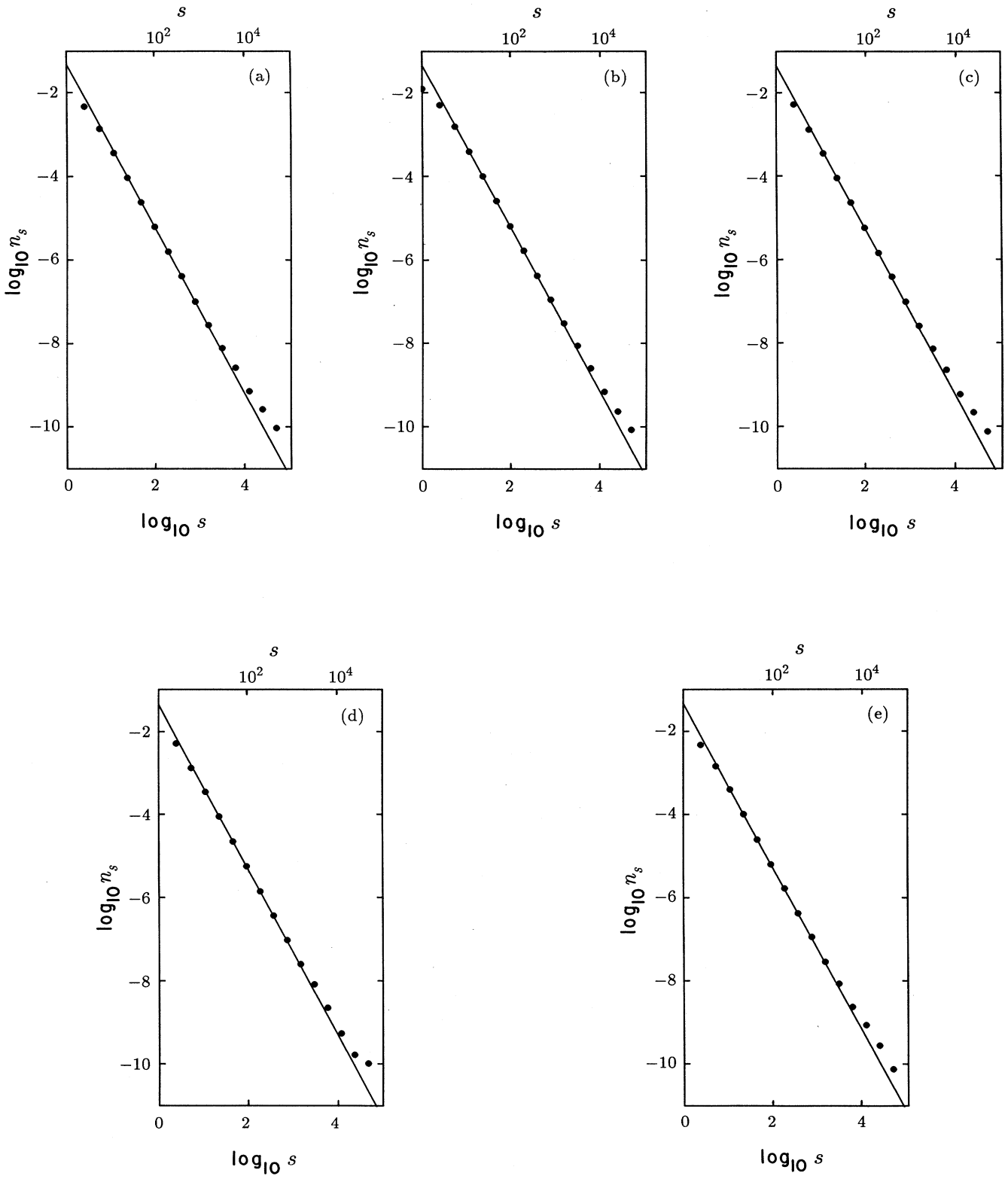


FIG. 3. $\mathrm{Log}_{10} n_s$ vs $\log_{10} s$ for the bond problem where $n_s$ is the number of clusters composed of $s$ bonds. The value $n_s$ is determined by taking the average of the data from 100 MC simulations in a lattice composed of approximately 400 000 bonds at $p = p_c$. (a) square; (b) kagomé; (c) dice; (d) Penrose; and (e) dual lattice of Penrose.

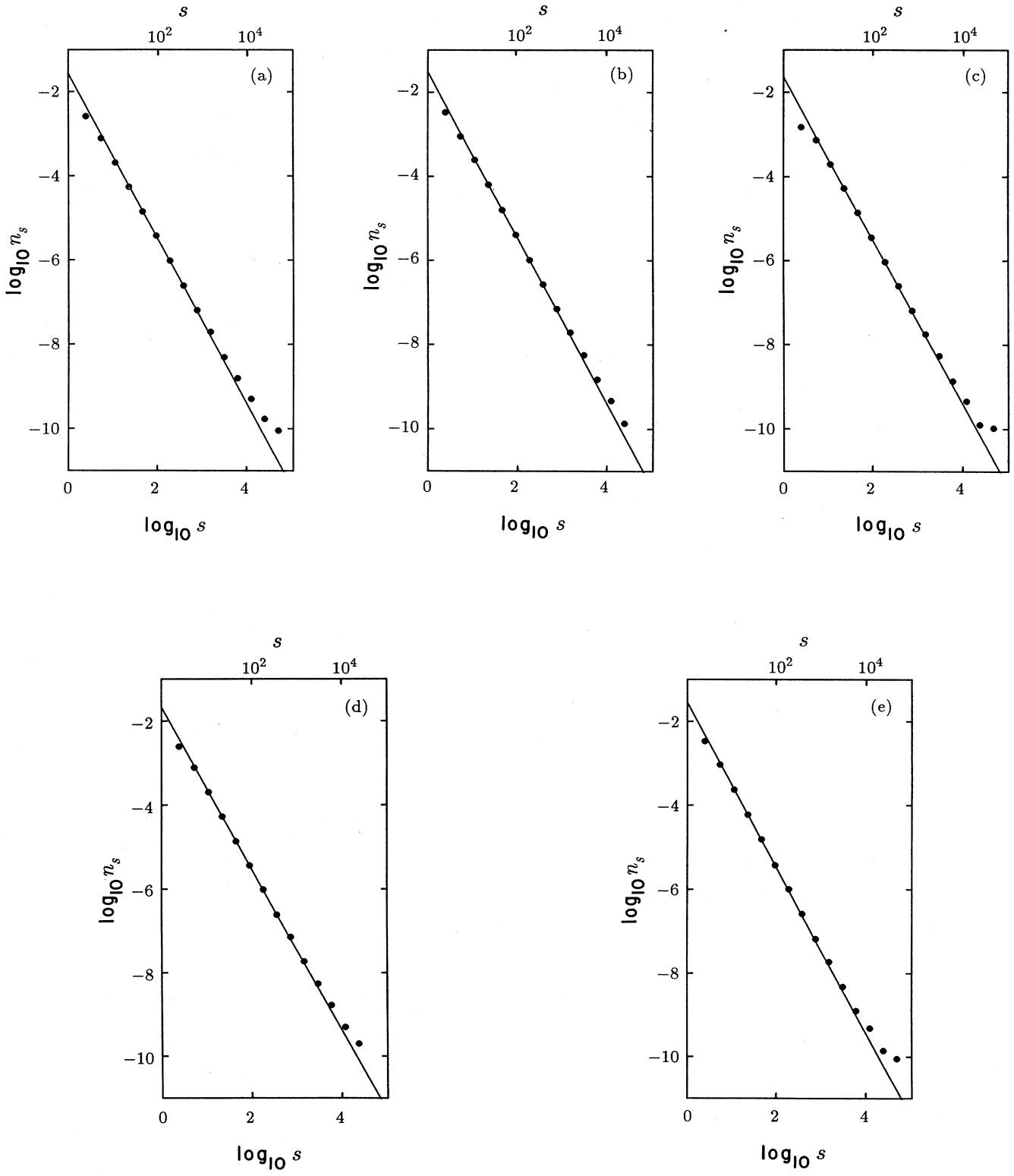


FIG. 4. $\mathrm{Log}_{10} n_s$ vs $\log_{10} s$ for the site problem where $n_s$ is the number of clusters composed of $s$ sites. The value $n_s$ is determined by taking the average of the data from 100 MC simulations in a lattice composed of approximately 200 000 bonds at $p = p_c$. (a) square; (b) kagomé; (c) dice; (d) Penrose; and (e) dual lattice of Penrose.

TABLE VI. Critical value for the effective coordination number.

| Lattice | $p_c$(bond) | $\eta_c = \bar{z}p_c$(bond) | $\eta_c/\eta_c$(emp) |
|---|---|---|---|
| square | 0.5001 | 2.00 | 1.00 |
| kagomé | 0.5244 | 2.10 | 1.05 |
| dice | 0.4760 | 1.90 | 0.95 |
| Penrose | 0.4770 | 1.91 | 0.96 |
| dual of Penrose | 0.5233 | 2.10 | 1.05 |
| triangular | 0.3473 | 2.08 | 1.04 |
| honeycomb | 0.6527 | 1.96 | 0.98 |

## IV. DIMENSIONAL INVARIANTS OF OTHER KINDS

Aside from the critical exponents which we have discussed in the preceding section, it has been empirically examined that there exists a dimensional invariant for bond percolation as well as for site percolation. As for bond percolation, the product of the threshold $p_c$(bond) and the coordination number $z$ seems to be given as[28]

$$\eta_c \equiv z p_c(\text{bond}) = \frac{d}{d-1} \simeq \begin{cases} 2 & (2D) \\ 1.5 & (3D)\,, \end{cases} \tag{4.1}$$

where $d$ is the dimension of the lattice; $\eta_c$ can be interpreted as the critical value for the effective coordination number. As for site percolation, the product of the threshold $p_c$(site) and the filling factor $f$ of the lattice seems to be a dimensional invariant,[7] satisfying

$$v_c \equiv f p_c(\text{site}) \simeq \begin{cases} 0.45 & (2D) \\ 0.15, & (3D)\,, \end{cases} \tag{4.2}$$

where $v_c$ is usually called the percolation volume fraction.

Although the relations (4.1) and (4.2) are found only empirically, these relations play important roles in the practical applications of the idea of percolation. For instance, Eq. (4.1) gives a good measure for the study of the resistance in a random network and Eq. (4.2) serves as a good guideline for the discussion of conductivity in the classical mixture of conducting and insulating constituents.

For our lattice, it would be approximate to substitute $z$ in Eq. (4.1) by the average coordination number $\bar{z}$ when $z$ is mixed valued. The products $\eta_c \equiv \bar{z}p$(bond) are given in Table VI where our thresholds $p_c$(bond) as obtained in I are used. Agreement with Eq. (4.1) is reasonable.

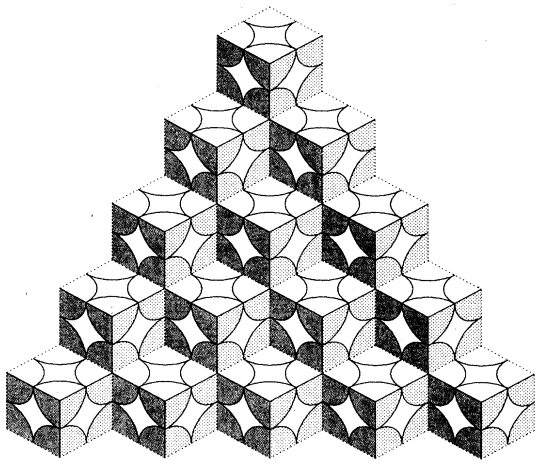

FIG. 5. A three-dimensional illustration of a dice lattice where the packing object at each vertex is defined by the cross-sectional view of a sphere at the lattice point of a simple-cubic lattice.

In order to calculate the critical percolation volume, it is necessary to know the filling factor of a lattice under consideration. When a lattice is periodic and simple as is the case for a square, triangular, honeycomb, and kegomé lattice, the filling factor $f$ is obtained by enumerating the area occupied by close-packed hard circles associated with lattice sites, because these close-packed hard circles do not change the connectivity of a lattice. This does not hold any longer when close-packed hard circles are allotted to lattice sites of a dice lattice, because the circles on the opposite corners along the short diagonal of a rhombus become in contact with each other despite the fact that these two sites are not topologically connected.

One way of avoiding this trouble in the case of a dice lattice is to take the cross-sectional view of a three-dimensional sphere on a three-dimensional simple-cubic lattice as illustrated in Fig. 5. The packing object at each vertex looks like either a three- or six-leaf clover. As is clearly demonstrated in the figure, the clover at each vertex does not violate the connectivity of the lattice.

A similar recipe of assigning an $n$-leaf clover with an $n$-bonded vertex is shown to work even in quasiperiodic and disordered lattices. Some detailed explanation is given in the Appendix. The filling factors thus determined for Penrose tiling and its dual lattice are, respec-

TABLE VII. Critical volume fraction.

| Lattice | $p_c$(site) | $f$ | $v_c = f p_c$(site) | $v_c/v_c$(emp) |
|---|---|---|---|---|
| square | 0.5930 | 0.7854 | 0.466 | 1.04 |
| kagomé | 0.6527 | 0.6802 | 0.444 | 0.99 |
| dice | 0.5851 | 0.7854 | 0.459 | 1.02 |
| Penrose | 0.5837 | 0.7854 | 0.459 | 1.02 |
| dual of Penrose | 0.6381 | 0.7106 | 0.453 | 1.01 |
| triangular | 0.5[a] | 0.9069 | 0.453 | 1.01 |

[a]Exact.

tively, about 0.7854 and 0.7106.

When these values are used in Eq. (4.2) to calculate $v_c$, we achieve the results as shown in Table VII, and we can see that they approximately satisfy the empirical prediction as expressed by Eq. (4.2).

In this way, the numerical analysis shows that both $v_c$ and $\eta_c$ are dimensional invariants for some two-dimensional systems both periodic and nonperiodic where the coordination could be single valued or mixed valued. This suggests that Eqs. (4.1) and (4.2) would hold generally even when the system is disordered.

## V. SUMMARY AND DISCUSSION

In this article, we have investigated the percolation problem using the idea of finite-size scaling and Monte Carlo simulations. We have studied five different kinds of two-dimensional lattices—square, kagomé, dice, Penrose, and dual lattice of Penrose. The reason why we have chosen these particular lattices is the following. The so-called "*universality*" concerning critical exponents has previously been studied only for periodic lattices where the coordination number is single valued. Since actual systems in which the concept of percolation plays an important role are mostly disordered, it is interesting to see if this "*universality*" survives even in systems without periodicity and/or without the single-valued requirement for coordination numbers. The above-mentioned five lattices meet our needs as clearly presented in Table II. The information obtained from our study is summarized as follows.

(1) We have evaluated critical exponents—$\nu$ for the correlation length, fractal dimension $D$, $\tau$ for the cluster number at $p=p_c$, $\alpha$ for the total number of clusters, $\beta$ for the strength of the percolating cluster, and $\gamma$ for the mean size of finite clusters. Each of them agrees well with the value predicted from the scaling theory. This means that these exponents are all dimensional invariants, and accordingly, "*universality*" is retained in the lattices studied in this paper.

(2) Empirical laws are fulfilled concerning the product $\eta_c \equiv \bar{z}p_c$(bond), $\bar{z}$ being the average coordination number and $p_c$(bond) being the bond percolation threshold and concerning the product $v_c \equiv fp_c$(site), $f$ being the packing fraction, and $p_c$(site) the site percolation threshold. The latter product is identical with the percolation volume fraction which has an important meaning especially when percolation at macroscopic level is a matter of concern.

Our final objects are disordered systems. From the analysis of our results, we are almost certain that there exists "*universality*" in disordered systems. Furthermore,

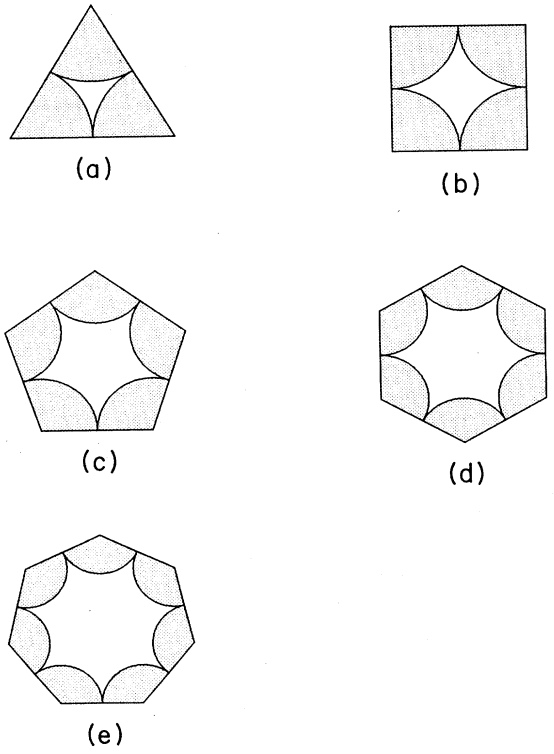


FIG. 6. Portions of packing spheres in regular polygons for (a) $n=3$, (b) $n=4$, (c) $n=5$, (d) $n=6$, and (e) $n=7$.

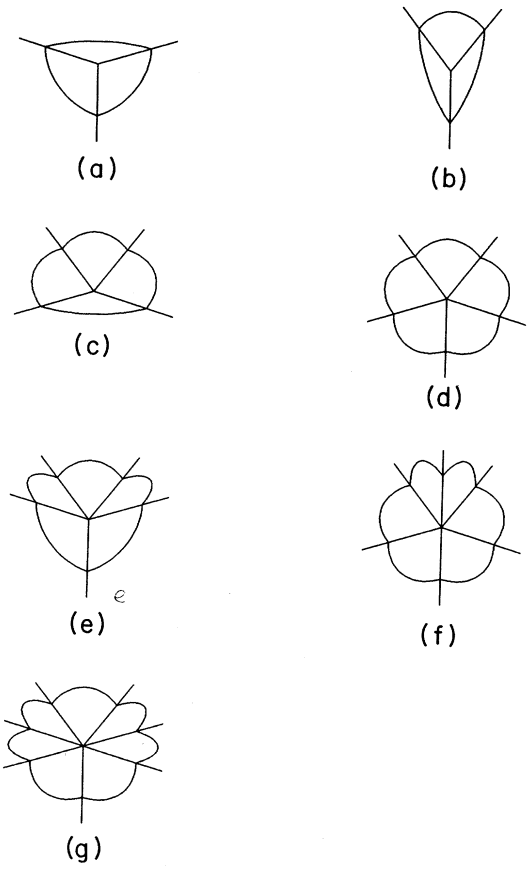


FIG. 7. Seven different clovers at seven different vertices in Penrose tiling.

TABLE VIII. Filling factors of regular polygons with $n$ edges for (a) $n=3$, (b) $n=4$, (c) $n=5$, (d) $n=6$, and (e) $n=7$. The corresponding figure polygons with packing objects are shown in Fig. 6.

| Regular polygon of $n$ edges | $n$ | Filling factor | |
|---|---|---|---|
| (a) | 3 | $\frac{\pi}{2\sqrt{3}}$ | =0.906 90 |
| (b) | 4 | $\frac{\pi}{4}$ | =0.785 40 |
| (c) | 5 | $\frac{3\pi}{10\cot\left[\frac{\pi}{5}\right]}$ | =0.684 75 |
| (d) | 6 | $\frac{\pi}{3\sqrt{3}}$ | =0.604 60 |
| (e) | 7 | $\frac{5\pi}{14\cot\left[\frac{\pi}{7}\right]}$ | =0.540 33 |

our prediction is that "*universality*" holds in three-dimensional systems as well.

## ACKNOWLEDGMENTS

The authors are grateful to Dr. S. Nosé, Dr. H. Nakanishi, and Ms. K. Aoki for valuable discussions. They are also indebted to Professor Y. Saitoh and Dr. H. Takano for critical readings of the manuscript.

## APPENDIX

When each pair of nearest-neighbor lattice sites in a system are connected by a bond, the whole space is filled with various kinds of polygons. For a simple and period-

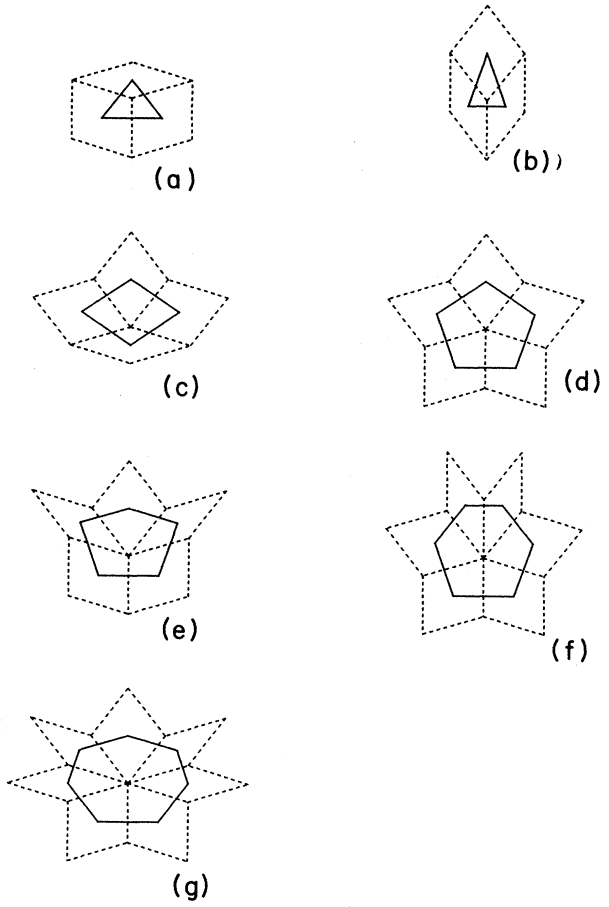


FIG. 8. Seven different vertices (dashed curves) (a)–(g) in two-dimensional Penrose tiling and the corresponding seven different polygons (solid curves) in the dual lattice of Penrose tiling.

TABLE IX. Volume fractions of packing in seven different polygons in a dual lattice of two-dimensional Penrose tiling.

| Polygons in a dual lattice of Penrose | $n$ | Volume of polygon | Frequency | Volume fraction of polygon | Volume fraction of packing |
|---|---|---|---|---|---|
| (a) | 3 | $\frac{\tau^2}{8}$ | $\tau^{-2}$ | $\frac{1}{8(3\tau-4)}$ | 0.1760 |
| (b) | 3 | $\frac{\sqrt{5}\tau}{8}$ | $\tau^{-4}$ | $\frac{7\tau-11}{8(3\tau-4)}$ | |
| (c) | 4 | $\frac{2\sqrt{5}\tau}{8}$ | $\tau^{-5}$ | $\frac{2(7\tau-11)}{8(3\tau-4)}$ | 0.0750 |
| (d) | 5 | $\frac{5\tau}{8}$ | $\frac{\tau^{-5}+\tau^{-7}}{\sqrt{5}}$ | $\frac{5(5-3\tau)}{8(3\tau-4)}$ | 0.3355 |
| (e) | 5 | $\frac{\tau^2}{8}$ | $\tau^{-3}$ | $\frac{\tau+1}{8(3\tau-4)}$ | |
| (f) | 6 | $\frac{\sqrt{5}\tau^4}{8}$ | $\tau^{-7}$ | $\frac{7-4\tau}{8(3\tau-4)}$ | 0.0467 |
| (g) | 7 | $\frac{3\sqrt{5}\tau^2}{8}$ | $\tau^{-6}$ | $\frac{3(7\tau-11)}{8(3\tau-4)}$ | 0.0774 |

ic lattice, there appears only one kind of unit polygon, e.g., a square in a square lattice, a triangle in a triangular lattice, and a hexagon in a honeycomb lattice. A kagomé lattice has two kinds of unit polygons—a triangle and a hexagon.

The filling factors for these lattices are evaluated by enumerating the area occupied by close-packed hard circles associated with lattice sites. If we look at the problem from a slightly different viewpoint, the filling factors can be obtained by calculating the shaded regions in each regular polygon as shown in Fig. 6.

When the three- and six-leaf clovers are associated as the packing objects with vertices in a dice lattice as in Fig. 5, the filling factor in each rhombus is $\pi/4$ which is identical with the filling factor in a square as shown in Table 8(b). The correspondence is clear because the leaves in each rhombus are the projections from the cross-sectional view of the spheres associated with the lattice points of a three-dimensional simple-cubic lattice, and this cross-sectional view in three dimensions is nothing but a regular square with four quarters of a circle.

A Penrose lattice is tiled with two kinds of rhombi. When we employ the same recipe as that used in a dice lattice; the filling factor of a Penrose lattice is identical with that of a square lattice, i.e., $f=\pi/4$. Seven different clovers on seven different vertices in a two-dimensional Penrose lattice are presented in Fig. 7.

Unit polygons in a dual lattice of Penrose are not necessarily regular as can be seen from Fig. 8, but here again we can generalize the above-described recipe such that the filling factor in each $n$ polygon is equivalent to the value described in Table VIII even if the polygon is not regular. This generalization is verified by the fact that Penrose tiling as well as its dual lattice are obtained by projecting the cross-sectional view of a five-dimensional hypercubic lattice into two-dimensional space. Now, what we have to do is to calculate the product of (a) the volume of each polygon in Fig. 8, (b) its frequency, and (c) the filling factor of the corresponding regular polygon in order to obtain the volume fraction of the filling object in the polygon under consideration. This product is given in the last column of Table IX for each polygon. The filling factor of a dual lattice of Penrose is given by the sum of these products to be 0.7106.

[1]F. Yonezawa, S. Sakamoto, and M. Hori, the preceding paper, Phys. Rev. B **40**, 636 (1989).
[2]H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University, New York, 1971).
[3]*Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1972).
[4]V. K. S. Shante and S. Kirkpatrick, Adv. Phys. **20**, 325 (1971).
[5]S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).
[6]J. W. Essam, Rep. Prog. Phys. **43**, 843 (1980).
[7]A. L. Efros, *Physics and Geometry of Disorder Percolation Theory* (MIR, Moscow, 1986).
[8]D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985).
[9]M. P. M. den Nijs, J. Phys. A **12**, 1857 (1979).
[10]B. Nienhuis, E. K. Riedel, and M. Schick, J. Phys. A **13**, L189 (1980).
[11]R. P. Pearson, Phys. Rev. B **22**, 2579 (1980).
[12]H. W. J. Blöte, M. P. Nightingale, and B. Derrida, J. Phys. A **14**, L45 (1981).
[13]M. F. Sykes, D. S. Gaunt, and M. Glen, J. Phys. A **9**, 97 (1976).
[14]P. J. Reynolds, H. E. Stanley, and W. Klein, Phys. Rev. B **21**, 1223 (1980).
[15]P. D. Eschbach, D. Stauffer, and H. J. Hermann, Phys. Rev. B **23**, 422 (1981).
[16]B. Derrida and L. De. Seze, J. Phys. (Paris) **43**, 475 (1982).
[17]A. Margolina, Z. Djordjevic, H. E. Stanley, and D. Stauffer, Phys. Rev. B **28**, 1652 (1983).
[18]A. Margolina, H. Nakanishi, D. Stauffer, and H. E. Stanley, J. Phys. A **17**, 1683 (1984).
[19]D. C. Rapaport, J. Phys. A **18**, L175 (1985).
[20]N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966).
[21]R. E. Peierls, Proc. Cambridge Philos. Soc. **32**, 477 (1936).
[22]R. E. Peierls, Proc. Annu. Inst. Henri Poincare **5**, 177 (1935).
[23]N. D. Mermin and H. Wagner, Phys. Rev. **176**, 250 (1968).
[24]N. F. Mott and W. D. Twose, Adv. Phys. **10**, 107 (1961).
[25]R. E. Borland, Proc. R. Soc. London **77**, 705 (1961).
[26]P. W. Anderson, Phys. Rev. **109**, 1492 (1985).
[27]E. Abtaham, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[28]H. Scher and R. Zallen, J. Chem. Phys. **53**, 3759 (1970).
[29]F. Yonezawa, S. Sakamoto, K. Aoki, S. Nosé, and M. Hori, a preliminary account of the preceding paper and this paper was reported at the 4th International Conference on Structure of Noncrystalline Materials, Oxnard, 1988 (unpublished).
[30]J. Hoshen and R. Kopelman, Phys. Rev. B **14**, 3428 (1976).
[31]S. Sakamoto (unpublished).