Electron-phonon relaxation in pure metals and superconductors at very low temperatures

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The electron-phonon relaxation processes in pure metals and superconductors are considered at low temperatures when electrons interact with both longitudinal and transverse phonons. The screening of the electron-ion interaction is treated in a gauge-invariant form, so that both longitudinal and transverse electromagnetic fields are taken into consideration. The results obtained are applied to the analysis of the heating effect of the electron system. The interaction of electrons with transverse phonons increases the electron-phonon relaxation rate by a factor of $\simeq 15$ in comparison with the case when only longitudinal phonons are taken into consideration. The peculiarity of the screening effects in a superconductor leads to the conclusion that at low temperatures $T \ll \Delta$ transverse phonons do not participate in the electron-phonon relaxation processes.

INTRODUCTION

In most theoretical treatments of kinetic phenomena in pure metals and superconductors associated with the electron-phonon interaction, only longitudinal phonons are taken into account. As was shown in Ref. 1 this is justified only under definite conditions, originating from the peculiarity of screening effects for the longitudinal and transverse electromagnetic fields which describe the electron-ion interaction. The conclusion of Ref. 1 is that the interaction of electrons with thermal transverse phonons is not important for temperatures such that

$$T_1, T_2 < T \quad \text{with} \quad T_2 = u_t l^{-1}, \quad T_1 = \frac{u_t}{c}(v_F \kappa^2 u_t)^{1/2} , \qquad \kappa^2 = 4\pi e^2 \nu_0, \quad \nu_0 = \frac{m p_F}{\pi^2} , \tag{1}$$

where u_t is the velocity of transverse sound, v_F is the Fermi velocity, p_F is the Fermi momentum, and ν_0 is the electron two-spin density of states at the Fermi surface. l is the electron mean free path $l = v_F \tau$, where τ is the electron momentum relaxation time due to the elastic electron-impurity scattering. The inequality $T_2 < T$ permits one to consider the metal as clean and to neglect the electron-impurity scattering.

In the present paper we consider the case $T_2 < T < T_1$. Thus, the electron-impurity scattering is neglected, but the interaction of electrons with transverse phonons is essential. In pure metals $T_1 \approx 0.1$ K and T_2 is very small $T_2 \approx 10^{-3}$ K for the impurity concentration $c_i = 0.01\%$. The results obtained are applied to recent experimental data for nonlinear effects in metals at very low temperatures[2,3] and to the analysis of relaxation processes in superconductors when $T \ll \Delta$, where Δ is the energy gap. The results are also pertinent to a proposed electron cooling experiment.[4]

NORMAL METAL

We are going to consider the problem of the electron energy relaxation at low temperatures $T < T_1$ when electrons interact with real longitudinal and transverse phonons. Note that for $T > T_1$ electrons interact only with real (thermal) longitudinal phonons. For the effective electron-electron interaction due to exchange of virtual phonons (see Ref. 5), also only longitudinal phonons are important.

We use the Keldysh-diagram technique for nonequilibrium processes[6] in which the electron and phonon Green's functions, along with the electron and phonon self-energies, are represented by matrices

$$\hat{G} = \begin{bmatrix} 0 & G^A \\ G^R & G^C \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} 0 & D^A \\ D^R & D^C \end{bmatrix} , \qquad \hat{\Sigma} = \begin{bmatrix} \Sigma^C & \Sigma^R \\ \Sigma^A & 0 \end{bmatrix}, \quad \hat{\Pi} = \begin{bmatrix} \Pi^C & \Pi^R \\ \Pi^A & 0 \end{bmatrix} . \tag{2}$$

When solving the problem of energy relaxation, electron Green's functions are

$$G^R(\mathbf{p},\varepsilon) = (\varepsilon - \xi_{\mathbf{p}} + i0)^{-1}, \quad \xi_{\mathbf{p}} = (p^2 - p_F^2)/2m , \tag{3}$$

$$G^C(\mathbf{p},\varepsilon) = S(\varepsilon)[G^A(\mathbf{p},\varepsilon) - G^R(\mathbf{p},\varepsilon)] , \qquad S(\varepsilon) = 2n(\varepsilon) - 1 = -\tanh(\varepsilon/2T) , \tag{4}$$

where $n(\varepsilon)$ can be interpreted as the electron energy distribution function, and T is the electron temperature. The electron-phonon collision integral equals

$$I_{e\text{-ph}}(\mathbf{p},\varepsilon) = -i\{\Sigma^C(\mathbf{p},\varepsilon) - S(\varepsilon)[\Sigma^A(\mathbf{p},\varepsilon) - \Sigma^R(\mathbf{p},\varepsilon)]\} \tag{5}$$

(e-ph denotes electron-phonon). According to Ref. 7 the electron energy relaxation time τ_ε is defined by the following kinetic equation:

$$\frac{1}{\tau_\varepsilon} = -\frac{\delta}{\delta n(\varepsilon)}\frac{dn(\varepsilon)}{dt} , \qquad \frac{dn(\varepsilon)}{dt} = \frac{1}{\pi \nu_0}\int \frac{d\mathbf{p}}{(2\pi)^3}\mathrm{Im}[G^A(\mathbf{p},\varepsilon)] I_{e\text{-ph}}(\mathbf{p},\varepsilon) . \tag{6}$$

Following Refs. 1 and 8, we treat the electron-phonon interaction as the interaction between electrons and ions by means of the intermediate electromagnetic field. The interaction between the electrons and the electromagnetic field is described by the Hamiltonian

$$H_{e\text{-}\gamma}=-\sum_{\mathbf{p}}\sum_{\mathbf{q}\,(\neq 0)}\varphi_{\mathbf{q}}C^{\dagger}_{\mathbf{p}+\mathbf{q},s}C_{\mathbf{p},s}+\sum_{\mathbf{p}}\sum_{\mathbf{q}\,(\neq 0)}\mathbf{A}_{\mathbf{q}}\cdot\frac{2\mathbf{p}+\mathbf{q}}{2mc}C^{\dagger}_{\mathbf{p}+\mathbf{q},s}C_{\mathbf{p},s}+\frac{1}{2mc^2}\sum_{\mathbf{p}}\sum_{\mathbf{q},\mathbf{q}'\,(\neq 0)}\mathbf{A}_{\mathbf{q}}\cdot\mathbf{A}_{\mathbf{q}'}C^{\dagger}_{\mathbf{p}+\mathbf{q}+\mathbf{q}',s}C_{\mathbf{p},s}\ , \tag{7}$$

where $\varphi_{\mathbf{q}}$, $\mathbf{A}_{\mathbf{q}}$ are the Fourier components of the scalar and vector potentials, $C^{\dagger}_{\mathbf{p},s}$ is the creation operator for an electron with momentum $\mathbf{p}$ and spin s, c is the velocity of light, and the absolute value of the electron charge is absorbed in the definition of the electromagnetic potentials (e-γ denotes electron-photon).

The electron-photon vertices corresponding to the Hamiltonian (7) are

$$a^0=-1,\quad \mathbf{a}^1=\frac{1}{mc}\left[\mathbf{p}+\frac{\mathbf{q}}{2}\right],\quad a^{11}=\frac{1}{2mc^2}\ . \tag{8}$$

From here on, the index 0 will refer to vertices describing interactions with scalar potentials, while 1 will denote interactions with vector potentials.

The electromagnetic field Green's function $V_{\mu\nu}$ in the Coulomb gauge (div $\mathbf{A}=0$) is diagonal. If screening is taken into consideration, $V_{\mu\mu}$ is given by

$$V^R_{00}(\mathbf{q},\omega)=\frac{4\pi e^2}{q^2-4\pi e^2P^R_{00}(\mathbf{q},\omega)}\ , \tag{9}$$

$$[V^R_{11}(\mathbf{q},\omega)]_{mn}=V^R_{11}(\mathbf{q},\omega)T_{mn}\ ,\qquad V^R_{11}(\mathbf{q},\omega)=\frac{4\pi e^2c^2}{\omega^2-c^2q^2-4\pi e^2c^2P^R_{11}(\mathbf{q},\omega)}\ , \tag{10}$$

where e is the electron charge and m,n stand for the Cartesian coordinates x,y,z. If the vector $\mathbf{q}$ is directed along the z axis, we have

$$T_{mn}=\delta_{mn}-\frac{q_mq_n}{q^2}\ . \tag{11}$$

$P_{\mu\mu}$ is the polarization operator (Fig. 1) with the vertices a^{μ}_x, where $a^0_x=-1$ and $a^1_x=(2p_x+q_x)/2mc$.

Vertices corresponding to the electron-boson interaction have tensor structure Q^k_{ij}, where the upper index is for bosons, the lower for electrons:

$$(a^{\mu})^k_{ij}=a^{\mu}K^k_{ij},\quad K^1_{ij}=\frac{1}{\sqrt{2}}\delta_{ij},\quad K^2_{ij}=\frac{1}{\sqrt{2}}(\sigma_x)_{ij}\ , \tag{12}$$

where σ_x is the usual 2×2 Pauli matrix.

Thus for $P^R_{\mu\mu}$ we have

$$P^R_{\mu\mu}(\mathbf{q},\omega)=-2i\int\int\frac{d\mathbf{p}\,d\varepsilon}{(2\pi)^4}(a^{\mu}_x)^2\times K^1_{ij}[G(\mathbf{p}+\mathbf{q},\varepsilon+\omega)]_{jl}\times K^2_{lk}[G(\mathbf{p},\varepsilon)]_{ki}+\frac{n_e}{mc^2}\delta_{\mu 1}\ , \tag{13}$$

where n_e is the electron density. Summation is implied by the pairs of repeated indices.

For $ql\gg 1$,

$$P^R_{00}(\mathbf{q},\omega)=-\nu_0\left[1+i\frac{\pi\omega}{2qv_F}\right],\qquad P^R_{11}(\mathbf{q},\omega)=-i\frac{\pi\omega\nu_0v_F}{4qc^2}\ . \tag{14}$$

To describe the coupling between electrons and longitudinal and transverse phonons on the basis of a single approach, we introduce the Hamiltonian for interaction of ions with the electromagnetic field:

$$H_{\text{ion-}\gamma}=\int d\mathbf{r}\left[\rho_{\text{ion}}(\mathbf{r})\varphi(\mathbf{r})-\frac{1}{c}\mathbf{j}_{\text{ion}}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\right], \tag{15}$$

where the charge density and current density are given by the equations

$$\rho_{\text{ion}}(r)=Z\sum_{\alpha}\delta(\mathbf{r}-\mathbf{R}_{\alpha}),\quad \mathbf{j}_{\text{ion}}=\frac{Z}{M}\sum_{\alpha}M\dot{\mathbf{R}}\delta(\mathbf{r}-\mathbf{R}_{\alpha})\ . \tag{16}$$

Here Z is the valence, M is the mass, $\mathbf{R}_{\alpha}$ is the position, and $\dot{\mathbf{R}}$ is the velocity of the ion. In carrying out the expansion near the equilibrium position of the ion in the lattice $\mathbf{R}_{\alpha 0}$, we retain only linear terms $\Delta\mathbf{R}_{\alpha}=\mathbf{R}_{\alpha}-\mathbf{R}_{\alpha 0}$ and $\Delta\dot{\mathbf{R}}_{\alpha}$. We then express $\Delta\mathbf{R}$ and $\Delta\dot{\mathbf{R}}$ in terms of phonon creation and annihilation operators $b^{\dagger}_{q\lambda}, b_{q\lambda}$ (λ is the index of a phonon branch):

$$\Delta\mathbf{R}=\sum_{\alpha}\frac{\mathbf{e}_{\lambda}e^{i\mathbf{q}\cdot\mathbf{r}}}{(2MN\omega_{q\lambda})^{1/2}}\phi^0_{q\lambda}\ ,\qquad \Delta\dot{\mathbf{R}}=-i\sum_{\alpha}\frac{\mathbf{e}_{\lambda}\omega_{q\lambda}e^{i\mathbf{q}\cdot\mathbf{r}}}{(2MN\omega_{q\lambda})^{1/2}}\phi^1_{q\lambda}\ , \tag{17}$$

where

$$\phi^0_{q\lambda}=b_{q\lambda}+b^{+}_{-q\lambda},\quad \phi^1_{q\lambda}=b_{q\lambda}-b^{+}_{-q\lambda}\ . \tag{18}$$

As a result we obtain the phonon-photon interaction Hamiltonian

$$H_{\text{ph-}\gamma}=\sum_{\alpha}(Q^0_{q\lambda}\phi^0_{q\lambda}\varphi_q+Q^1_{q\lambda}\phi^1_{q\lambda}\mathbf{A}_q)\ , \tag{19}$$

where

$$Q^0_{q\lambda}=i\frac{(\mathbf{e}_{\lambda}\cdot\mathbf{q})ZN}{(2MN\omega_{q\lambda})^{1/2}},\quad Q^1_{q\lambda}=-i\frac{\mathbf{e}_{\lambda}\omega_{q\lambda}ZN}{c(2MN\omega_{q\lambda})^{1/2}}\ . \tag{20}$$

The vertices Q^{μ} are written in (20) for absorption of a phonon; vertices for emission are $\tilde{Q}^{\mu}=\{Q^{\mu}\}^*$, where $\{\ \}^*$ means the complex conjugate. In the Keldysh-

diagram technique vertices Q^{μ} acquire the factor $1-\delta_{kl}$, where k,l are the boson indices. In order to include the two kinds of operators ϕ^0 and ϕ^1, we introduce the matrix Green's function for phonons,

$$D_{\mu\nu}(\mathbf{q},\lambda,t)=-i\langle \mathcal{T}_t(\hat{\phi}^{\mu}_{q\lambda}(t)\hat{\phi}^{\nu}_{q\lambda}(0))\rangle, \quad \mu,\nu=0,1 \tag{21}$$

where $\hat{\phi}^{\mu}_{q\lambda}(t)$ is the operator in the Heisenberg representation and $\mathcal{T}_t$ denotes time ordering. As will be shown below, we will require only the functions D_{00} and D_{11}, for which we have the following Fourier representation:

$$D^R_{00}(\mathbf{q},\omega)=-D^R_{11}(\mathbf{q},\omega)=(\omega-\omega_{q\lambda}+i0)^{-1} -(\omega+\omega_{q\lambda}+i0)^{-1} . \tag{22}$$

The electron self-energy, which corresponds to the electron-phonon interaction, is shown in Fig. 1. The presence of three ω-dependent boson propagators leads to a complicated structure of the collision integral. In describing the nonequilibrium case when electrons and the lattice have different temperatures, we will consider the electron Green's function G^C to depend on the electron temperature T and the boson functions V^C and D^C to depend on the temperature of the lattice Θ:

$$D^C_{\mu\mu}(\mathbf{q},\omega)=[2N_{\Theta}(\omega)+1][D^R_{\mu\mu}(\mathbf{q},\omega)-D^A_{\mu\mu}(\mathbf{q},\omega)] , \tag{23}$$

$$V^C_{\mu\mu}(\mathbf{q},\omega)=[2N_{\Theta}(\omega)+1][V^R_{\mu\mu}(\mathbf{q},\omega)-V^A_{\mu\mu}(\mathbf{q},\omega)] . \tag{24}$$

As a result we get the following kinetic equation:

$$\frac{\partial n(\varepsilon)}{\partial t}=-\frac{1}{\pi\nu_0}\int d\mathbf{p}\int d\mathbf{q}\int d\omega\frac{1}{(2\pi)^7}\mathrm{Im}[G^A(\mathbf{p},\varepsilon)]\mathrm{Im}[G^A(\mathbf{p}+\mathbf{q},\varepsilon+\omega)] \times\mathrm{Im}\{(a^{\mu})_m[V^R_{\mu\mu}(q,\omega)]_{mn}(Q^{\mu})_n D^R_{\mu\mu}(q,\omega)(\tilde{Q}^{\mu})_k[V^R_{\mu\mu}(\mathbf{q},\omega)]_{kl}(a^{\mu})_l\}R_{\Theta}(\varepsilon,\omega) , \tag{25}$$

where m,n,k,l are the Cartesian indices and

$$R_{\Theta}(\varepsilon,\omega)=[2N_{\Theta}(\omega)+1][S(\varepsilon+\omega)-S(\varepsilon)] -S(\varepsilon+\omega)S(\varepsilon)+1 . \tag{26}$$

Note that the nondiagonal propagator $(a^{\mu})_m(V_{\mu\mu})_{mn}(Q^{\mu})_n D_{\mu\nu}(\tilde{Q}^{\nu})_k(V_{\nu\nu})_{kl}(a^{\nu})_l$ gives zero after the angular integration. The analytical structures of V_{00} and V_{11} are different. While for V_{00} the condition of strong screening,

$$\frac{4\pi e^2}{q^2}|P^R_{00}(\mathbf{q},\omega)|\gg 1 , \tag{27}$$

is always valid for $q\ll\kappa$ (and so $V^R_{00}\approx-\nu^{-1}$), the condition of strong screening for transverse electromagnetic fields,

$$\frac{4\pi e^2c^2}{|\omega^2-c^2q^2|}|P^R_{11}(q,\omega)|\gg 1 , \tag{28}$$

depends on the characteristic values of q and ω. Performing the ω integral in (25), and being interested in the electron-phonon interaction, we take into account only the poles of the function $D_{\mu\mu}(\mathbf{q},\omega)$ and hence $\omega=\pm\omega_{q\lambda}\sim T,\Theta$, which means that only thermal phonons are important. It is easy to see that the condition of strong screening of transverse electromagnetic field is valid for $T\ll T_1$ and

$$V^R_{11}(\mathbf{q},\omega)\approx-i\frac{4qc^2}{\nu_0\omega_{qt}v_F} . \tag{29}$$

Thus for one longitudinal and two transverse phonon modes we have

$$\frac{\partial n(\varepsilon)}{\partial t}=-\frac{\pi\beta_l}{8(p_Fu_l)^2}\left[1+\frac{16}{\pi^2}\left[\frac{u_l}{u_t}\right]^4\right] \times\int_0^{\infty}dz\,z^2[R_{\Theta}(\varepsilon,z)-R_{\Theta}(\varepsilon,-z)] ,$$

$$\beta_l=\left[\frac{p_Fv_F}{3}\right]^2\frac{\nu_0}{2MNu_l^2} \tag{30}$$

where β_l is the dimensionless constant which characterizes the interaction between electrons and longitudinal phonons. For $T=\Theta$ and $\varepsilon=0$,

$$\frac{1}{\tau_{e\text{-ph}}}=\frac{7\pi\zeta(3)\beta_lT^3}{(p_Fu_l)^2}\left[1+\frac{16}{\pi^2}\left[\frac{u_l}{u_t}\right]^4\right] , \tag{31}$$

where $\zeta(x)$ is the zeta function, $\zeta(3)\approx1.2$.

Usually $u_l>u_t$ and hence the interaction of electrons with transverse phonons is very important for the electron energy relaxation at low temperatures $T<T_1$. For a cubic lattice $u_l=\sqrt{3}u_t$, and $\tau^{-1}_{e\text{-ph}}$ increases by a factor of ≈15 in comparison with the case when only longitudinal phonons are taken into account. Note that under the condition of strong screening of the transverse electromagnetic field, i.e., for $T<T_1$, the velocity of light does not enter the expression for $\tau_{e\text{-ph}}$, while for $T_1<T$ (weak screening) the contribution to $\tau^{-1}_{e\text{-ph}}$ from transverse phonons is negligible.

The phonon-electron relaxation may be considered by means of the phonon-electron collision integral

$$I_{\text{ph-}e}(\mathbf{q},\omega,\lambda)=-i\{\Pi^C_{\mu\mu}(\mathbf{q},\omega) -[2N_{\Theta}(\omega)+1] \times[\Pi^R_{\mu\mu}(\mathbf{q},\omega)-\Pi^A_{\mu\mu}(\mathbf{q},\omega)]\} . \tag{32}$$

The phonon self-energy is shown in Fig. 1, where we extract one electron loop in which the electron Green's functions G^C depend on the electron temperature T. The functions V^C depend on the lattice temperature Θ. Π_{00} corresponds to longitudinal phonons, and Π_{11} to transverse phonons. As a result, we get the following kinetic equations:

$$\frac{\partial N(\omega_{q\lambda})}{\partial t}=\tfrac{1}{2}I_{\text{ph-}e}(\mathbf{q},\omega_{q\lambda})=-2\int d\mathbf{p}\int d\mathbf{q}\int d\varepsilon\frac{1}{(2\pi)^7}(a^\mu)_m[V^A_{\mu\mu}(\mathbf{q},\omega_{q\lambda})]_{mn}(Q^\mu)_n(\tilde{Q}^\mu)_k[V^R_{\mu\mu}(\mathbf{q},\omega_{q\lambda})]_{kl}(a^\mu)_l$$
$$\times\text{Re}[G^A(\mathbf{p},\varepsilon)G^R(\mathbf{p}+\mathbf{q},\varepsilon+\omega_{q\lambda})]R_\Theta(\omega_{q\lambda},\varepsilon)\ , \tag{33}$$

and so obtain the phonon relaxation times, which coincide with Pippard's results:[9]

$$\frac{1}{\tau_{\lambda\text{ph-}e}}=-\frac{\delta}{\delta N(\omega_{q\lambda})}\frac{\partial N(\omega_{q\lambda})}{\partial t}\ ,$$
$$\frac{1}{\tau_{l\text{ph-}e}}=\frac{\pi Zm}{6M}qv_F,\quad \frac{1}{\tau_{t\text{ph-}e}}=\frac{4Zm}{3\pi M}qv_F\ . \tag{34}$$

The heat flow from electrons to phonons for $\Theta<T<T_1$ may be expressed in two equivalent forms:

$$P=-\int d\varepsilon\,\varepsilon\nu(\varepsilon)\frac{\partial n(\varepsilon)}{\partial t}=\sum_\lambda\int d\mathbf{q}\frac{1}{(2\pi)^3}\omega_{q\lambda}\frac{\partial N(\omega_{q\lambda})}{\partial t}\ , \tag{35}$$

where $\nu(\varepsilon)$ is the electron density of states, $\nu_0=\nu(0)$. Carrying out calculation, we find

$$P=\frac{\beta_l}{2\pi v_F u_l^2}\left[1+\frac{16}{\pi^2}\left[\frac{u_l}{u_t}\right]^4\right]\Gamma(5)\zeta(5)(T^5-\Theta^5)\ . \tag{36}$$

For small heating $T-\Theta\ll\Theta$ we represent P in the form

$$P=\frac{C_e(T-\Theta)}{\tau'_{e\text{-ph}}},\quad C_e=\frac{\pi^2}{3}T\nu_0\ ,$$
$$\frac{1}{\tau'_{e\text{-ph}}}=\frac{180\zeta(5)}{7\pi^2\zeta(3)}\frac{1}{\tau_{e\text{-ph}}}\simeq\frac{2.5}{\tau_{e\text{-ph}}}\ , \tag{37}$$

where C_e is the electronic specific heat. Equation (35) may be expressed in the form of the energy balance equation $C_e/\tau_{e\text{-ph}}\simeq C_{\text{ph}}/\tau_{\text{ph-}e}$, which is useful for estimates; C_{ph} is the phonon specific heat.

The description of the nonequilibrium electron system by means of the Fermi distribution function with an effective temperature T is justified only if the relaxation in the electron system is faster than the relaxation between electrons and the lattice $\tau_{e\text{-}e}<\tau_{e\text{-ph}}$, where $\tau_{e\text{-}e}$ is the electron-electron relaxation time. As was shown in Ref. 5, the main mechanism of electron-electron relaxation in pure metals at low temperatures is the interaction of electrons with transverse vector photons (e-$v\gamma$ is electron–vector-photon and e-$s\gamma$ is electron–scalar-photon),

$$\frac{1}{\tau_{e\text{-}v\gamma}}=\frac{\pi}{6}\left[\frac{\kappa}{mc}\right]^2 T\ln\left[\frac{T}{T_3}\right]\ ,$$
$$T_3<T<T_4,\quad T_3=\frac{1}{\tau}\left[\frac{c}{v_F^2\kappa\tau}\right]^2,\quad T_4=\frac{\kappa v_F^2}{c} \tag{38}$$
$$\frac{1}{\tau_{e\text{-}v\gamma}}=\frac{2\sqrt{2}-1}{\sqrt{3}}\Gamma(\tfrac{3}{2})\zeta(\tfrac{3}{2})\left[\frac{\kappa}{mc}\right]^2(\varepsilon_F\tau)(T\tau)^{1/2}T\ ,$$
$$T<T_3\ . \tag{39}$$

For the same concentration of impurities as before $c_i=0.01\%$, we have $T_3\simeq 0.01$ K and $T_4\simeq 10^3$ K. So from (31), (38), and (39) we have $\tau_{e\text{-}v\gamma}<\tau_{e\text{-ph}}$ for $T_3<T<T_1$ and for $T<T_3$. Thus, in this regime the concept of the electron temperature is valid. Note that at such low temperatures in pure enough metals $\tau_{e\text{-}v\gamma}$ is more important than the energy relaxation time due to

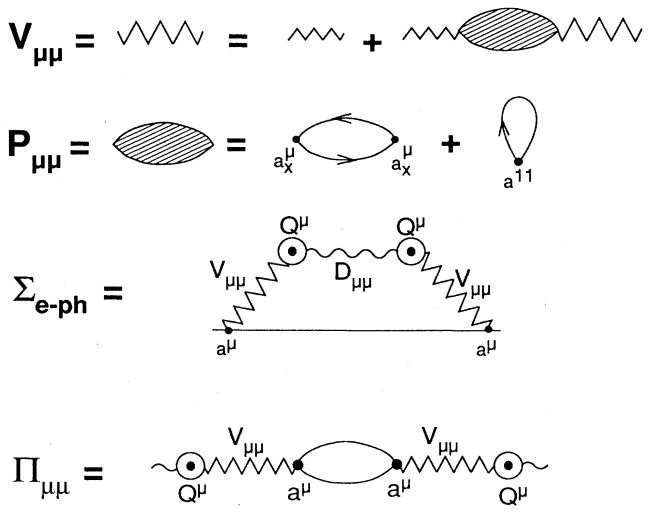

FIG. 1. The electromagnetic field Green's function is $V_{\mu\mu}$. $P_{\mu\mu}$ is the photon polarization operator. $\Sigma_{e\text{-ph}}$ ($\Sigma_{\text{electron-phonon}}$) is the electron self-energy corresponding to the electron-phonon interaction. $\Pi_{\mu\mu}$ is the phonon self-energy.

the electron–scalar-photon interaction with small energy transfer,[10,11] given by

$$\frac{1}{\tau_{e\text{-}s\gamma}}=\frac{3\sqrt{3}(2\sqrt{2}-1)\pi\zeta(3/2)}{32}\frac{(T\tau)^{1/2}T}{(\varepsilon_F\tau)^2},\quad T\ll\frac{1}{\tau}\ . \tag{40}$$

SUPERCONDUCTOR

The electron Green's function in a clean superconductor in the Nambu representation has the form

$$\hat{G}^R(\mathbf{p},\varepsilon)=\frac{-\xi_p\hat{\sigma}_z-\varepsilon\hat{1}+\Delta\hat{\sigma}_x}{(\xi_p-\xi_\varepsilon-i0)(\xi_p+\xi_\varepsilon+i0)}\ , \tag{41}$$

where

$$\xi_\varepsilon=(\varepsilon^2-\Delta^2)^{1/2}\mathrm{sgn}(\varepsilon),\quad |\varepsilon|>\Delta\ . \tag{42}$$

Vertex a^0 carries a factor $\hat{\sigma}_z$ and vertex a^1 carries $\hat{1}$.

The kinetic equation which describes the energy relaxation in a superconductor is

$$\frac{\partial n(\varepsilon)}{\partial t}=-\frac{i}{2\pi\nu_0}\frac{\xi_\varepsilon}{\varepsilon}\mathrm{Tr}\int d\mathbf{p}\frac{1}{(2\pi)^3}\mathrm{Im}[G^A(\mathbf{p},\varepsilon)]\{\hat{\Sigma}^C(\mathbf{p},\varepsilon)-S(\varepsilon)[\Sigma^A(p,\varepsilon)-\Sigma^R(p,\varepsilon)]\}\ . \tag{43}$$

After some transformation, this takes the form

$$\frac{\partial n(\varepsilon)}{\partial t}=-\frac{1}{\pi\nu_0}\frac{\xi_\varepsilon}{\varepsilon}\mathrm{Tr}\sum_\mu\int d\mathbf{p}\int d\mathbf{q}\int d\omega\frac{1}{(2\pi)^7}\mathrm{Im}[G^A(\mathbf{p},\varepsilon)](\hat{a}^\mu)_m\mathrm{Im}[\hat{G}^A(\mathbf{p}+\mathbf{q},\varepsilon+\omega)](\hat{a}^\mu)_l$$
$$\times\mathrm{Im}\{[V^R_{\mu\mu}(\mathbf{q},\omega)]_{mn}(Q^\mu)_n D^R_{\mu\mu}(\mathbf{q},\omega)(\tilde{Q}^\mu)_k[V^R_{\mu\mu}(\mathbf{q},\omega)]_{kl}\}R(\varepsilon,\omega)\ , \tag{44}$$

where $\hat{a}^0=a^0\hat{\sigma}_x$, $\hat{a}^1=a^1\hat{1}$, $R(\varepsilon,\omega)\equiv R_T(\varepsilon,\omega)$, and we let $T=\Theta$. Functions $V^R_{\mu\mu}(\mathbf{q},\omega)$ in a superconductor are defined by the same expressions (9) and (10), with the polarization operators

$$P^R_{\mu\mu}(\mathbf{q},\omega)=-i\,\mathrm{Tr}\int d\mathbf{p}\int d\varepsilon\frac{1}{(2\pi)^4}\hat{a}^\mu_x K^1_{ij}\hat{G}_{jk}(\mathbf{p},\varepsilon)$$
$$\times\hat{a}^\mu_x K^2_{kl}\hat{G}_{li}(\mathbf{p},\varepsilon)+\frac{n_e}{mc^2}\delta_{\mu 1}\ . \tag{45}$$

The function $\mathrm{Re}V^R_{00}$ is practically the same in the superconducting and normal states; hence the condition of strong screening (27) is still valid. For this reason the kinetic phenomena associated with longitudinal phonons may be described by the effective vertex $g_{ql}=a^0V_{00}Q^0$, $|g_{ql}|^2=\beta_l\omega_{ql}/\nu_0$. Transforming the collision integral for longitudinal phonons from the electronic representation to the quasiparticle representation, we get the following expressions for the scattering and recombination relaxation times:

$$\frac{1}{\tau_S(\varepsilon,T)}=-\frac{1}{2\pi^2v_F}\int_0^\infty dq\,q|g_q|^2\int_\Delta^\infty d\omega\frac{\varepsilon(\varepsilon+\omega)-\Delta^2}{\varepsilon[(\varepsilon+\omega)^2-\Delta^2]^{1/2}}[N(\omega)+n(\varepsilon+\omega)]\mathrm{Im}[D^R(\omega,\mathbf{q})]\ , \tag{46}$$

$$\frac{1}{\tau_R(\varepsilon,T)}=-\frac{1}{2\pi^2v_F}\int_0^\infty dq\,q|g_q|^2\int_{2\Delta}^\infty d\omega\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{\varepsilon[(\omega-\varepsilon)^2-\Delta^2]^{1/2}}[N(\omega)+n(\omega-\varepsilon)]\mathrm{Im}[D^R(\omega,\mathbf{q})]\ . \tag{47}$$

In our case $T<T_1\ll\Delta$, and for $\varepsilon=\Delta$ we get

$$\frac{1}{\tau_S(\Delta,T)}=\frac{\beta_l\pi T^3}{4(p_Fu_l)^2}\left[\frac{2T}{\Delta}\right]^{1/2}\Gamma(\tfrac{7}{2})\zeta(\tfrac{7}{2})\ , \tag{48}$$

$$\frac{1}{\tau_R(\Delta,T)}=\frac{2\beta_l}{\pi}\left[\frac{\Delta}{p_Fu_l}\right]^2(2\pi\Delta T)^{1/2}\exp(-\Delta/T)\ . \tag{49}$$

Expressions (48) and (49) coincide with the results of Ref. 12.

For transverse phonons the situation is different. After some algebra we have

$$P^R_{11}(\mathbf{q},\omega)=-i\frac{\nu_0}{16v_F}\left[\frac{v_F}{c}\right]^2\left[2\int_\Delta^\infty d\varepsilon[S(\varepsilon)-S(\varepsilon+\omega)]\frac{\varepsilon(\varepsilon+\omega)+\Delta^2}{(\varepsilon^2-\Delta^2)^{1/2}[(\varepsilon+\omega)^2-\Delta^2]^{1/2}}\right.$$
$$\left.+\Theta(\omega-2\Delta)\int_\Delta^{\omega-\Delta}d\varepsilon[S(\varepsilon-\omega)-S(\varepsilon)]\frac{\varepsilon(\omega-\varepsilon)-\Delta^2}{(\varepsilon^2-\Delta^2)^{1/2}[(\omega-\varepsilon)^2-\Delta^2]^{1/2}}\right]\ , \tag{50}$$

where $\Theta(x)$ is the Heaviside function. The first term in (50) describes the scattering processes of quasiparticles and the second refers to recombination processes. For the scattering processes typical value of ω is T. For such ω and for $T \ll \Delta$,

$$P_{11}^{R}(\mathbf{q},\omega) \simeq -i\frac{\pi^{3/2}\nu_0}{4qv_F}\left[\frac{v_F}{c}\right]^2 \Delta \exp(-\Delta/T) . \quad (51)$$

The small exponential factor implies that it is impossible to satisfy the condition of strong screening of the transverse electromagnetic fields for the scattering processes. For the recombination processes, frequencies satisfying $\omega - 2\Delta > T$ are important, and because $T \ll \Delta$ we get $\omega \gg T_1$. Hence the condition of strong screening cannot be satisfied. Thus, the contribution to the electron-phonon collision integral from transverse phonons is negligible in comparison with longitudinal phonons.

SUMMARY

Equations (31) and (36) provide a description of the heating or cooling of the electron system in a pure metal at low temperatures $T_2 < T < T_1$. The heating of the electrons at temperatures 25–320 mK was investigated in Ref. 2, and the inelastic electron-phonon relaxation time $\tau_\varepsilon = \alpha^{-1}T^{-3}$, $\alpha = 9\times 10^7$ sec^{-1}K^{-3} was obtained. The thickness of the Cu films $d \approx 1000$ Å and the resistance ratio $R(273)/R(1) \approx 3.75$ imply an electron mean free path $l \approx 1500$ Å. For this data $T_1 \approx T_2 \approx 100$ mK, and the expression (36) which is strictly speaking valid for $T_2 < T < T_1$ fits the experimental data rather well. For $T < T_2$ the metal should be considered as dirty and for this case[13,14]

$$\frac{1}{\tau_{e\text{-ph}}} = \frac{\pi^4\beta_l}{5}\frac{(p_F l)T^4}{(p_F u_l)^3}\left[1+\frac{3}{2}\left[\frac{u_l}{u_t}\right]^5\right] . \quad (52)$$

Using the electron-phonon collision integral derived in Ref. 14,

$$\frac{dn(\varepsilon)}{dt} = -\frac{\beta_l\tau}{5\pi^2 u_l^3}\left[1+\frac{3}{2}\left[\frac{u_l}{u_t}\right]^5\right] \times \int_0^\infty d\omega\,\omega^3[R_\Theta(\varepsilon,\omega) - R_\Theta(\varepsilon,-\omega)] \quad (53)$$

and formula (35), we get the following expression for the energy flow:

$$P = \frac{32\pi^4}{315}\frac{\beta\tau}{(u_l)^3}\left[1+\frac{3}{2}\left[\frac{u_l}{u_t}\right]^5\right](T^6-\Theta^6) . \quad (54)$$

In a dirty metal the electron-phonon relaxation time becomes longer by a factor $(q_T l)^{-1}$, where $q_T = T/u$ is the thermal-phonon wave vector, in comparison with a clean metal at the same temperature.

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