Strong mode locking in systems far from chaos

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Based on a simple model of a driven oscillator without inertia, we demonstrate that *entirely* nonchaotic systems in general may develop a rich mode-locking structure in an appropriate range of control parameters. We show that for this structure the scaling properties are similar to those for a quasiperiodic system near the transition of chaos. Nevertheless, the structure originates from a *different* type of transition. In particular, the analysis serves to resolve the origin of the "close-to-chaotic" behavior in overdamped charge-density-wave systems. Also, this work provides a natural basis for studies of overdamped spatially extended systems.

One of the puzzles in nonlinear dynamics that has received much attention recently is the empirical observation of a surprisingly rich mode-locking structure in systems that are far from chaotic.[1-4] These include systems as diverse as charge-density waves (CDW),[2] the Belousov-Zhabotinsky reaction,[3] and relaxation oscillators.[4] A study of the scaling properties in these systems often gives results that are remarkably similar to those of a quasiperiodic system at the transition to chaos. It was suggested that this behavior may be caused by an underlying integrate-and-fire phenomenon[5] that leads to a separation of the transition to complete mode locking and the transition to chaos.

In this paper we demonstrate that even completely nonchaotic systems with two frequencies in general may exhibit a strong mode-locked structure with scaling properties similar to those of a quasiperiodic system near the transition to chaos. We probe the origin of this behavior, and clarify the connection to the integrate-and-fire mechanism.[5] As an example, we apply our results to explain the strong mode-locking structure observed in overdamped CDW systems.[2]

We begin with a simple driven oscillator,

$$G\dot{\phi}+U(\phi)=E+A\sin t \ , \tag{1a}$$

where the field $U(\phi)$ is periodic and has the Fourier series

$$U(\phi)=\sum_{m=1}^{\infty} b_m \sin(m\phi) \ , \tag{1b}$$

with the constraint that the maximal value of $U(\phi)$ is 1. Since (1a) is a first-order differential equation, *no* chaotic motion can evolve. By the beating of the higher-order terms in (1b) with the applied ac drive, subharmonic steps are created where the rotation number $R=\langle\dot{\phi}\rangle$ takes on rational values ($\langle\ \rangle$ denotes time average). We emphasize that the pure sinusoidal form of $U(\phi)$ with only b_1 nonzero represents a nongeneric situation, where the subharmonic steps vanish.

Consider the skew field $U_4(\phi)$ defined by the four nonzero values, $b_1=0.754$, $b_2=-0.337$, $b_3=0.189$, and $b_4=-0.076$ (Fig. 1). If we further choose G and A to be of order 1, we obtain an R-E characteristic as shown in Fig. 2. *In contrast to the pure sinusoidal field, a multitude of distinct subharmonic steps appears.* In order to understand what causes this strong mode-locked structure to appear, we next consider the extreme skew field: the sawtooth field

$$U_s(\phi)=(\phi/\pi)-(2p+1) \quad \text{if } 2\pi p \le \phi < 2\pi(p+1) \ . \tag{2}$$

By a change of variable from ϕ to U_s ($\dot{U}_s=\dot{\phi}$), the system is mapped to a driven integrate-and-fire relaxation oscillator: The field U_s builds up from the lower threshold $T_0=-1$ to the firing threshold $T_1=1$, where U_s is reset to T_0. The mode-locking structure of relaxation oscillators is described in Ref. 5. A transition to complete mode locking is obtained along the line $E=A+1$. Moreover, the scaling properties at this critical line are *identical* to those for the quasiperiodic transition to chaos. For R irrational and $E<A+1$ a (vertical) gap is present in the Poincaré map, defined as the function

$$U_s(t=n/2\pi)\rightarrow U_s[t=(n+1)/2\pi] \ .$$

Let us now return to the nonchaotic system (1) with $U=U_4$. For the parameters in Fig. 2 and $R\simeq(\sqrt{5}-1)/2$ (the golden mean) we have traced out the Poincaré map $\phi_n\rightarrow\phi_{n+1}$, where $\phi_n=\phi(2\pi n)$ (Fig. 3). We find that a steep part appears in the Poincaré map, where ϕ_{n+1} changes rapidly with ϕ_n. To this end we note that the value of E is less than $A+1=3$. From the foregoing considerations, the steep part can be understood as reminiscent of the gap that occurs in a "sawtooth" system, where $U(\phi)$ resets discontinuously. Since the scaling structure is determined by the form of the Poincaré map, we expect this to be similar to the structure obtained for a sawtooth system near the transition to complete mode locking.

In general, one has to follow a very special curve in pa-

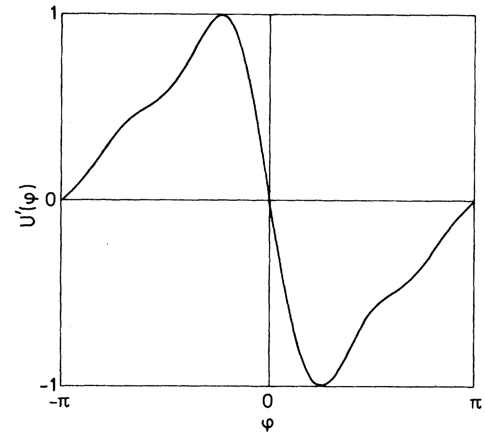

FIG. 1. The skew field $U_4(\phi)$ of the pinning potential used in the model (1).

rameter space to obtain the scaling properties at the transition to complete mode locking correctly. However, one might hope that a scaling region still survives, following a curve close by, e.g., a curve with many subharmonic steps. For instance, by adding up the total length $S(r)$ of steps larger than size r, this might give an estimate of the fractal dimension D, defined by the proportionality

$$N(r)\equiv[l-S(r)]/r \propto (r^{-1})^D , \tag{3}$$

where l is the total length of the E interval considered. Figure 4 shows the result along the R-E curve in Fig. 2 using 40 step sizes. As an intriguing result we find a scaling region where D has the value $D=0.87$ found for the quasiperiodic transition.[6] At smaller values of r, the

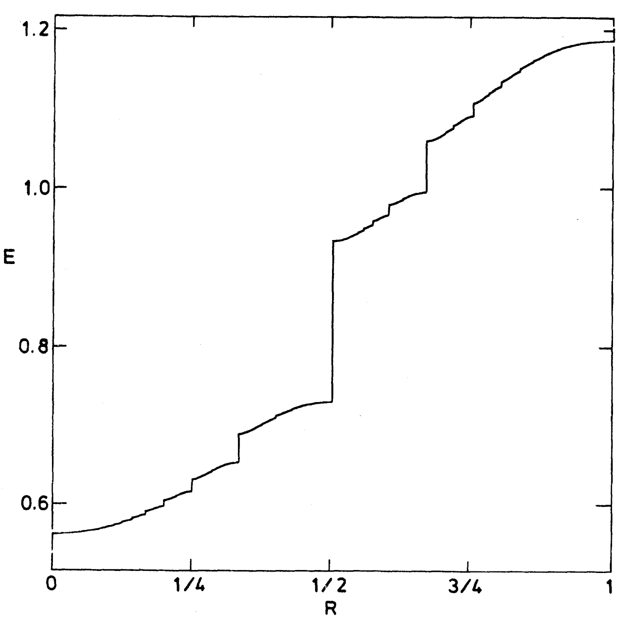

FIG. 2. R-E characteristic for Eqs. (1) with $G=1.42$ and $A=2$, obtained by use of analog computer (Electronics Associates, Inc. 680).

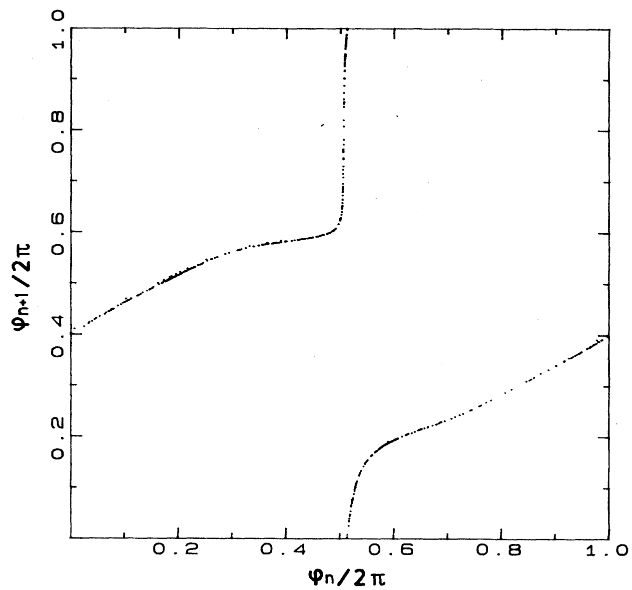

FIG. 3. Poincaré map obtained for the point on the characteristic in Fig. 2 at which R has the golden mean value.

finite slope of the steep part of the Poincaré map causes an eventual change of the slope in Fig. 4 to the true value $D=1$. The position of the "crossover" from a slope less than one to the slope one is dependent on the values of A and G: For smaller values of A or larger values of G, the steep part of the Poincaré map decreases in size, leading to a general decrease of the subharmonic step sizes, and to a crossover to $D=1$ at larger values of r.

As an application of the preceding general results, we consider the CDW system $NbSe_3$. When an external oscillatory field is applied, this system shows a rich mode-locking structure where the average phase velocity $\langle\dot{\phi}\rangle$

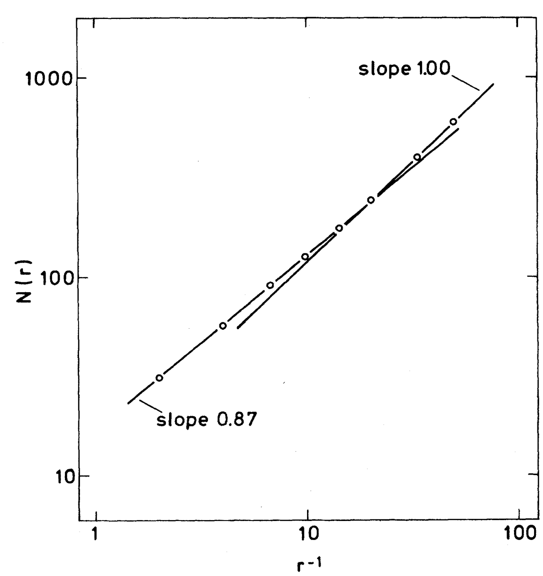

FIG. 4. Double-logarithmic plot of $N(r)$ vs r^{-1} [Eq. (3)]. The slope determines the fractal dimension D.

of the CDW locks to the external frequency ω with a rational ratio or rotational number R.[2] As regards scaling, the analysis (3) for the fractal dimension was carried out along an I-V curve, where I is the dc current and proportional to R, and V is the external dc voltage. The value $D=0.91\pm0.03$ was obtained by fitting.

In order to understand the basic physics that creates this structure, the normalized equation of motion for the phase ϕ of the CDW,

$$\ddot{\phi}+\gamma\dot{\phi}+U(\phi)=E+A\sin\omega t\ , \tag{4}$$

has been used. Here, γ is a damping factor, and $U(\phi)$ is the gradient of the periodic pinning potential; the amplitude of $U(\phi)$ is normalized to 1. The external dc and ac amplitudes E and A are measured in units of the pinning threshold E_T, and the time in units of

$$\tau_c\equiv\sqrt{m/2k_F eE_T}\ ,$$

where k_F is the Fermi wave vector, and m and e are the mass and charge of the CDW.

The mode-locking structure for Eq. (4) has been studied in great detail,[6,7] in particular when γ and ω are both of order unity, in which case a multitude of subharmonic mode-locked steps in the R-E characteristics appears. The mode-locking features, including the quasiperiodic transition to complete mode locking and chaos, have been successfully described in terms of "critical" circle maps with *zero* slope inflection points. However, one main problem that has plagued investigators in this field is the observation that the response of the CDW is strongly *overdamped,*[8] which means that the value of γ in Eq. (4) is very large and *not* of order unity. In the frequency range where the rich mode-locking structure is observed ($R\sim1$) the inertial term $\ddot{\phi}$ becomes *negligible,* since $\ddot{\phi}\sim\omega\dot{\phi}\ll\gamma\dot{\phi}$. This implies that no chaos will occur. Although this is in *agreement* with experiments, it has also raised the important question of why subharmonic steps are observed at all,[2] since they typically first become visible *close* to the onset of chaos.[6,7] As a consequence, this further calls into question whether the CDW system at all can be described in terms of "close-to-critical" circle maps which do show mode locking.

In order to resolve the foregoing questions, we observe that neglecting $\ddot{\phi}$ in Eq. (4), and renormalizing time to ω^{-1}, we obtain Eq. (1a) with a renormalized damping factor $G\equiv\gamma\omega$. To this end we notice that the overdamped response of the CDW was observed from measurements of the low-field frequency-dependent conductivity.[8] From the experiments one finds that the damping factor γ has a value of several hundred. Moreover, one finds that the frequency τ_c^{-1} is several GHz. Since the rich mode-locking structure is observed at frequencies in the MHz range, *we find that while γ is very large, $\gamma\omega$ is of order unity.* Thus, although the system of an overdamped CDW does not develop chaos, it is, as far as *scaling* properties are concerned, well described by close-to-critical circle maps. The strong mode locking is encountered where the renormalized damping factor is of order 1 and can be understood as a consequence of a skew pinning field.[9] The underlying integrate-and-fire mechanism gives rise to a steep part in the Poincaré map, in contrast to the zero slope inflection point that has been central in studies of circle-map systems with inertia as Eq. (4).[6,7] The result is that a rich mode-locked structure emerges with a fractal dimension D apparently smaller than one. However, in accordance with the observation that the overdamped CDW system is nonchaotic, a crossover should occur at small scales to the true value $D=1$.

We emphasize that the equivalency of an overdamped CDW system and a system described by a close-to-critical circle map constitutes a *basis* for understanding the more complicated dynamics, e.g., when spatial degrees of freedom have to be taken into account. In particular, this equivalency shows that a coupled system of circle maps is an excellent candidate for modeling an overdamped spatially extended CDW system. Recently, such a model[10] has indeed described several complicated dynamical features observed in $NbSe_3$, e.g., (i) the depinning of CDW has a critical behavior different from that of a simple tangential bifurcation,[11] and (ii) the mode-locking regions at almost all rational rotation numbers $R=P/Q$ have a lower resistance than the normal electronic background,[2] i.e., although there is a strong tendency to lock, the locking is not complete; moreover, the tendency to lock is weaker for larger denominators Q.

In conclusion, we have shown from a simple but generic model that an overdamped system without inertia in general develops a rich mode-locking structure in an appropriate range of parameters. In this range, the strength of the nonlinearity obtained from the Poincaré map and scaling analysis show that systems which *never* become chaotic, *still* can be similar to systems modeled by close-to-critical circle maps. In particular, when spatial degrees of freedom are taken into account, our description gives an explanation of why the behavior of the overdamped CDW in $NbSe_3$ in all respects is similar to that for coupled invertible circle maps, including the rich mode-locking structure, the critical behavior at the pinning threshold, and the observation that only the major mode-locking regions develop true steps.

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