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Thermodynamics of Ising models with layered randomness: Exact solutions on square and triangular lattices

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Two-dimensional Ising models are studied on square and triangular lattices with layered disordered bonds (McCoy-Wu-type models). It is shown that exact solutions exist for diluted exponential distributions of couplings. In the ferromagnetic case the free energy exhibits an essential singularity (slightly different from the result of McCoy and Wu) at the transition point. If frustration is present the transitions occurs at $T=0$, and the specific heat is linear.

I. INTRODUCTION

In the present paper we study the thermodynamics of 2D Ising models on a square or triangular lattice with layered disorder. We make use of exact methods. First Grassman variables are used in order to obtain recurrence relations for quantities determining the free energy of finite systems. Then integral equations are derived in the thermodynamic limit under the assumption that randomness is independent with the same distribution in each column. It is shown that these integral equations can be reduced to recurrence relations with nonrandom coefficients if one assumes disorder to be distributed exponentially.

If all couplings are ferromagnetic, these systems have a phase transition at some positive temperature. Our result for the essential singularity occurring there differs slightly from the one predicted by McCoy and Wu. At low temperatures the specific heat is, of course, exponentially small.

If there is frustration, the phase transition occurs at $T=0$, and the specific heat is linear in our model. It is further shown that the behavior on the triangular lattice is very close to the behavior on the square lattice, the main difference being a redefinition of relevant parameters.

The approach of the present paper may be extended to systems where the random value of the couplings in a certain row is correlated to the value in the previous row. This would describe a crystal grown from one side by attaching new layers, which themselves are homogeneous but may differ at random from the previous layer. An exactly soluble form of the correlation probabilities exists and can be studied with exactly the same methods.

The square lattice with layered disorder has also been studied by Shankar and Murthy. These authors discuss in particular Griffiths singularities. An exact calculation of these quantities was performed by Forgacs *et al.* in a 2D Ising model with infinite pinning fields pointing up or down and fully correlated along lines at random positions. A full understanding of these singularities in general 2D Ising models with correlated disorder is still lacking, however.

Since the original work of Onsager on the two-dimensional Ising model,[1] physicists have devoted a lot of effort to finding exactly solvable 2D systems. Three major methods have been successful in this goal, namely the transfer matrix techniques developed by Baxter,[2] the Coulomb gas technique of Nienhuis,[3] and the conformal invariance.[4]

On the other hand, two-dimensional *disordered* systems have resisted much more exact solutions, and only few models have been successfully investigated. First, the McCoy and Wu (denoted MW in the following) model[5] is a model where all vertical bonds are identical, whereas horizontal bonds within the same row are random variables. The main result of MW was that disorder smoothens the transition, transforming the standard power-law singularities of usual phase transitions into smooth exponential essential singularities. Second, the presumably exact critical behavior of the 2D random-bond Ising model was derived by Dotsenko and Dotsenko;[6] they showed that for weak disorder, at criticality, the system is equivalent to a zero component Gross-Neveu model, which can be exactly analyzed by perturbative renormalization group calculations. Their conclusions are again that disorder smoothens the critical singularities, e.g., transforming the log singularity of the pure Ising model into a weaker log(log) singularity. Furthermore, it was shown that the other critical exponents are modified by logarithms.[7]

In both cases, since it is bond disorder, according to the Harris criterion,[8] the relevance of the disorder to the critical behavior is governed by the exponent α of the pure system. Since α is zero in the 2D Ising model (with a logarithmic singularity), the disorder is expected to be marginally relevant, which is indeed verified in these models.

It should be noted that in the latter example, the techniques make extensive use of replicas, small momentum expansions, and thus although very plausible, the results of Refs. 6 and 7 should be taken with caution.

On the field of 1D random-field Ising chains (RFIC) many solvable models have been found and solved (Derrida and Hilhorst,[9] Nieuwenhuizen and Luck,[10] denoted NL). The basic technique developed by one of the authors is to solve the corresponding Dyson-Schmidt integral equation,[11] and exact solutions were found[12] at all temperatures for diluted exponential distributions of the disorder.

The exact location of the critical point for layered Ising models has been studied by various groups. The result for $\mathcal{J}_2$ random was first found by McCoy and Wu.[5(a)] The situation where both $\mathcal{J}_1$ and $\mathcal{J}_2$ are random was solved by Barouch[13] and by Au-Yang and McCoy.[14] The critical point of systems where disorder occurs periodically in arbitrarily large unit cells was studied in detail by Wolff, Zittartz, and Hoever.[15]

The aim of this paper is twofold. First we recalculate the MW operations in the more elegant way of RFIC of NL: We show at first that these two problems are identical, except for an extra angle θ, over which one has to integrate at the very end (Secs. II and III). This angle is the remnant of the direction along which the system is translationally invariant. Also, we show that one of the shortcomings of MW, namely the necessity of a temperature-dependent bond distribution, can be overcome. We show in Sec. IV that there exists a temperature-independent distribution of bonds (diluted exponential) for which the model can be solved exactly. A similar exact solution is also found on the triangular lattice (Sec. V). In the case of a ferromagnetic transition, we find a slightly different essential singularity at T_c (see Secs. III C and III D).

The second aspect of this study is the case when frustration (and disorder) is present. Then, the analytic properties are much more intricate, but we find that the critical temperature is displaced to $T=0$. Furthermore, as often occurs in disordered systems, the specific heat is a linear function of T at low temperature (which signals the absence of a gap in the excitation spectrum of the system) and the zero-temperature entropy vanishes, as one expects in systems with quasi-one-dimensional frustration.[16] (See Sec. IV C.)

Similar work was initiated by Shankar and Murthy,[17] who also noted the mapping to NL (Ref. 10) and presented results on Griffiths singularities, the ferromagnetic transition, and correlation functions.

II. PARTITION SUMS AND GRASSMANN VARIABLES

In this section we shall calculate explicit forms for the partition sum of finite lattices with a one-dimensional type of disorder. We use Grassmann variables for evaluating the spin sums. In later sections we will study these results in the thermodynamical limit, thereby confining ourselves to diluted exponential distributions of the random couplings.

A. The square lattice

We consider an Ising model on a square lattice with sites (n,m) with $1\le n\le N$ and $1\le m\le M$, with periodic boundary conditions in the m direction. It is assumed that couplings depend on n but not on m. The Hamiltonian therefore has the form

$$\mathcal{H}=-\sum_{n,m}[\mathcal{J}_1(n)\sigma_{n,m}\sigma_{n,m+1}+\mathcal{J}_2(n)\sigma_{n,m}\sigma_{n+1,m}] \,. \quad (2.1)$$

The partition sum can be written as

$$\begin{aligned} Z&\equiv\sum_{\{\sigma\}}e^{-\beta H}\\ &=\left[\prod_{n,m}2\cosh\beta\mathcal{J}_1(n)\cosh\beta\mathcal{J}_2(n)\right]Z_p \,, \end{aligned} \quad (2.2)$$

where Z_p has only contributions from closed polygons

$$\begin{aligned} Z_p=2^{-NM}\sum_{\{\sigma\}}\prod_{n,m}&[1+z_1(n)\sigma_{n,m}\sigma_{n,m+1}]\\ &\times[1+z_2(n)\sigma_{n,m}\sigma_{n+1,m}] \,. \end{aligned} \quad (2.3)$$

We have defined

$$\begin{aligned} z_1(n)&=\tanh\beta\mathcal{J}_1(n) \,,\\ z_2(n)&=\tanh\beta\mathcal{J}_2(n) \,. \end{aligned} \quad (2.4)$$

In order to calculate Z_p we do not follow the method of McCoy and Wu,[5] but use the method of Grassmann variables, see Berezin,[18] Samuel,[19] and Dotsenko and Dotsenko.[6] We introduce four anticommuting variables at each site (n,m) according to Fig. 1.

The function Z_p can be shown to be equal to

$$Z_p=\int D\psi_1 D\psi_2 D\psi_3 D\psi_4 \exp B \,, \quad (2.5)$$

where

$$D\psi_i=\prod_{(n,m)}d\psi_i(n,m)$$

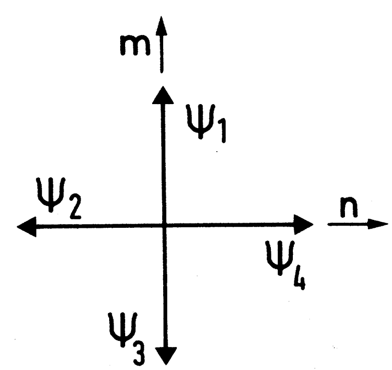

FIG. 1. Labeling of Grassmann variables on the square lattice.

and

$$B=\sum_{n,m}[z_1(n)\psi_4(n,m)\psi_2(n,m+1)+z_2(n)\psi_3(n,m)\psi_1(n+1,m)+(\psi_1\psi_2+\psi_3\psi_4+\psi_1\psi_4+\psi_2\psi_3+\psi_4\psi_2+\psi_3\psi_1)(n,m)]\ . \qquad (2.6)$$

The equivalence between (2.3) and (2.6) can be established by expanding in powers of z_1 and z_2. Since the problem is translationally invariant in the m direction one performs a Fourier transformation

$$\psi_i(n,\theta)=\frac{1}{\sqrt{M}}\sum_m \psi_i(n,m)e^{im\theta}\ , \qquad (2.7)$$

where $\theta=2\pi l/M$ with integer l ($-\frac{1}{2}M<l\le\frac{1}{2}M$). It leaves the integration measure invariant, and B becomes

$$B=\sum_n\sum_{\theta>0}\{z_1(n)[e^{-i\theta}\psi_4^*(n)\psi_2(n)-e^{i\theta}\psi_2^*(n)\psi_4(n)]+z_2(n)[\psi_3^*(n)\psi_1(n+1)-\psi_1^*(n+1)\psi_3(n)]+[\psi_1^*\psi_2-\psi_2^*\psi_1+\psi_3^*\psi_4-\psi_4^*\psi_3+\psi_1^*\psi_4-\psi_4^*\psi_1+\psi_2^*\psi_3-\psi_3^*\psi_2+\psi_4^*\psi_2-\psi_2^*\psi_4+\psi_3^*\psi_1-\psi_1^*\psi_3](n)\}\ , \qquad (2.8)$$

where the argument θ of the ψ_i has been omitted.

For $\theta>0$, $\psi_i(n,\theta)$ and $\psi_i^*(n,\theta)$ are independent variables. One can verify that

$$\int d\psi_3^* d\psi_2 e^{\psi_3^*(\psi-\psi_2)+\varphi^*\psi_2}=e^{\varphi^*\psi}\ . \qquad (2.9)$$

In other words, the ψ_3^* integral yields $\delta(\psi-\psi_2)$. Using this we obtain from (2.8)

$$Z_p=\int D\psi_1 D\psi_4 \exp B_1 \qquad (2.10)$$

with

$$B_1=\sum_{\theta>0}\sum_n (z_1(n)(e^{-i\theta}-e^{i\theta})\psi_4^*(n)\psi_4(n)+\psi_4^*(n)\{z_1(n)e^{-i\theta}[z_2(n)\psi_1(n+1)+\psi_1(n)]-\psi_1(n)+z_2(n)\psi_1(n+1)\}-\{z_1(n)e^{i\theta}[z_2(n)\psi_1^*(n+1)+\psi_1^*(n)]-\psi_1^*(n)+z_2(n)\psi_1^*(n+1)\}\psi_4(n)+z_2(n)[\psi_1^*(n)\psi_1(n+1)-\psi_1^*(n+1)\psi_1(n)])\ . \qquad (2.11)$$

The next step is to perform the integrations over ψ_4 and ψ_4^*. The final result is

$$Z_p=\left[\prod_n\prod_{\theta>0} z_1(n)(e^{-i\theta}-e^{i\theta})\right]\int D\psi_1^* D\psi_1 \exp\left[-\sum_{\theta>0}\sum_{n,n'}\psi_1^*(n)A_{nn'}(\theta)\psi_1(n)/(e^{-i\theta}-e^{i\theta})\right]=\prod_{\theta>0}\left[\prod_n z_1(n)\Delta_{N+1}(\theta)\right]\ , \qquad (2.12)$$

where

$$A_{n,n+1}=A_{n+1,n}=+z_2(n)[z_1(n)-1/z_1(n)]\ ,$$
$$A_{n,n}=z_2^2(n-1)[z_1(n-1)+1/z_1(n-1)+2\cos\theta]+z_1(n)+1/z_1(n)-2\cos\theta\ , \qquad (2.13)$$

all other elements being zero. Further

$$\Delta_{N+1}(\theta)=\det A(\theta)\ . \qquad (2.14)$$

Combining (2.2) and (2.12) one finds for the free energy per site

$$-\beta F=\frac{1}{N}\sum_n[\ln 2\cosh K_2(n)+\tfrac{1}{2}\ln\tfrac{1}{2}\sinh 2K_1(n)]+\frac{1}{M}\sum_{\theta>0}\Omega(\theta)\ , \qquad (2.15)$$

where

$$\Omega(\theta)=\frac{1}{N}\ln\Delta_{N+1}(\theta)=\frac{1}{N}\sum_{n=1}^{N}\ln[\Delta_{n+1}(\theta)/\Delta_n(\theta)] \qquad (2.16)$$

holds because $\Delta_1=1$. The Δ_n satisfy the recurrence relations

$$\Delta_{n+1}=\{z_2^2(n)[z_1(n)+1/z_1(n)+2\cos\theta]+z_1(n+1)+1/z_1(n+1)-2\cos\theta\}\Delta_n-z_2^2(n)[z_1(n)-1/z_1(n)]^2\Delta_{n-1}\ . \qquad (2.17)$$

The expression (2.15) is very complicated because it depends on all couplings $\mathcal{J}_1(n)$, $\mathcal{J}_2(n)$. In Sec. III we shall proceed by going to the thermodynamic limit and assuming that the couplings are independent random variables with common distribution functions.

B. The hexagonal and the triangular lattice

In this section we use Grassman variables for calculating the partition sum of a layered random Ising model on a hexagonal lattice. Via a duality transformation[19,20] this determines also the result on a triangular lattice.

We start considering a brick lattice, such as depicted in Fig. 2. The lattice sites are present on sites (m,n) and $(m+\frac{1}{2},n+\frac{1}{2})$ with $0\le m\le M$, $0\le n\le N$, and has $2NM$ sites. This lattice is topologically equivalent to the hexagonal lattice. The Ising Hamiltonian to be considered is

$$\mathcal{H}=-\sum_{n,m}[\mathcal{J}_1(nm)S_{mn}S_{m+1/2,n+1/2}+\mathcal{J}_2(nm)S_{mn}S_{m-1/2,n+1/2}+\mathcal{J}_3(n,m)S_{m,n}S_{m-1/2,n-1/2}]\ . \tag{2.18}$$

It leads to the partition sum

$$Z_{N,M}=\sum_{\{S_{nm},S_{n+1/2,m+1/2}\}}e^{-\beta\mathcal{H}}=2^{2NM}\prod_{n,m}[\cosh K_1(nm)\cosh K_2(nm)\times\cosh K_3(nm)]Z_p \tag{2.19}$$

where Z_p is the polygon sum,

$$Z_p=4^{-NM}\sum_{\{S_{mn},S_{m+1/2,n+1/2}\}}\prod_{nm}[1+S_{mn}S_{m+1/2,n+1/2}z_1(nm)][1+S_{mn}S_{m-1/2,n+1/2}z_2(n,m)]\times[1+S_{mn}S_{m-1/2,n-1/2}z_3(n,m)]\ , \tag{2.20}$$

where

$$z_j=\tanh K_j(n,m)\equiv e^{-2L_j(n,m)} \tag{2.21}$$

with $K_j=\beta\mathcal{J}_j$ as usual, and L_j denote the couplings on the dual triangular lattice. The polygon sum can be written as the following Grassmann integral:

$$\begin{aligned}Z_p=\int D\psi\exp\sum_{mn}(\psi_3\psi_1+\psi_4\psi_2+\psi_1\psi_2+\psi_3\psi_4+\psi_1\psi_4+\psi_2\psi_3)(m,n)\\ \times\exp\sum_{mn}(\psi_3\psi_1+\psi_4\psi_2+\psi_1\psi_2+\psi_3\psi_4+\psi_1\psi_4+\psi_2\psi_3)(m+\tfrac{1}{2},n+\tfrac{1}{2})\\ \times\exp\sum_{mn}[z_1(nm)\psi_3(m+\tfrac{1}{2},n+\tfrac{1}{2})\psi_1(mn)+z_2(nm)\psi_4(m-\tfrac{1}{2},n+\tfrac{1}{2})\psi_2(mn)\\ +z_3(nm)\psi_3(mn)\psi_1(m-\tfrac{1}{2},n-\tfrac{1}{2})]\ ,\end{aligned} \tag{2.22}$$

where $D\psi$ denotes integration over $\psi_1\cdots\psi_4$ at all sites. Next it is useful to define

$$x_j(m,n)=\psi_j(m,n),\ y_j(m,n)=\psi_j(m+\tfrac{1}{2},n+\tfrac{1}{2})\ . \tag{2.23}$$

As a first step the integrals over x_4 and x_2, and also over y_2 and y_3 can be performed using (2.9). The result is

$$\begin{aligned}Z_p=\int Dx_1Dx_3Dy_1Dy_4\\ \times\exp\sum_{m,n}(x_1x_3+y_1y_4)(m,n)\exp\sum_{mn}\{z_1(nm)[y_1(mn)+y_4(mn)]x_1(mn)+z_2(nm)y_4(m-1,n)[x_1(mn)+x_3(mn)]\\ +z_3(nm)x_3(mn)y_1(m-1,n-1)\}\ ,\end{aligned} \tag{2.24}$$

where $Dx_1=\prod_{m,n}dx_1(m,n)$, etc. Now we specialize to the situation where the couplings $z_j(n,m)\equiv z_j(n)$ only vary in the vertical direction. Then a Fourier transformation can be made in the m direction, like in (2.7). This introduces $x_i(n,\theta)$ and $y_i(n,\theta)$ with $-\pi<\theta<\pi$. We shall again omit the θ variable and write $\bar{x}_i(n)$ when $\theta<0$. In this notation (2.24) takes the form

$$Z_p=\int Dx_1Dx_2Dy_1Dy_4\exp B$$

with

$$B=\sum_n \sum_{\theta>0} \{[x_1^*(n)x_3(n)-x_3^*(n)x_1(n)+y_1^*(n)y_4(n)-y_4^*(n)y_1(n)]$$
$$+z_1(n)[y_1^*(n)x_1(n)-x_1^*(n)y_1(n)+y_4^*(n)x_1(n)-x_1^*(n)y_4(n)]$$
$$+z_2(n)[e^{i\theta}y_4^*(n)x_1(n)-e^{-i\theta}x_1^*(n)y_4(n)+e^{i\theta}y_4^*(n)x_3(n)-e^{-i\theta}x_3^*(n)y_4(n)]$$
$$+z_3(n)[e^{i\theta}x_3^*(n)y_1(n-1)-e^{-i\theta}y_1^*(n-1)x_3(n)]\} \; . \qquad (2.25)$$

The integration measure is invariant under Fourier transformation,

$$Dx_1=\prod_{m,n} dx_1(m,n)=\prod_n \prod_{\theta>0} dx_1(n,\theta)dx_1^*(n,\theta) \; . \qquad (2.26)$$

Now the integrations with respect to x_1^* and x_3, and with respect to x_1 and x_3^* can be performed, yielding

$$Z_p=\int Dy_1 Dy_4 \exp B_1 \; , \qquad (2.27)$$

with

$$B_1=\sum_n \sum_{\theta>0} ([z_1(n)z_2(n)\eta_4^*(n)y_4(n)]$$
$$-\eta_4^*\{[1-z_1(n)z_2(n)e^{i\theta}]y_1(n)-[z_1(n)e^{i\theta}+z_2(n)e^{2i\theta}]z_3(n)y_1(n-1)\}$$
$$+\{[1-z_1(n)z_2(n)e^{-i\theta}]y_1^*(n)-[z_1(n)e^{-i\theta}+z_2(n)e^{-2i\theta}]z_3(n)y_1^*(n-1)\}\eta_4(n)$$
$$+2i\sin\theta\, z_1(n)z_3(n)[e^{i\theta}y_1^*(n)\eta_1(n-1)-e^{-i\theta}y_1^*(n-1)\eta_1(n)]) \; . \qquad (2.28)$$

Here we introduced

$$\eta_4^*=y_4^*2i\sin\theta, \quad \eta_1=y_1/(2i\sin\theta) \; . \qquad (2.29)$$

Also the y_4 and η_4^* integral and then the final y_1 and η_1 integrations can be performed. The result is

$$Z_p=\prod_{\theta>0}\left[\Delta_N(\theta)\prod_n z_1(n)z_2(n)\right] \qquad (2.30)$$

with

$$\Delta_n=\det\Gamma_{nm} \; ,$$

where Γ is a tridiagonal matrix with elements

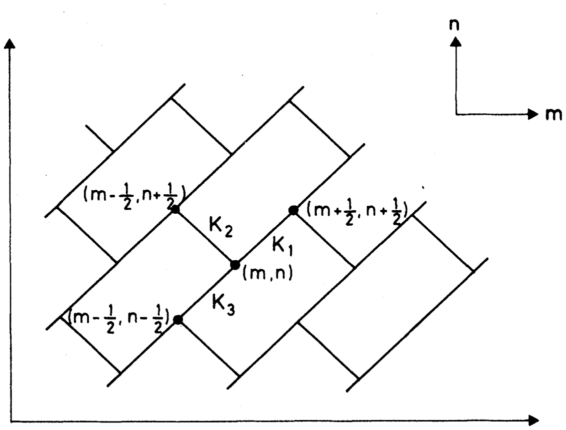

FIG. 2. Labeling of lattice sites and couplings on the brick lattice (topologically equivalent to a hexagonal lattice).

$$\Gamma_{n,n}=z_1(n)z_2(n)+\frac{1}{z_1(n)z_2(n)}-2\cos\theta$$
$$+z_3^2(n+1)\left[\frac{z_1(n+1)}{z_2(n+1)}+\frac{z_2(n+1)}{z_1(n+1)}+2\cos\theta\right] , \qquad (2.31)$$

$$\Gamma_{n,n-1}=\Gamma_{n-1,n}^*$$
$$=-z_3(n)e^{i\theta}\left[\frac{1}{z_2(n)}-z_2(n)+e^{i\theta}\left[\frac{1}{z_1(n)}-z_1(n)\right]\right] . \qquad (2.32)$$

As usual Δ_{N+1} satisfies a recurrence relation

$$\Delta_{N+1}=\Gamma_{N,N}\Delta_N-\Gamma_{N,N-1}\Gamma_{N-1,N}\Delta_{N-1} \; . \qquad (2.33)$$

It is useful to define a variable R_N which only depends on random couplings $K_j(n)$ with $n\le N$

$$R_N=\frac{\Delta_N}{2\Delta_{N-1}}-e^{-4L_3(N+1)}\{\cosh[2L_1(N+1)-2L_2(N+1)]+\cos\theta\} \; , \qquad (2.34)$$

where L_j is related to K_j by (2.21). The recursion for R_N is

$$R_N=\cosh[2L_1(N)+2L_2(N)]-\cos\theta-e^{-4L_3(N)}\frac{\sinh^2 2L_1(N)+\sinh^2 2L_2(N)+2\cos\theta\sinh 2L_1(N)\sinh 2L_2(N)}{e^{-4L_3(N)}\{\cosh[2L_1(N)-2L_2(N)]+\cos\theta\}+R_{N-1}} . \tag{2.35}$$

The free energy per pair of sites follows from

$$-\beta F=\frac{1}{NM}\ln Z_{N,M}=\frac{1}{N}\sum_n \ln[\sinh 2K_1(n)\sinh 2K_2(n)\cosh K_3(n)]+\frac{1}{M}\sum_{\theta>0}[\Omega(\theta)+\ln 2] \tag{2.36}$$

with

$$\begin{aligned}\Omega(\theta)&=\frac{1}{N}\ln\Delta_N(\theta)\\&=\frac{1}{N}\sum_{n=1}^{N}\ln\frac{\Delta_n(\theta)}{\Delta_{n-1}(\theta)}\\&=\frac{1}{N}\sum_{n=2}^{N+1}\ln(R_{n-1}+e^{4L_3(n)}\{\cosh[2L_1(n)-2L_2(n)]+\cos\theta\}) .\end{aligned} \tag{2.37}$$

The partition sum on a triangular lattice (Fig. 3) is related to the one on the hexagonal lattice by the duality relation

$$Z_{\mathrm{tr}}(L_j)=Z_{\mathrm{hex}}(K_j)\prod_{\text{triangles}}(\tfrac{1}{2}\sinh 2L_1\sinh 2L_2\sinh 2L_3)^{1/4} \tag{2.38}$$

and

$$-\beta F_{\mathrm{tr}}=-\beta F_{\mathrm{hex}}+\frac{1}{4N}\sum_{\text{triangles}}\ln(\tfrac{1}{2}\sinh 2L_1\sinh 2L_2\sinh 2L_3) \tag{2.39}$$

with relations (2.21). The notation $\prod_{\text{triangles}}$ ($\sum_{\text{triangles}}$) means a sum (product) over all triangles of the triangular lattice.

III. INTEGRAL EQUATIONS AND THEIR SOLUTIONS

The recurrence relations for $\Delta_N(\theta)$ derived in the previous section have the same structure as in random one-dimensional problems. In the thermodynamical limit one can derive a Dyson-Schmidt[11] integral equation for distribution of the ratios $\Delta_N(\theta)/\Delta_{N-1}(\theta)$. This equation is analogous to the Chapman-Kolmogorov equation in stochastic processes, the role of time in our problem being played by a space variable. The method we shall follow is very close to the one introduced by Nieuwenhuizen and Luck[10] to be referred to as NL.

A. The square lattice

We start with Eq. (2.17) and define

$$S_n=\Delta_{n+1}/\Delta_n-z_1(n+1)-1/z_1(n+1)+2\cos\theta . \tag{3.1}$$

It satisfies the recurrence relation

$$S_n=\frac{S_{n-1}z_2^2(n)[z_1(n)+1/z_1(n)+2\cos\theta]+4z_2^2(n)\sin^2\theta}{z_1(n)+1/z_1(n)-2\cos\theta+S_{n-1}} \tag{3.2}$$

and depends only on random variables at site n or to the left of site n. In order to study its distribution function we define

$$D_n(u)=\langle\ln(S_n-u)\rangle . \tag{3.3}$$

We assume that all $\vartheta_1(n)$ are independent random variables with a common distribution $\rho_1(\vartheta_1)$; similarly $\rho_2(\vartheta_2)$ for $\vartheta_2(n)$. Then using (3.2) D_n can be expressed in terms of D_{n-1}. It is an integral equation because the integrals over $\vartheta_1(n)$ and $\vartheta_2(n)$ still have to be performed. In the limit $n\to\infty$ the limit function $D(u)$, therefore, satisfies

$$D(u)=\int\rho_1(\vartheta_1)d\vartheta_1\rho_2(\vartheta_2)d\vartheta_2\times\left[D\left(\frac{u(z_1+1/z_1-2\cos\theta)-4z_2^2\sin^2\theta}{z_2^2(z_1+1/z_1+2\cos\theta)-u}\right)-D(2\cos\theta-z_1-1/z_1)+\ln[z_2^2(z_1+1/z_1+2\cos\theta)-u]\right] , \tag{3.4}$$

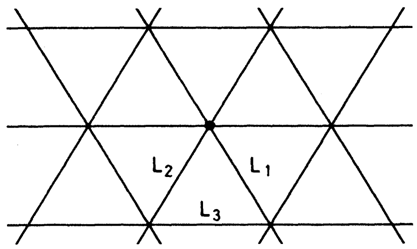

FIG. 3. The triangular lattice and its couplings. The couplings L_3 depend on n).

where $z_{1,2}=\tanh\beta\mathcal{J}_{1,2}$. The quantity of interest is Ω, defined in (2.16) for a given finite system. In the thermodynamic limit it has a chain-independent value with probability one (the free energy is a self-averaging quantity). Hence we calculate its average

$$\Omega(\theta)=\lim_{n\to\infty}\langle \ln\Delta_{n+1}(\theta)/\Delta_n(\theta)\rangle$$
$$=\int\rho_1(\mathcal{J}_1)d\mathcal{J}_1 D(2\cos\theta-z_1-1/z_1) \quad (3.5)$$

according to (3.1). This integral is just the constant term in (3.4).

B. Exact solutions: the McCoy-Wu model

In order to study Eq. (3.4) one has to specify the distributions $\rho_1(\mathcal{J}_1)$, $\rho_2(\mathcal{J}_2)$. We first consider the situation studied by McCoy and Wu,[5] and show that their results can be obtained in a rather elegant manner with methods introduced in NL.

McCoy and Wu consider the problem where z_1 is fixed $[\rho_1(\tilde{\mathcal{J}}_1)=\delta(\tilde{\mathcal{J}}_1-\mathcal{J}_1)]$ and where z_2 is random with a power-law distribution. We will slightly generalize their choice by allowing that only a fraction $q\equiv 1-p$ of the horizontal bonds is random:

$$r(z_2)=q2Nz_2^{2N-1}z_0^{-2N}+p\delta(z_2-z_0)\quad(0\le z_2\le z_0) \quad (3.6)$$
$$=0 \quad \text{else}$$

with $N>0$. If we define dual couplings K_j by

$$z_1=\tanh\beta\mathcal{J}_1=e^{-2K_2},$$
$$z_2\equiv\tanh\beta\mathcal{J}_{2,x}\equiv e^{-2K_{1,x}}, \quad (3.7)$$

it follows that the dual couplings are distributed exponentially

$$K_{1,x}=K_1+K_r x$$

with $K_{1,0}\equiv K_1$ and x having the density

$$\rho(x)=qe^{-x}+p\delta(x)\quad(x\ge 0)$$
$$=0\quad(x<0) \quad (3.8)$$

provided one identifies

$$K_r=(4N)^{-1}. \quad (3.9)$$

However, the dual couplings are temperature dependent, and so is the distribution of disorder (3.6). This unphysical property led McCoy and Wu to study the limit of weak disorder ($N\to\infty$), where this temperature dependence can be neglected close to the critical point. In the next section we shall show that for random couplings in the vertical direction a temperature-independent distribution of disorder can be solved exactly.

The way to proceed was introduced by one of us;[12] see also Ref. 10. One writes Eqs. (3.4) and (3.5) as

$$D(u)=q\int_0^\infty e^{-x}dx[D(\Phi(u,x))+\ln\Psi(u,x)]$$
$$+pD(\Phi(u,0))+p\ln\Psi(u,0)-\Omega, \quad (3.10)$$

where

$$\Phi(u,x)=[2u(\cosh 2K_2-\cos\theta)$$
$$-4e^{-4K_1-4K_r x}\sin^2\theta]/\Psi(u,x),$$
$$\Psi(u,x)=2e^{-4K_1-4K_r x}(\cosh 2K_2+\cos\theta)-u. \quad (3.11)$$

Next one performs a partial integration with respect to x. It gives a boundary term at $x=0$ and an integral involving $\partial\Phi(u,x)/\partial x$. It follows that

$$\frac{\partial\Phi(u,x)}{\partial x}=4uK_r\frac{\partial\Phi(u,x)}{\partial u}. \quad (3.12)$$

Therefore the latter integral is proportional to a derivative with respect to u of the integral occurring in (3.10). Hence it may be eliminated, yielding a difference-differential equation equivalent to (3.10):

$$(1-4K_r u\partial_u)[D(u)-pD(v)-p\ln w]$$
$$=qD(v)+q\ln w-4qK_r-\Omega, \quad (3.13)$$

where

$$v=\Phi(0,u),\quad w=\Psi(u,0). \quad (3.14)$$

This equation can be cast in a standard form by going to new variables. The computation is lengthy, and we only give the results. One defines parameters $\mu\ge 0$, ν, and η by

$$\cosh\mu=(\cosh 2K_2\cosh 2K_1-\sinh 2K_1\cos\theta)/\sinh 2K_2,$$
$$\sinh\nu=(\cosh 2K_1\cos\theta-\sinh 2K_1\cosh 2K_2)/\sin\theta,$$
$$\cosh\nu=\sinh\mu\sinh 2K_2/\sin\theta,$$
$$\eta=2K_r/\cosh\nu. \quad (3.15)$$

We note that θ is in the interval $0\le\theta\le\pi$.

Further one introduces a new variable and a new function

$$u=2e^{-4K_1}(\cosh 2K_2+\cos\theta)$$
$$-2e^{-2K_1}\sinh 2K_2(e^{\mu}+ye^{-\mu-\nu})/(1+ye^{-\nu}),$$
$$G(y)=D(u)+\ln(1+ye^{-\nu}), \quad (3.16)$$

where relation between $D(u)$ and $G(y)$ is such that $G(y)$ is regular at $y=-e^{\nu}$. In these units Eq. (3.14) takes the form

$$[1+\eta(e^{\nu}+y)(e^{-\nu}-y)\partial_y][G(y)-pG(ye^{-2\mu})] =qG(ye^{-2\mu})-q\eta y-q\eta e^{\nu}-\Omega_r \ , \quad (3.17)$$

where

$$\Omega_r(\theta)=\Omega(\theta)-\Omega_{\text{pure}}(\theta) =\Omega(\theta)-(\mu-2K_1+\ln 2\sinh 2K_2) \quad (3.18)$$

with Ω_{pure} denoting its value in the model without disorder, that is to say, for $q=0$ or for $K_r=0$. In both these limits $G=\Omega_r=0$ solves Eq. (3.17) in a trivial way. In the general situation one inserts

$$G(y)=G(0)-\sum_{k=1}^{\infty}\frac{qC_k}{k(1-pe^{-2k\mu})}y^k \quad (3.19)$$

in (3.17). One finds the recursion relations

$$C_{k-1}+2\sinh\nu\, C_k-C_{k+1}=\frac{1-e^{-2k\mu}}{\eta k(1-pe^{-2k\mu})}C_k \quad (3.20)$$

for $k=1,2,3,\dots$, and where $C_0\equiv 1$. The terms proportional to y^0 give a simple expression for Ω_r

$$\Omega_r=q\eta(C_1-e^{\nu}) \ . \quad (3.21)$$

Assuming that one knows how to solve the recurrence relations (3.20), one only needs the expression for the free energy. The ensemble average of Eq. (2.15) is

$$-\beta F=\langle \ln 2\cosh\beta\mathcal{J}_2\rangle+\tfrac{1}{2}\ln\tfrac{1}{2}\sinh 2\beta\mathcal{J}_1+\int_0^{\pi}\frac{d\theta}{2\pi}\Omega(\theta) \equiv-\beta F_{\text{pure}}-\beta F_r \ , \quad (3.22)$$

where

$$-\beta F_{\text{pure}}=\tfrac{1}{2}\ln 2\sinh 2K_{2,0}+\int_0^{\pi}\frac{d\theta}{2\pi}\mu(\theta) \quad (3.23a)$$

is the Onsager expression for the pure system and where

$$-\beta F_r=\tfrac{1}{2}q\int_0^{\infty}e^{-x}dx\,\ln\frac{\sinh(2K_1)}{\sinh(2K_1+2K_r x)} +q\int_0^{\pi}\frac{d\theta}{2\pi}\eta(\theta)[C_1(\theta)-\sinh\nu(\theta)] \equiv-\beta F_{r1}-\beta F_{r2} \ . \quad (3.23b)$$

The second term occurring in (3.24) is exactly the angular average of the expression for the random part of the free energy for the closely related random-field Ising chain, see Eq. (4.12) of Ref. 10.

In the case that ν and η have opposite signs Eq. (3.20) can be solved in terms of an infinite continued fraction

$$C_1=1/(C_0/C_1)=\frac{1}{\rho_1+}\,\frac{1}{\rho_2+}\,\frac{1}{\rho_3+}\,\frac{1}{\rho_4+}\cdots \ , \quad (3.24a)$$

where

$$\rho_k=-2\sinh\nu+\frac{1-e^{-2k\mu}}{\eta k(1-pe^{-2k\mu})} \ . \quad (3.24b)$$

For an algorithm which can be used numerically in cases where ν and η have the same sign, see NL.[10] An analytical method for $\nu\to+\infty$, $\eta>0$ is described in the next section.

C. Critical behavior of the McCoy-Wu model

The main result of the detailed study by McCoy and Wu[5] was that the free energy has an essential singularity at the ferromagnetic phase transition. This showed explicitly that the usual power-law critical behavior of ordered systems may be suppressed by disorder.

In the present section we give an elegant derivation of this singular behavior, starting from Eqs. (3.20) and (3.24). It is caused by the behavior at small θ. According to Eq. (3.15) the parameter ν goes to plus infinity for $\theta\to 0$. The leading behavior of C_k is obtained from a singular perturbation expansion, introduced in Ref. 10. One defines

$$C_k=e^{k\nu}a_k \quad (3.25)$$

and drops all terms of relative order $e^{-2\nu}\sim\theta^2$ from (3.20). One gets

$$a_{k+1}=\Sigma_k a_k+e^{-2\nu}a_{k-1} \quad (3.26)$$

with

$$\Sigma_k=1-\frac{1-e^{-2k\mu}}{4K_r k(1-pe^{-2k\mu})} \ . \quad (3.27)$$

The first step is to consider these equations for all complex k and to define $a(k)=a_k$ and $\Sigma(k)=\Sigma_k$. Then the solution of (3.26) with $e^{-2\nu}=0$ is

$$a_0(s)=C\prod_{n\geq 0}\Sigma(n)/\Sigma(s+n) \ . \quad (3.28)$$

However, the term $e^{-2\nu}a(s-1)$ cannot be omitted if $s\equiv s_0$ is such that $\Sigma(s_0)=0$. Since the normalization is defined by $C_0=a_0=1$ the problem shows up for the first time when, for $\theta=0$,

$$\Sigma(0)=1-\frac{\mu}{2qK_r}=0 \ . \quad (3.29)$$

Here μ is needed for $\theta=0$. From (3.15) it follows that $\mu=2|\beta\mathcal{J}_1-K_1|$ at $\theta=0$. [Note that in the present section K_1 denotes the dual of the coupling $\mathcal{J}_2$, see (3.7)]. For our situation with $K_r>0$ the relevant sign leads to

$$K_1+qK_r\equiv\langle K_{1,x}\rangle=\beta\mathcal{J}_1 \ . \quad (3.30)$$

This condition is just the average of the Onsager condition for the critical temperature; it has a simple form for lattices with 1D type of disorder.[5,15]

In order to proceed with the solution of (3.26) we define

$$b(s)=a(s)C^{-1}e^{2\nu s}\prod_{n\geq 1}\Sigma(n)/\Sigma(n+s) \quad (3.31)$$

and substitute (3.28) in the lhs, because this does not become singular in the region of interest. It is found that $b(s)$ satisfies

$$b(s)+b(s-1)=L(s)=e^{2vs}\frac{\Sigma(0)}{\Sigma(s)}\prod_{n\geq 1}\frac{\Sigma(n)^2}{\Sigma(n+s)} \quad (3.32)$$

and its solution is

$$b(s)=\int_{0<\mathrm{Re}t<1}\frac{dt}{2i\sin\pi t}L(s+t) . \quad (3.33)$$

This quantity is needed for $s=0$ (in order to fix the normalization) and for $s=1$. The leading contributions come from the pole at $t=0$ and from the zeroes s_j of Σ. The latter ones are irrelevant for $b(s=1)$. We end up with the following expression for the second term in (3.23b)

$$-\beta F_{r2}\simeq -qK_r+2qK_r\int_0^{\pi}\frac{d\theta}{\pi}\frac{\Sigma(0)}{1+\sum_j \frac{\pi}{\sin\pi s_j}\frac{e^{2vs_j}\Sigma(0)}{\Sigma'(s_j)}\prod_{n\geq 1}\frac{\Sigma(n)^2}{\Sigma(n+s_j)^2}} , \quad (3.34)$$

where the prime denotes differentiation. It will turn out that zeroes of $\Sigma(n+s)$ can be neglected.

To leading order the variable θ only occurs in $v\sim\ln 1/\theta$. It can be checked that the zeroes s_j $(-\infty<j<\infty)$ occur in complex conjugate pairs $(s_j=s_{-j}^*)$ and that only $s_0=s_0^*\sim T-T_c$ may have a positive real part. The real part of the other s_j is bounded by a negative constant, also for $T=T_c$. Hence only the $j=0$ term determines the leading ferromagnetic singularity in (3.34). In later sections we will consider situations where all terms are relevant.

For studying the leading critical behavior of (3.34) we insert

$$\mu(\theta=0)\equiv 2qK_r(1-\tau) \quad (3.35a)$$

where

$$\tau=(T-T_c)(qK_r-T_c^2)^{-1}\times(\mathcal{J}_1+\mathcal{J}_2/\sinh 2\beta\mathcal{J}_2)+O((T-T_c)^2) \quad (3.35b)$$

is a temperature variable. It is then found that $\Sigma(s_0)=0$ for

$$s_0=\frac{-\tau}{2K_r(1+p)}\left[1+\frac{4\tau(q^2+3p)}{3(2p+q)^2}+O(\tau^2)\right] . \quad (3.36)$$

We further need

$$\frac{\pi}{\sin\pi s_0}\frac{\Sigma(0)}{\Sigma'(s_0)}=\frac{\Sigma(s_0-s_0)}{s_0\Sigma'(s_0)}=-1+\tfrac{1}{2}s_0\frac{\Sigma''(s_0)}{\Sigma'(s_0)}$$
$$=-1+\tfrac{1}{2}s_0\frac{\Sigma''(0)}{\Sigma'(0)}$$
$$\equiv-(1+2\alpha_1 s_0) , \quad (3.37)$$

where equality signs hold up to order s_0^2 and where

$$\alpha_1=K_r(2q^2/3+4p)/(q+2p) . \quad (3.38)$$

We also introduce α_2 by

$$\prod_{n\geq 1}\frac{\Sigma(n)^2}{\Sigma(n+s_0)^2}=1+2\alpha_2 s_0+O(s_0^2) \quad (3.39)$$

so that

$$\alpha_2=-\sum_{n=1}^{\infty}\Sigma'(n)/\Sigma(n) . \quad (3.40)$$

With these definitions the $j=0$ contribution to (3.34) may be written, to leading order in s_0,

$$-\beta F_{r2}=-qK_r+2qK_r\int_0^{\pi}\frac{d\theta}{\pi}\frac{-s_0\Sigma'(0)}{1-e^{(v+\alpha_1+\alpha_2)2s_0}} . \quad (3.41)$$

A change of variables gives

$$-\beta F_{r2}=-qK_r+\frac{4}{\pi}qK_r^2(1+p)\sinh 2qK_r\times\sinh 2K_2\, e^{\alpha_1+\alpha_2}\tilde{I}(\delta) , \quad (3.42)$$

where

$$\delta\equiv-2s_0 \quad (3.43)$$
$$=(T-T_c)[(1-p^2)K_r^2T_c^2]^{-1}(\mathcal{J}_1+\mathcal{J}_2/\sinh 2\beta\mathcal{J}_2)$$

is the deviation from the critical point. It is twice as large as the variable δ defined by MW [Ref. 5(c); see Eq. (4.19) of Chap. XIV]. This is useful for having a simple argument in the function $\tilde{I}$

$$\tilde{I}(\delta)=\int_0^{\varphi_0}d\varphi\frac{\delta}{1-\varphi^{\delta}} . \quad (3.44)$$

The upper limit φ_0 is small ($\varphi_0\ll 1$) but independent of s_0. The nonanalytical behavior of the free energy is totally contained in $\tilde{I}(\delta)$. It was noted by McCoy and Wu[5] that $\tilde{I}$ has the same singularities as

$$I(\delta)=\int_0^1 d\varphi\left[\frac{\delta}{1-\varphi^{\delta}}+\frac{1}{\ln\varphi}\right]$$
$$=\delta+I(-\delta)$$
$$=\tfrac{1}{2}\delta+\sum_{n=1}^{\infty}B_{2n}\frac{(\delta)^{2n}}{2n}$$
$$=\ln\delta-\psi(1/\delta) , \quad (3.45)$$

where B_{2n} are Bernoulli numbers. So all derivatives of the free energy (3.42) are finite at T_c. Nevertheless the free energy is nonanalytic at $T-T_c$ or $s_0=0$. This can be seen, for instance, from the fact that the series (3.45) is divergent for all $\delta\neq 0$.

Our result for the free energy follows from (3.22), (3.23), and (3.42). It will not confirm the expression for the specific heat derived by McCoy and Wu.[5] We have a prefactor e^{α_1,α_2} arising from the correction terms in (3.37) and (3.39). Nevertheless, both results show an essential singularity in the free energy at the ferromagnetic phase transition.

D. Relation to the results of McCoy and Wu

From (3.22), (3.23), and (3.42), the singular part of the free energy is

$$-\beta F_{\text{sing}} = \frac{4}{\pi}(1-p^2)K_r^2 \sinh 2qK_r \times \sinh 2K_2 e^{\alpha_1+\alpha_2} I(\delta) . \tag{3.46}$$

We can compare it with the prediction for the specific heat, derived by McCoy and Wu in Chap. XIV of their book on the 2D Ising model.[5] These authors study the case of no dilution of randomness ($p=0$) and define parameters

$$\begin{aligned} &a = -2\sin\theta/c, \quad b = (1/z_1 - z_1)/c , \\ &c = 1/z_1 + z_1 + 2\cos\theta , \\ &N = (4K_r)^{-1}, \quad B = e^{-\mu} \end{aligned} \tag{3.47}$$

and variables

$$\lambda_n = z_2(n)^2, \quad x_n = ac\lambda_n/S_n \tag{3.48}$$

with S_n defined in (3.1). The recursion (3.2) for S_n leads to the MW recursion (2.17) for x_n. MW define a distribution function $X(\eta)$, which is a transform of the stationary distribution of the x_n. This function $X(\eta)$ is closely related to our function $G(y)$. We show this by noting that, from the definition (3.16), (3.15), and (3.3), it follows that

$$G(y) = G(0) + \lim_{n\to\infty} \langle \ln(1 - yY_n) \rangle , \tag{3.49a}$$

where

$$G(0) = \lim_{n\to\infty} \langle \ln[S_n + z_0 c(be^{\mu} - z_0)] \rangle \tag{3.49b}$$

and

$$Y_n = e^{-v} \frac{-S_n + z_0 c(z_0 - be^{-\mu})}{S_n + z_0 c(-z_0 + be^{\mu})} . \tag{3.49c}$$

Here $z_0 \equiv \tanh\beta\mathcal{J}_{2,0} \equiv e^{-2K_1}$ is the maximal value of z_2, cf. Eqs. (3.6) and (3.7). From (3.49a) the stationary distribution function of Y_n can be obtained as

$$\begin{aligned} H(y) &= \lim_{n\to\infty} \text{Prob}(Y_n < y) \\ &= 1 - \frac{1}{\pi}\text{Im}G(1/y - i0) . \end{aligned} \tag{3.50}$$

The MW function $X(\eta)$ is related to H as

$$X(\eta) = \frac{d}{d\eta} H(e^{2\mu+v}\eta) . \tag{3.51}$$

It can be verified that the range of the Y_n is $0 \le Y_n \le e^{v}$. Consequently, $X(\eta)$ vanishes unless $0 < \eta < B^2 = \exp(-2\mu)$, in agreement with MW. From Eq. (3.17) we find an exact equation for H:

$$H(ye^{2\mu}) - H(y) = -\eta(y + e^{-v})(y - e^{v}) \times [H'(y) - pe^{2\mu}H'(ye^{2\mu})] . \tag{3.52}$$

[Our parameter η, defined in (3.15), should not be confused with the MW variable η.] In the limit $p \to 0$ Eq. (3.52) reduces to the MW equation (3.24) for $X(\eta)$.

McCoy and Wu assume that the distribution of disorder has a narrow width, viz. $N = (4K_r)^{-1} \gg 1$. They derive the singular behavior of the specific heat near the critical point by expanding the nonlocal term $H[y\exp(2\mu)]$ of (3.52) in a certain way. This is done in the region of small angles θ, where $\mu \approx 2K_r$ is also small. Instead of (3.52), a first-order differential equation is obtained, which is solved by quadratures. The final MW result involves the singular part of the specific heat in the limit $N \to \infty$, $(T - T_c)N^2 = $fixed. Note that our variable $\delta \equiv -2s_0$, defined in (3.43), remains fixed in this limit. We write the MW result as

$$C_{\text{sing}} = \frac{16\beta^2}{\pi}(\mathcal{J}_1 + \mathcal{J}_{2,0}/\sinh 2\beta\mathcal{J}_{2,0})^2 \times (\sinh 2\beta\mathcal{J}_1)^{-1} I''(\delta) . \tag{3.53}$$

It follows from Eq. (4.40) of MW, after replacing the integral $\tilde{I}(\delta)$ by $I(\delta)$ (see also previous section) and after inserting

$$(T_c)_{\text{random}} = (T_c)_{\text{pure}} + O(1/N)$$

in the prefactor. Note that (3.53) is finite in the limit under consideration. For comparing Eq. (3.53) with our Eq. (3.46) we set $p=0$ and also take $K_r \to 0$. In this limit the parameter α_1, defined in (3.38), vanishes. From (3.40) we calculate

$$\alpha_2 \simeq -\sum_{n=1}^{N} \frac{\Sigma'(n)}{\Sigma(n)} - \int_N^{\infty} dn \frac{\Sigma'(n)}{\Sigma(n)} , \tag{3.54}$$

where we choose $N \gg 1$, $K_r N \ll 1$ and use that $\mu \approx 2qK_r$ in the region of interest. This leads to

$$\alpha_2 = \ln[2(1+p)K_r] - \gamma_e \quad (K_r \to 0) , \tag{3.55}$$

where Euler's constant γ_e appears as the finite part of the logarithmically divergent sum in (3.54). Inserting this in (3.46) and calculating the singular part of the specific heat we find the exact result, for any value of p,

$$C_{\text{sing}} = \frac{16\beta^2}{\pi}(\mathcal{J}_1 + \mathcal{J}_{2,0}/\sinh 2\beta\mathcal{J}_{2,0})^2 \times (\sinh 2\beta\mathcal{J}_1)^{-1} I''(\delta) e^{-\gamma_e} \tag{3.56}$$

in the MW limit $K_r \to 0$, $(T - T_c)/K_r^2$ fixed. Note that our exact result differs from the MW prediction by a factor $\exp(-\gamma_e)$. It has probably disappeared from the MW calculations because of going to the continuum limit in an analogon of (3.52). MW suggested an improvement of their method at the end of Sec. XIV. Indeed, it is to be expected that the factor $\exp(-\gamma_e)$ will be approximated the better, the more derivatives are taken into account in their analysis. Note that our analysis, on the other hand, solves the problem exactly.

We conclude that, in the limit of narrow distributions, we find the MW result up to a prefactor. For general power-law distributions, the singularity still involves the function $I(\delta)$; its prefactor is more complicated, but still explicit. This confirms the claim of McCoy[5(d)] that the singular behavior of the free energy will be proportional to $I(\delta)$ for general distributions of disorder.

IV. EXACTLY SOLUBLE MODEL WITH TEMPERATURE-INDEPENDENT DISTRIBUTION OF DISORDER

In the preceding section we have considered the singular behavior of the free energy in the original McCoy-Wu model. In the present section we shall consider a model on a square lattice where the vertical bonds are random, instead of the horizontal ones. In doing so we will find out that for this case a temperature-independent, diluted exponential distribution of vertical couplings allows for an exact solution. The reason is that we are considering the dual of the problem in Sec. II B. There it was found that the dual of the horizontal couplings is distributed exponentially; this corresponds to the vertical bonds in the present case.

In terms of the notation of Sec. III A we shall consider the case of sure couplings $\mathcal{J}_2$ $[\rho_2(y)=\delta(y-\mathcal{J}_2)]$. The couplings $\mathcal{J}_1$ decompose as

$$\mathcal{J}_1 \equiv \mathcal{J}_{1,x} = \mathcal{J}_{1,0} + \mathcal{J}_r x \tag{4.1}$$

with x distributed by

$$\rho(x) = p\delta(x) + qe^{-x} \quad (x \geq 0)$$
$$= 0 \quad (x < 0) \; . \tag{4.2}$$

We shall take $\mathcal{J}_{1,0}$ positive and $\mathcal{J}_r$ either positive leading to a ferromagnetic phase transition or negative leading to frustration and therefore causing a transition at $T=0$. We denote $\beta\mathcal{J}_{1,0}$ by K_1, $\beta\mathcal{J}_r$ by K_r, and $\beta\mathcal{J}_2$ by K_2 everywhere in Sec. IV.

There are two ways to calculate the free energy for the present model. The simplest way is to use a general connection between the partition sums on original and dual lattices[18]

$$Z = Z_1 2^{-NM} \prod_{\langle ij \rangle} \frac{e^{\beta \mathcal{J}_{ij}}}{\cosh\beta\mathcal{J}^*_{ij}} \; , \tag{4.3}$$

where $\langle ij \rangle$ denotes all different bonds and $\mathcal{J}^*_{ij}$ is the dual of $\mathcal{J}_{ij}$

$$e^{-2\beta\mathcal{J}^*_{ij}} = \tanh\beta\mathcal{J}_{ij} \; . \tag{4.4}$$

The partition sum Z_1 is the one which occurred in Sec. III B. From (4.3) one gets

$$-\beta F = -\beta F_{\rm MW} + \langle K_1 \rangle + K_2 - \ln\cosh K_1^* - \langle \ln 2\cosh K_2^* \rangle$$
$$= \int \frac{d\theta}{2\pi}(\mu + \Omega_r) + qK_r + \tfrac{1}{2}\ln 2\sinh 2K_2 \; , \tag{4.5}$$

where $-\beta F_{\rm MW}$ is given by (3.22).

This duality transformation is not perfectly satisfactory if couplings are negative. The second way to derive (4.5) is to start from the integral equation for $D(u)$; see Sec. III A. This by far more laborious method will be treated in detail in Sec. IV B. The physical content of Eq. (4.5) for the case of only ferromagnetic couplings or mixed ferro- and antiferromagnetic couplings will be discussed in Secs. IV C and IV D, respectively.

A. Derivation of the free energy from the integral equation

In this section we derive the exact solution of the integral equation (2.4) for the distribution (4.1). We follow the ideas of Sec. III B and write (3.4) and (3.5) as

$$D(u) = q\int_0^\infty e^{-x}dx[D(\Phi(u,x)) + \ln\Psi(u,x)] + pD(\Phi(u,0)) + p\ln\Psi(u,0) - \Omega \; , \tag{4.6}$$

where now

$$\Phi(u,x) = \{u[\tanh(K_1+K_rx) + \coth(K_1+K_rx) - 2\cos\theta] - 4\tanh^2K_2\sin^2\theta\}/\Psi(u,x) \; ,$$
$$\Psi(u,x) = \tanh^2K_2[\tanh(K_1+K_rx) + \coth(K_1+K_rx) + 2\cos\theta] - u \; . \tag{4.7}$$

Straightforward algebra shows that this integral equation may be reduced to the following differential-difference equation:

$$[1 - A(u)\partial_u][D(u) - pD(v) - p\ln w] = qD(v) + q\ln w + qK_r(-u\coth^2K_2 + 2\cos\theta) - \tilde{\Omega} - 2qK_r \; , \tag{4.8}$$

where $v = \Phi(u,0)$, $w = \Psi(u,0)$, and where

$$\tilde{\Omega} = \Omega + q\int_0^\infty e^{-x}dx \ln\frac{\sinh(2K_1+2K_rx)}{\sinh 2K_1} - 2qK_r \; ,$$
$$A(u) = K_r\coth^2K_2[(u - 2\tanh^2K_2\cos\theta)^2 - 4\tanh^4K_2] \; . \tag{4.9}$$

The appropriate change of variables now is

$$u = \tanh^2K_2(\tanh K_1 + \coth K_1 + 2\cos\theta) - \tanh K_2(\coth K_1 - \tanh K_1)f(y) \tag{4.10}$$

with

$$f(y) = (e^{\mu} - ye^{-\nu^*})(1 - ye^{\mu-\nu^*})$$

and

$$G(y) = D(u) + \ln(1 - ye^{\mu-\nu^*}) \; . \tag{4.11}$$

The parameters μ and ν^* are defined by

$$\cosh\mu = (\cosh 2K_2\cosh 2K_1 - \sinh 2K_1\cos\theta)/\sinh 2K_2 \; ,$$
$$\sinh\nu^* = (\cosh 2K_1^*\cos\theta - \sinh 2K_1^*\cosh 2K_2^*)/\sin\theta \; , \tag{4.12a}$$
$$\cosh\nu^* = \sinh\mu\,\sinh 2K_2^*/\sin\theta \; .$$

We recall that

$$e^{-2K_1^*}=\tanh K_2,\quad e^{-2K_2^*}=\tanh K_1\ . \tag{4.12b}$$

Lengthy calculations yield a simple equation for $G(y)$,

$$[1+\eta(e^{\nu}+y)(e^{-\nu}-y)\partial_y][G(y)-pG(ye^{-2\mu}]$$
$$=qG(ye^{-2\mu})-q\eta(e^{\nu}+y)-\Omega_r\ , \tag{4.13}$$

where ν and η are defined by

$$\sinh\nu=(\cosh 2K_1\cos\theta-\sinh 2K_1\cosh 2K_2)/\sin\theta\ ,$$
$$\cosh\nu=\sinh\mu\,\sinh 2K_2/\sin\theta\ , \tag{4.14}$$
$$\eta=2K_r/\cosh\nu\ .$$

The relations for ν are dual to the ones in (4.12a), and μ is self-dual. In the same way (4.13) is dual to (3.17). The solution to (4.13) is

$$G(y)=G(0)-\sum_{k=1}^{\infty}\frac{qC_k}{k(1-pe^{-2k\mu})}y^k \tag{4.15}$$

and results in the recursion relations

$$C_{k-1}+2\sinh\nu\,C_k-C_{k+1}=\frac{1-e^{-2k\mu}}{\eta k(1-pe^{-2k\mu})}C_k \tag{4.16}$$

with $C_0\equiv 1$. The relation to the free energy follows from (2.15), (4.9), and

$$\tilde{\Omega}\equiv\Omega_{\text{pure}}+\Omega_r+2qK_r\ . \tag{4.17}$$

The result is

$$-\beta F=-\beta F_{\text{pure}}-\beta F_r\ , \tag{4.18}$$

where

$$-\beta F_{\text{pure}}=\tfrac{1}{2}\ln 2\sinh 2K_2+\int_0^{\pi}\frac{d\theta}{2\pi}\mu(\theta) \tag{4.19}$$

and

$$-\beta F_r=q\int_0^{\pi}\frac{d\theta}{2\pi}\eta(\theta)[C_1(\theta)-\sinh\nu(\theta)]\ . \tag{4.20}$$

These equations are equivalent to Eq. (4.5), because $\Omega_r=q\eta(C_1-e^{\nu})$. In conclusion, we have verified (4.5) by derivation from Eq. (3.4). The method used here did not depend on the sign of K_r and therefore is valid both for $K_r>0$ and $K_r<0$.

B. Ferromagnetic disorder

In this section we consider the case $\mathcal{J}_{1,0}>0$, $\mathcal{J}_r>0$, $\mathcal{J}_2>0$, so that all couplings are ferromagnetic. Consequently one expects the very same exponential singularity near the phase transition as in the McCoy-Wu model of Sec. III B. Indeed, the description of the present case is fully analogous to the problem analyzed in Sec. III B. The final result (3.42) holds for $-\beta F_r$ defined in (4.20). We conclude that the McCoy-Wu conjecture that their nonanalytic behavior near T_c be valid for broad distributions of disorder, is correct in the situation under consideration. Here we only note that the phase transition occurs at temperatures such that

$$K_1+qK_r\equiv\langle\beta\mathcal{J}_{1,x}\rangle=K_1^*\ . \tag{4.21}$$

Another point of interest is the behavior at small temperatures. From Eq. (4.14) one sees that $\nu\to-\infty$ as $T\to 0$. Hence one can use the simple continued fraction expansion (3.24), and the first approximant $C_1\simeq 1/\rho_1$ gives

$$-\beta F=K_1+qK_r+K_2+[p+q/(4K_r+1)]e^{-4K_1-4K_2} \tag{4.22}$$

which is just the average

$$-\beta F=\langle\beta\mathcal{J}_{1,x}\rangle+\beta\mathcal{J}_2+\langle e^{-4\beta\mathcal{J}_{1,x}-4\beta\mathcal{J}_2}\rangle \tag{4.23}$$

with respect to the distribution of disorder (4.1).

C. Square lattice with frustrated layered bonds: Linear specific heat

Since we have an exact solution for a temperature-independent distribution of disorder we can study the interesting low-temperature behavior of frustrated two-dimensional lattices. We now consider the situation where $\mathcal{J}_{10}>0$, $\mathcal{J}_r<0$, $\mathcal{J}_2>0$, so that part of the vertical bonds are antiferromagnetic. It was already noted by Shankar and Murthy[17] that frustrated loops are present if one row of vertical bonds has ferromagnetic bounds, while an adjacent row has antiferromagnetic bonds. Therefore the ferromagnetic phase is destroyed, and there is only a phase transition at $T=0$.

The behavior for low T can be studied with the tools of preceding sections. From Eq. (4.4) it is clear that $\nu\to-\infty$ for $T\to 0$, but $\eta<0$ has the same sign. Hence the continued fraction (3.24) does not converge, and one needs the singular perturbation expression (3.34) for $-\beta F_r$ defined in (4.20). There are some minor differences, however. Because $\nu\to-\infty$, contrary to $\nu\to+\infty$ in Sec. III C, the equivalent of (3.25) is $C_k=(-1)^k e^{-k\nu}a_k$. This introduces

$$\Sigma(s)=1+\frac{1-e^{-2\mu s}}{4K_r s(1-pe^{-2\mu s})}\ . \tag{4.24}$$

From (4.14) it follows that $\mu=2K_1$ for $T\to 0$. Hence $\Sigma(s)$ may be written

$$\Sigma(s)=\frac{1+4K_r s}{4K_r s(1-pe^{-4K_1 s})}[1-f(\beta s)]\ , \tag{4.25}$$

where

$$f(s)=e^{-4\mathcal{J}_{10}s}(1+4p\mathcal{J}_r s)/(1+4\mathcal{J}_r s) \tag{4.26}$$

is nothing but the average

$$f(s)=\int\rho_1(\mathcal{J}_1)d\mathcal{J}_1 e^{-4\mathcal{J}_1 s}=\langle e^{-4\mathcal{J}_1 s}\rangle \tag{4.27}$$

with respect to the distribution of disorder (4.1) and (4.2). The special role of $f(s)$ was noted by Derrida and Hilhorst.[9] Nieuwenhuizen and Luck[10] observed that the low-temperature behavior of some frustrated random-field Ising chains is governed by all (complex) solutions of the equation $f(s)=1$. The very same aspect shows up in the present two-dimensional case. For a related problem where the relation $f(s)=1$ determines quantities of physical interest, see de Calan *et al.*[21]

From (4.25) it is seen that the zeroes of $\Sigma(s)$ occur at $s_j=T\alpha_j$ where α_j $(-\infty<j<\infty)$ is a complex solution of the equation $f(\alpha)=1$. These solutions occur in complex conjugate pairs, $\alpha_j=\alpha^*_{-j}$, except for α_0, which is real. We obtain from the analogon of (3.34)

$$-F_r=q\mathcal{J}_r -2(\mathcal{J}_{10}+q\mathcal{J}_r)\times\int_0^\pi\frac{d\theta}{\pi}\left[1-\sum_j A_j(2\sin\theta)^{-2\alpha_jT}\right]^{-1}, \tag{4.28a}$$

where

$$A_j=\frac{\pi\alpha_jT}{\sin\pi\alpha_jT}B_j\prod_{n\geq1}\frac{\Sigma^2(n)}{\Sigma^2(n+\alpha_jT)} \tag{4.28b}$$

depends on T also through $\Sigma(x)\simeq1-T/(4\mathcal{J}_rx)$ for the arguments needed, and involves a temperature-independent factor,

$$B_j=-(\mathcal{J}_{10}+q\mathcal{J}_r)\mathcal{J}_{10}^{-1}[1+q+(1+p)4\mathcal{J}_r\alpha_j+p(4\mathcal{J}_r\alpha_j)^2]^{-1}e^{4\mathcal{J}_2\alpha_j}. \tag{4.28c}$$

Expanding (4.28) in powers of temperature one finds for the free energy

$$F=F_0+TF_1+T^2F_2+\cdots \tag{4.29}$$

with

$$F_0=-\mathcal{J}_{10}-q\mathcal{J}_r-\mathcal{J}_2+2(\mathcal{J}_{10}+q\mathcal{J}_r)\left[1-\sum_j B_j\right]^{-1}. \tag{4.30}$$

The coefficient F_1 is proportional to

$$\int_0^\pi d\theta\ln(2\sin\theta)=0 \tag{4.31}$$

and vanishes. This implies that, in general, the zero-temperature entropy vanishes. Finally, F_2 is given by

$$F_2=4(\mathcal{J}_{10}+q\mathcal{J}_r)\gamma\frac{\pi^2}{6} \tag{4.32a}$$

with

$$\gamma=\sum_j B_j\alpha_j\left[\alpha_j+\frac{1}{4\mathcal{J}_r}\right]\left[1-\sum_j B_j\right]^{-2}+\left[\sum_j B_j\alpha_j\right]^2\left[1-\sum_j B_j\right]^{-3}. \tag{4.32b}$$

Here we note a surprising conspiracy of mathematics, yielding the familiar[10] factor $\pi^2/6$ in F_2, arising here from three different origins

$$\frac{d^2}{dx^2}\frac{\pi x}{\sin\pi x}\bigg|_{x=0}=\sum_{n=1}^\infty\frac{1}{n^2}=\frac{2}{\pi}\int_0^\pi d\theta[\ln(2\sin\theta)]^2=\frac{\pi^2}{6}. \tag{4.33}$$

From (4.29) it follows that the specific heat is linear at low temperatures

$$C=-2F_2T+O(T^2). \tag{4.34}$$

This linear behavior of the specific heat was already found in the frustrated Ising chain with exponential distributions of random fields[10] and also in the van Hemmen mean-field spin-glass model,[22] with a constant density of the random couplings ξ_j and η_j at $\xi_j=\eta_j=0$. Note that also in our model the density of couplings $\mathcal{J}_1(n)$ is constant at $\mathcal{J}_1(n)=0$. The question of what happens if this density is linear is the topic of current investigations.

In order to elucidate the structure of (4.29)–(4.32) we consider the limit of strong ferromagnetic coupling $\mathcal{J}_2\gg1$. The equation of $f(\alpha_j)=0$ determines the values of α_j. There are two situations: $\langle\mathcal{J}_{1x}\rangle=\mathcal{J}_{10}+q\mathcal{J}_r>0$ and $\mathcal{J}_{10}+q\mathcal{J}_r<0$. In the first case (weak disorder) all α_j have a negative real part, except for $\alpha_0>0$. Hence the leading behavior of (4.28) for $\mathcal{J}_2\to\infty$ leads to

$$F=-\mathcal{J}_{10}-q\mathcal{J}_r-\mathcal{J}_2-A_+\exp(-4\mathcal{J}_2\alpha_0) \tag{4.35}$$

with

$$A_+=-2(\mathcal{J}_{10}+q\mathcal{J}_r)(A_0e^{-4\mathcal{J}_2\alpha_0})^{-1}\int_0^\pi\frac{d\theta}{\pi}(2\sin\theta)^{2\alpha_0T}. \tag{4.36}$$

The first terms are equal to $-\langle\mathcal{J}_{1x}\rangle-\mathcal{J}_2$ and the correction is exponentially small. A similar behavior was found in the random-field Ising chain by Derrida and Hilhorst.[9] In the second case ($\langle\mathcal{J}_{1x}\rangle<0$, strong disorder), all roots α_j have a negative real part and α_0 still has the largest. Hence from (4.28) it follows in the limit $\mathcal{J}_2\to\infty$

$$F=\mathcal{J}_{10}+q\mathcal{J}_r-\mathcal{J}_2-A_-\exp[-4(\mathcal{J}_{10}+\mathcal{J}_2)|\alpha_0|] \tag{4.37}$$

with

$$A_-=2(\mathcal{J}_{10}+q\mathcal{J}_r)A_0e^{-4(\mathcal{J}_{10}+\mathcal{J}_2)\alpha_0}\times\int_0^\pi\frac{d\theta}{\pi}(2\sin\theta)^{2|\alpha_0|T}. \tag{4.38}$$

Again, this is equal to $-|\langle\mathcal{J}_{1x}\rangle|-\mathcal{J}_2$ plus an exponentially small correction. Finally there remains the more subtle case $\langle\mathcal{J}_{1x}\rangle=\mathcal{J}_{10}+q\mathcal{J}_r\to0$. Then A_0 and α_0 in (4.28) approach unity, the other A_j going to zero. The result is

$$F=-\mathcal{J}_2-2\mathcal{J}_{10}^2\frac{1+p}{1-p}\int_0^\pi\frac{d\theta}{\pi}\frac{1}{2\mathcal{J}_{10}+2\mathcal{J}_2-T\ln(2\sin\theta)+(\alpha_1+\alpha_2)T} \tag{4.39}$$

with

$$\alpha_1 = -\frac{\Sigma''(0)}{\Sigma'(0)} ,$$
$$\alpha_2 = -T \sum_{n=1}^{\infty} \Sigma'(n) = \frac{T\pi^2}{24\mathcal{J}_r} \quad (4.40)$$

both proportional to T. Equation (4.33) behaves linearly for $\mathcal{J}_2 \to \infty$ but is also valid at finite $\mathcal{J}_2$ and small T. Equations (4.29)–(4.34) all behave like $F = F_0 + F_2 T^2 + O(T^3)$ for low T. In fact, they are closely related to results for the random-field Ising chain.[10]

In that study also resonances were found, where the zero-point entropy is finite, that is to say, the factor F_1 in (4.29) is nonvanishing. In the present model there is an infinity of values of θ for which the integrand in (4.28a) has a *finite* contribution proportional to T for low T. However, upon integration this vanishes: In the present two-dimensional model on a square lattice, there is no zero-point entropy, with or without disorder. The specific heat is linear in T, see (4.34), and (4.32), because of the presence of frustrated bonds.

V. EXACT SOLUTION FOR THE TRIANGULAR LATTICE

The Ising model on the triangular (and hexagonal) lattice was solved and studied by Houtappel.[23] In the frustrated antiferromagnetic case, he found that the critical temperature drops to zero as soon as two of the couplings become equal. Furthermore, in the case of three identical couplings, there is a finite-zero-temperature entropy (which vanishes when the couplings become unequal), but the specific heat has an exponential essential singularity, of the type $e^{-\Delta/T}$, indicating that there is a gap between the ground states and the first excited states of the system. It is interesting to study how these results translate in the case of the frustrated MW model on a triangular lattice.

In Sec. II B we have derived the expression for the free energy of hexagonal and triangular lattices. This was done for arbitrary values of the layered random couplings. In the present section we investigate the question whether in the thermodynamic limit exact solutions are present for appropriate distributions of disorder for these lattices.

It turns out that such solutions are only present on the triangular lattice, if the couplings $\mathcal{J}_3$ have diluted exponential distributions

$$\mathcal{J}_3(n) \equiv \mathcal{J}_{30} + \mathcal{J}_r x(n) \equiv \mathcal{J}_{3x(n)} \quad (5.1)$$

with all $x(n)$ distributed by Eq. (3.8). The related hexagonal system with random couplings $K_3(n)$ has exact solutions for temperature-dependent distribution of disorder. This is totally the same as in the McCoy-Wu case of the soluble systems on the square lattice, see Sec. III.

In order to derive the solution for the triangular lattice we follow the method of Sec. III. We define $\tanh\beta\mathcal{J}_{30} \equiv \exp(-2L_3)$ and introduce the function

$$D_N(u) = \langle \ln(R_N - u) \rangle \quad (5.2)$$

and use (2.35) to express it into D_{N-1}. The limit function D satisfies

$$D(u) = \int \rho(x)dx\, D(e^{-4L_r x} v(u)) + \ln[\cosh(2L_1+2L_2) - \cos\theta - u] - \Omega \quad (5.3)$$

with ρ defined in (3.8) and

$$v(u) = e^{-4L_3} \frac{[u\cosh(2L_1-2L_2) - \sin^2\theta]}{[\cosh(2L_1+2L_2) - \cos\theta - u]} \quad (5.4)$$

and Ω equal to the ensemble average of the $n = N \to \infty$ term of (2.37). Of course, the free energy of a given system is self-averaging, so we can safely calculate its average. The differential-difference equation related to (5.3) is

$$[1 - A(u)\partial_u][D(u) - pD(v) - \ln w] = qD(v) - \Omega , \quad (5.5)$$

where

$$A(u) = \frac{4L_r[u + \cos\theta - \cosh(2L_1+2L_2)][u\cosh(2L_1-2L_2) + u\cos\theta - \sin^2\theta]}{\sinh^2 2L_1 + \sinh^2 2L_2 + 2\sinh 2L_1 \sinh 2L_2 \cos\theta} ,$$
$$w = \cosh(2L_1+2L_2) - \cos\theta - u . \quad (5.6)$$

Again the main problem now is to find the adequate transformation of variables. The connections are

$$\cosh\mu = (C_1C_2C_3 + S_1S_2S_3 - S_3\cos\theta)/W ,$$
$$\sinh\nu = -(C_1C_2S_3 + S_1S_2C_3 - C_3\cos\theta)/\sin\theta ,$$
$$\eta = 2L_r/\cosh\nu , \quad (5.7)$$
$$W = (S_1^2 + S_2^2 + 2S_1S_2\cos\theta)^{1/2}$$

and

$$u = \cosh(2L_1+2L_2) - \cos\theta - e^{-2L_3} W(1 + ye^{-\nu})/(e^{-\mu} + ye^{\mu-\nu}) ,$$
$$D(u) = G(y) - \ln(1 + ye^{2\mu-\nu}) ,$$

where $C_j \equiv \cosh 2L_j$, $S_j \equiv \sinh 2L_j$. This transforms (5.5) into

$$[1 + \eta(e^{\nu}+y)(e^{-\nu}-y)\partial_y][G(y) - pG(ye^{-2\mu})] = qG(ye^{-2\mu}) - q\eta(e^{\nu}+y) - \Omega_r \quad (5.8)$$

with

$$\Omega_r = \Omega - \Omega_{\text{pure}} ,$$
$$\Omega_{\text{pure}} = \mu + \ln W - 2L_3 . \tag{5.9}$$

Equation (5.8) has exactly the same form as Eq. (3.17). The free energy follows from (2.39) as

$$F = F_{\text{pure}} + F_r ,$$
$$-\beta F_{\text{pure}} = -L_3 + \int_0^{\pi} \frac{d\theta}{2\pi} (\mu + \ln W) , \tag{5.10}$$
$$-\beta F_r = \int_0^{\pi} \frac{d\theta}{2\pi} \Omega_r$$

with Ω_r determined by (3.20) and (3.21).

Again there are at least two limits of interest: the behavior near the phase transition in the ferromagnetic case ($\mathcal{J}_j > 0$, $\mathcal{J}_r > 0$) and the low-T behavior if the phase transition occurs at $T=0$ due to frustration in the system (some or all couplings are negative). The calculations near the ferromagnetic transition are again exactly the same as in Sec. III C. The difference between the square and the triangular lattice only shows up in the precise definition of the parameters μ, ν, and η in terms of the original couplings, cf. (5.7) and (4.14). In the same way the frustrated square lattice behaves like the frustrated triangular lattice. In particular the linear specific heat (4.34) also occurs in the latter model. It seems of interest to investigate the various possibilities (there are four couplings $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_r$, each of which may be positive or negative) in more detail.

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