

Spin-polarized Fermi liquids: Applications to liquid ^3He

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In this article, we discuss the thermodynamics and Landau theory of spin-polarized Fermi systems. Then, we review the “nearly metamagnetic” model of Bedell and Sanchez-Castro for the magnetic equation of state for polarized liquid ^3He . The predictions of this model, as well as the comparison to other existing models, are presented.

I. INTRODUCTION

With the introduction of the rapid melting technique for producing highly polarized ^3He proposed by Castaing and Nozières,¹ there has been a renewed interest in the theory of polarized Fermi liquids. One of the more interesting quantities of a polarized Fermi liquid is the magnetic equation of state (MEOS). From this a number of magnetic-field-dependent properties, e.g., depression of the melting curve, magnetostriction, sound velocity, etc., can be obtained. In this paper, we will be concerned with the “nearly metamagnetic” model of Bedell and Sanchez-Castro² and an extension of this MEOS to finite temperatures, $T \leq 100$ mK. A similar MEOS was obtained by Stringari³ and it has been used extensively by him to study the liquid-gas equilibrium of spin-polarized ^3He . Before we go on to describe this model and the steps leading to it, we will first give a brief qualitative description of the various models for the MEOS of ^3He .

In addition to the “nearly metamagnetic” model, there are at least three other models for the MEOS of liquid ^3He . The earliest of these models is the paramagnon or “nearly ferromagnetic” model which has been extended by Béal-Monod and Daniel⁴ to finite polarization. Another model in this class of “nearly . . .” models is the “nearly localized” model of Anderson and Brinkman⁵ which was extended by Vollhardt⁶ to finite polarization. While the physics of these models is quite different, they all assume the normal state of ^3He is a Fermi liquid with a well-defined Fermi surface. However, the most recent model for liquid ^3He , the quantum glass of Bouchaud and Lhuillier,⁷ does not even retain this feature of a normal Fermi liquid.

The MEOS of these models can be summarized by the magnetic field or polarization dependence of the susceptibility, X_n . In the “nearly ferromagnetic” model, the low-field susceptibility is large due to the nearness of a ferromagnetic instability. As a function of the polarization, $\Delta = m/n$, where m is the magnetization density and n is the density, the susceptibility decreases as Δ increases.⁴ Thus, in this model, a magnetic field drives one further away from the magnetic instability.

In the “nearly localized” model, the large zero-field susceptibility is due to the nearly localized quasiparticles that give rise to a large density of states. In the presence of a magnetic field, X_n increases with increasing Δ and diverges at a critical value $\Delta_c < 1$, e.g., $\Delta_c \approx 0.4$ at the melting pressure.⁶ This divergence can be viewed as a field-induced localization.

The next in this list of models is the “nearly metamagnetic” model.² For this model, the large zero-field susceptibility is an input; thus, the model does not offer any microscopic explanation for the large value of the susceptibility. In this model, we find that X_n initially increases with increasing Δ , reaching a maximum at $\Delta_{\text{max}} < 1$, after which it steadily drops to zero as Δ goes to one. Here the initial increase in X_n comes about as the system moves closer to a magnetic instability with increasing magnetic field. At the same time, the total density of states is decreasing and it eventually wins out, thus causing X_n to decrease. Here we can see the physical difference in the increase in X_n in the “nearly localized” and “nearly metamagnetic” model. Mainly, in the former there is a field-induced localization and in the latter we have the system approaching a field-induced magnetic transition, i.e., a metamagnetic phase.⁸

The last of the models we discuss is the quantum glass picture of Bouchaud and Lhuillier.⁷ In this model, the normal state of ^3He consists of atoms strongly coupled in pairs with parallel spin. Since these coupled spin pairs are bosonlike excitations, there is no Fermi surface. In this theory, the ground-state energy has only been calculated for $\Delta=0$ and 1; however, X_n at $\Delta=0$ has not been calculated. An extrapolation for the energy density, $\epsilon(n, m)$, was made by keeping terms up to fourth order in the magnetization and using the values $\epsilon(n, 0)$ and $\epsilon(n, 1)$ to fix the coefficients. This gives rise to a X_n that decreases with increasing Δ . Unfortunately, since X_n at zero field was not calculated and higher-order terms in m , in particular terms of the order of m^6 , were ignored, little can be said about the MEOS that follows from this quantum glass model.

The models described above, with the exception of the “nearly metamagnetic” one, are microscopic models.

The model of Bedell and Sanchez-Castro² is phenomenological in origin. Here, they made an ansatz for the polarization dependence of the Landau parameters and used a number of general results from thermodynamics and Fermi-liquid theory to extract these parameters.

This paper is organized as follows: In Sec. II, we develop a self-contained treatment of the thermodynamics of a spin-polarized Fermi system. In Sec. III, we review the Landau theory of a spin-polarized Fermi liquid⁹⁻¹⁴ and calculate various thermodynamical properties. Section IV discusses the kinetic equation and shows how to treat longitudinal and transverse excitations. Section V deals with collective modes in the polarized Fermi liquid. Section VI contains a calculation of the scattering amplitudes and a discussion of the different sum rules valid here. Section VII presents the "nearly metamagnetic" model and Section VIII deals with the predictions of that model.

II. THERMODYNAMICS

The thermodynamics of a paramagnetic Fermi liquid^{14,15} follows from the expression for the internal energy change

$$dE = T dS - P dV + \mu dN + H dM, \quad (2.1)$$

where E , T , S , P , V , μ , N , H , and M are the internal energy, the temperature, the entropy, the pressure, the volume, the chemical potential, the number of particles, the magnetic field, and the total magnetization, respectively. In many applications, it is more convenient to work with an intensive function, the internal energy density ϵ , defined by

$$E = V\epsilon(s, n, m), \quad (2.2)$$

where $s = S/V$, $n = N/V$, $m = M/V$ are the entropy density, the particle density and the magnetization density, respectively. To obtain the internal energy density differential $d\epsilon$, we use Eqs. (2.1) and (2.2) as follows:

$$\begin{aligned} T &= \left[\frac{\partial E}{\partial S} \right]_{V, N, M} = \left[\frac{\partial \epsilon}{\partial s} \right]_{n, m}, \\ \mu &= \left[\frac{\partial E}{\partial N} \right]_{S, V, M} = \left[\frac{\partial \epsilon}{\partial n} \right]_{s, m}, \\ H &= \left[\frac{\partial E}{\partial M} \right]_{S, V, N} = \left[\frac{\partial \epsilon}{\partial m} \right]_{s, n}, \\ P &= - \left[\frac{\partial E}{\partial V} \right]_{S, N, M} = -\epsilon - V \left[\left[\frac{\partial \epsilon}{\partial s} \right]_{n, m} \left[\frac{\partial s}{\partial V} \right]_{S, N, M} \right. \\ &\quad \left. + \left[\frac{\partial \epsilon}{\partial n} \right]_{s, m} \left[\frac{\partial n}{\partial V} \right]_{S, N, M} \right. \\ &\quad \left. + \left[\frac{\partial \epsilon}{\partial m} \right]_{s, n} \left[\frac{\partial m}{\partial V} \right]_{S, N, M} \right] \\ &= -\epsilon + Ts + \mu n + Hm. \end{aligned} \quad (2.3)$$

Thus, we find that

$$d\epsilon = T ds + \mu dn + H dm. \quad (2.4)$$

It also proves extremely useful to define other thermodynamical potentials that have different sets of natural variables. In particular, we will subsequently use the following:

$$F = E - TS, \quad (2.5a)$$

$$G = E + PV - TS, \quad (2.5b)$$

$$F_M = E - TS - MH. \quad (2.5c)$$

Their differentials, calculated using Eqs. (2.5) and (2.1), are

$$dF = -S dT - P dV + \mu dN + H dM, \quad (2.6a)$$

$$dG = -S dT + V dP + \mu dN + H dM, \quad (2.6b)$$

$$dF_M = -S dT - P dV + \mu dN - M dH. \quad (2.6c)$$

A class of thermodynamic identities, called Maxwell relations, is obtained from these differentials by requiring that the mixed second derivatives of a potential be independent of the order in which the differentiation is carried out. Three Maxwell relations we will subsequently use, obtained from Eqs. (2.6a), (2.6b), and (2.6c), respectively, are

$$- \left[\frac{\partial S}{\partial M} \right]_{T, V, N} = \left[\frac{\partial H}{\partial T} \right]_{V, N, M}, \quad (2.7a)$$

$$- \left[\frac{\partial S}{\partial P} \right]_{T, N, M} = \left[\frac{\partial V}{\partial T} \right]_{P, N, M}, \quad (2.7b)$$

$$\left[\frac{\partial S}{\partial H} \right]_{T, V, N} = \left[\frac{\partial M}{\partial T} \right]_{V, H, N}. \quad (2.7c)$$

In practice, one is interested in different thermodynamical differential coefficients, as the compressibility, the magnetic susceptibility, and the heat capacity. Different compressibilities can be measured by fixing any three of the variables (M, H, N, μ, T, S). Two of the most commonly used are

$$K_{M, N, T} = - \frac{1}{V} \left[\frac{\partial V}{\partial P} \right]_{M, N, T}, \quad (2.8a)$$

$$K_{H, N, T} = - \frac{1}{V} \left[\frac{\partial V}{\partial P} \right]_{H, N, T}. \quad (2.8b)$$

Similarly, different heat capacities per unit volume can be defined by fixing any three of the variables (P, V, M, H, N, μ). In this work, we will use

$$C_{V, H, N} = \frac{T}{V} \left[\frac{\partial S}{\partial T} \right]_{V, H, N}. \quad (2.9)$$

The magnetic susceptibility can be measured at fixed values of any three of the variables (P, V, N, μ, T, S). Here, we will consider

$$X_{T, n} = X_{T, n}(T, n, m) = \frac{1}{V} \left[\frac{\partial M}{\partial H} \right]_{N, V, T}. \quad (2.10)$$

We will adopt the following notation:

$$X_n = X_n(n, m) = X_{T,n}(T=0 \text{ K}, n, m)$$

and

$$X_n^0 = X_n(n, m=0).$$

In this work, we are concerned with two different applications of thermodynamics to the study of polarized ^3He . One involves being able to predict how various properties, e.g., the compressibility and the heat capacity, will change for small magnetizations from their zero-field values. The field dependence of these quantities can be expressed in terms of density or temperature derivatives of the zero-field magnetic susceptibility. Another application deals with the extension to low temperatures of a model for the magnetic behavior of the Fermi liquid at $T=0 \text{ K}$. This will require a detailed knowledge of the entropy of the system at low temperatures.

To obtain the field dependence of the compressibility, we need to obtain first the pressure change due to magnetization. At $T=0 \text{ K}$ and small magnetizations, the energy density can be expanded as¹

$$\begin{aligned} \epsilon(s=0, n, m) = \epsilon(s=0, n, m=0) &+ \frac{1}{2X_n^0} m^2 \\ &+ b(n)m^4 + \dots, \end{aligned} \quad (2.11)$$

where $X_n^0 = 3n/(2T_{\text{SF}})$ [T_{SF} is known as the spin-fluctuation temperature $T_{\text{SF}} = (1 + F_0^2)T_F$, $T_F = k_F^2/2m^*$] is the zero field and $T=0 \text{ K}$ magnetic susceptibility and $b(n)$ is a function of the density. Using Eqs. (2.3) and (2.11), we obtain that the pressure change at $T=0 \text{ K}$ and fixed density due to magnetizations is¹⁴

$$\begin{aligned} P(T=0, m, n) = P(T=0, m=0, n) \\ + \frac{1}{3} \left[\frac{dT_{\text{SF}}}{dn} \right] m^2 + \dots \end{aligned} \quad (2.12)$$

The spin fluctuation derivative, dT_{SF}/dn , is found experimentally to be negative for all pressures in ^3He (see Table I), e.g., the predicted pressure drop is 1.5 mbar at melting pressure in a 10-T field. We can now obtain the change in the compressibility $K_{M,N,T=0 \text{ K}}$ from Eqs. (2.8) and (2.12) by using^{2,16}

$$\frac{1}{K_{M,N,T=0 \text{ K}}} = -V \left[\frac{\partial P}{\partial V} \right]_{M,N,T=0 \text{ K}} = n \frac{\partial}{\partial n} P(T=0 \text{ K}, m, n) \Big|_{M,N} = n \frac{\partial P}{\partial n}(T=0, m, n) \Big|_m + m \frac{\partial P}{\partial m}(T=0, m, n) \Big|_n. \quad (2.13)$$

We obtain,

$$\frac{1}{K_{M,N,T=0 \text{ K}}} = \frac{1}{K_{M=0,N,T=0 \text{ K}}} + \frac{m^2}{3} \left[n \frac{d^2 T_{\text{SF}}}{dn^2} + 2 \frac{dT_{\text{SF}}}{dn} \right]. \quad (2.14)$$

Now, we will show how the field dependence of the heat capacity can be characterized using the temperature dependence of the zero-field susceptibility, $X_{T,n}(T, n, m=0)$, which is known to be of the form^{2,17}

$$X_{T,n}(T, n, m=0) = X_n^0 [1 - a(n)T^2]. \quad (2.15)$$

To analyze this effect, we use the Maxwell relation

$$\frac{\partial}{\partial H} (C_{V,H,N})_{T,V,N} = T \left[\frac{\partial^2 m}{\partial T^2} \right]_{V,H,N}, \quad (2.16)$$

which is obtained from Eqs. (2.7c) and (2.9) by differentiation with respect to T . From Eqs. (2.15) and (2.16), we obtain, after integration with respect to H , that

the field dependence of the specific heat is given by²

$$C_{V,H,N} = C_{V,H=0,N} - \frac{aT}{X_n^0} m^2. \quad (2.17)$$

Another important application of the Maxwell relations is the calculation of the leading temperature correction to a model for the $T=0 \text{ K}$ magnetic susceptibility at arbitrary magnetizations X_n . To obtain this correction, we use a Maxwell relation, Eq. (2.7a) and the low-temperature expression for the entropy density

$$s(n, m, T) = \gamma(n, m)T. \quad (2.18)$$

The coefficient γ , for ^3He , is given by²

$$\gamma = \frac{\pi^2}{3} [N_{\uparrow}(0) + N_{\downarrow}(0)],$$

where $N_{\uparrow}(0)[N_{\downarrow}(0)]$ is the density of states of the up (down) spins at their Fermi surface. Integration of Eq. (2.7a) with respect to T gives

TABLE I. Experimental inputs as well as the results for the coefficients at four different pressures.

| P (bars) | n ($1/\text{\AA}^3$) | dT_{SF}/dn ($K \text{\AA}^3$) | d^2T_{SF}/dn^2 ($K \text{\AA}^6$) | α ($1/K^2$) | b_0 | b_1 | c_1 | d_0 | d_1 | d_2 | d_3 | g_1 | g_3 |
|------------|--------------------------|--|--|----------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0.00 | 0.016 | -60.7 | 8745.0 | 3.20 | 0.21 | 0.06 | 0.17 | -0.38 | 0.32 | 0.64 | -1.58 | -0.95 | -0.05 |
| 21.00 | 0.022 | -27.6 | 3907.0 | 8.60 | 0.03 | -0.01 | 0.05 | -0.44 | 1.49 | 0.73 | -2.78 | -2.03 | 1.03 |
| 30.00 | 0.023 | -23.5 | 3313.0 | 9.16 | 0.024 | -0.01 | 0.04 | -0.47 | 1.58 | 0.76 | -2.86 | -2.06 | 1.06 |
| 34.36 | 0.024 | -21.8 | 3068.0 | 9.49 | 0.02 | -0.01 | 0.04 | -0.49 | 1.58 | 0.78 | -2.87 | -2.04 | 1.04 |

$$H(T, V, N, M) - H(T=0, V, N, M) = - \frac{\partial}{\partial m} \gamma(n, m) \Big|_n \frac{T^2}{2} . \quad (2.19)$$

This expression is now differentiated with respect to M at constant V, N, T and gives

$$\frac{1}{X_{T,n}} - \frac{1}{X_n} = - \frac{\partial^2}{\partial m^2} \gamma(n, m) \Big|_n \frac{T^2}{2} , \quad (2.20)$$

where we have used Eq. (2.10). Up to quadratic order in the temperature, the magnetic susceptibility $X_{T,n}$ is given by

$$X_{T,n} = X_n \left[1 + X_n \frac{\partial^2}{\partial m^2} \gamma(n, m) \Big|_n \frac{T^2}{2} \right] . \quad (2.21)$$

Similarly, we can obtain the leading temperature correction to the compressibility. Integration of Eq. (2.7b) with respect to T gives

$$V(T, P, N, M) - V(T=0, P, N, M) = - \frac{\partial}{\partial P} (\gamma V) \Big|_{T, N, M} \frac{T^2}{2} . \quad (2.22)$$

Differentiating this expression with respect to P at fixed T, N, M gives the desired correction,

$$K_{M, N, T} - K_{M, N, T=0} = \frac{1}{V} \frac{\partial^2 (\gamma V)}{\partial P^2} \Big|_{T, N, M} \frac{T^2}{2} . \quad (2.23)$$

III. LANDAU THEORY OF A SPIN-POLARIZED FERMION LIQUID

A spin-polarized Fermi liquid is described by a quasiparticle distribution function (density matrix) given by¹⁸

$$\delta \varepsilon = \varepsilon' - \varepsilon_0 = \sum_{\mathbf{p}, \alpha, \beta} [\varepsilon_{\mathbf{p}}^0]_{\alpha\beta} [\delta n_{\mathbf{p}}]_{\beta\alpha} + \frac{1}{2} \sum_{\mathbf{p}, \mathbf{p}', \alpha, \beta, \alpha', \beta'} [\tilde{f}_{\mathbf{p}, \mathbf{p}'}]_{\alpha\beta, \alpha'\beta'} [\delta n_{\mathbf{p}}]_{\beta\alpha} [\delta n_{\mathbf{p}'}]_{\beta'\alpha'} , \quad (3.3)$$

where

$$[\varepsilon_{\mathbf{p}}^0]_{\alpha\beta} = \varepsilon_{\mathbf{p}\alpha}^0 \delta_{\alpha\beta}$$

is the equilibrium quasiparticle energy matrix and $[\tilde{f}_{\mathbf{p}, \mathbf{p}'}]$ is the quasiparticle interaction function. The quasiparticle interaction function for finite polarizations has the form^{9,10}

$$[\tilde{f}_{\mathbf{p}, \mathbf{p}'}]_{\alpha\beta, \alpha'\beta'} = \phi_1(\mathbf{p}, \mathbf{p}') + \phi_2(\mathbf{p}, \mathbf{p}') \sigma_{z\alpha\beta} \delta_{\alpha'\beta'} + \phi_2'(\mathbf{p}, \mathbf{p}') \delta_{\alpha\beta} \sigma_{z\alpha'\beta'} + \phi_3(\mathbf{p}, \mathbf{p}') \sigma_{z\alpha\beta} \sigma_{z\alpha'\beta'} + \phi_4(\mathbf{p}, \mathbf{p}') (\sigma_{x\alpha\beta} \sigma_{x\alpha'\beta'} + \sigma_{y\alpha\beta} \sigma_{y\alpha'\beta'}) , \quad (3.4)$$

where $\phi_2'(\mathbf{p}, \mathbf{p}') = \phi_2(\mathbf{p}', \mathbf{p})$ and the functions ϕ_1, ϕ_3, ϕ_4 are symmetric under interchange of \mathbf{p} and \mathbf{p}' . In the limit of zero polarization, $\phi_3 = \phi_4, \phi_2 = \phi_2' = 0$, thus restoring the spin isotropy of the interaction.

The energy density change, Eq. (3.3), simplifies considerably for a longitudinal fluctuation in the distribution function

$$[\delta n_{\mathbf{p}}]_{\sigma\sigma'} = \delta n_{\mathbf{p}} \delta_{\sigma\sigma'} + \delta m_{\mathbf{p}} \tau_{z\sigma\sigma'} = \delta n_{\mathbf{p}\sigma} \delta_{\sigma\sigma'} , \quad (3.5)$$

where $\delta n_{\mathbf{p}\sigma} = \delta n_{\mathbf{p}} + \sigma \delta m_{\mathbf{p}}$. To see this we first note that

$$[n_{\mathbf{p}}] = n_{\mathbf{p}}(\mathbf{r}, t) 1 + \mathbf{m}_{\mathbf{p}}(\mathbf{r}, t) \cdot \boldsymbol{\tau} , \quad (3.1)$$

where

$$2 \sum_{\mathbf{p}} n_{\mathbf{p}}(\mathbf{r}, t) = n(\mathbf{r}, t)$$

is the quasiparticle density at (\mathbf{r}, t) ,

$$2 \sum_{\mathbf{p}} \mathbf{m}_{\mathbf{p}}(\mathbf{r}, t) = \mathbf{m}(\mathbf{r}, t)$$

is the local spin-polarization density, and $1, \tau_i$ are the unit matrix and the i th Pauli matrix in 2×2 spin space, respectively. The quasiparticle distribution function for the ground state with a given polarization

$$\Delta = (n_{\uparrow} - n_{\downarrow}) / (n_{\uparrow} + n_{\downarrow})$$

due to a constant magnetic field $H \hat{\mathbf{e}}_z$ is

$$[n_{\mathbf{p}}^0]_{\sigma\sigma'} = n_{\mathbf{p}\sigma}^0 \delta_{\sigma\sigma'} , \quad (3.2)$$

where $\sigma = 1 (-1)$ corresponds to spin up (down) quasiparticles, $n_{\mathbf{p}\sigma}^0 = \Theta(k_F^{\sigma} - p)$ are step functions and the Fermi radii k_F^{σ} are given by

$$n_{\sigma} = \frac{(k_F^{\sigma})^3}{6\pi^2} .$$

To describe the interaction between these quasiparticles, we must consider the change in the energy density functional due to a change in the distribution function $[\delta n_{\mathbf{p}}] = [n_{\mathbf{p}}] - [n_{\mathbf{p}}^0]$. This change is given by^{18,19}

from Eq. (3.4) it follows that the corresponding change in the quasiparticle energy matrix is also diagonal and is given by

$$[\delta \varepsilon_{\mathbf{p}}]_{\alpha\beta} = \sum_{\mathbf{p}', \alpha', \beta'} [\tilde{f}_{\mathbf{p}, \mathbf{p}'}]_{\alpha\beta, \alpha'\beta'} [\delta n_{\mathbf{p}'}]_{\beta'\alpha'} = \sum_{\mathbf{p}', \alpha'} \tilde{f}_{\mathbf{p}, \mathbf{p}'}^{\alpha\alpha'} \delta n_{\mathbf{p}'\alpha'} \delta_{\alpha\beta} , \quad (3.6)$$

where

$$\begin{aligned}\tilde{f}_{pp'}^{\uparrow\uparrow} &\equiv [\tilde{f}_{pp'}]_{\uparrow\uparrow, \uparrow\uparrow} = \phi_1 + \phi_2 + \phi_2' + \phi_3, \\ \tilde{f}_{pp'}^{\downarrow\downarrow} &\equiv [\tilde{f}_{pp'}]_{\downarrow\downarrow, \downarrow\downarrow} = \phi_1 - \phi_2 - \phi_2' + \phi_3, \\ \tilde{f}_{pp'}^{\uparrow\downarrow} &\equiv [\tilde{f}_{pp'}]_{\uparrow\downarrow, \downarrow\uparrow} = [\tilde{f}_{pp'}]_{\downarrow\uparrow, \uparrow\downarrow} \\ &= \tilde{f}_{p'p}^{\downarrow\uparrow} = \phi_1 + \phi_2 - \phi_2' - \phi_3\end{aligned}$$

are the longitudinal components of the quasiparticle interaction. Now using Eqs. (3.3), (3.5), and (3.6), we get that the change in the energy density is^{18,19}

$$\delta\varepsilon = \sum_{p\sigma} \varepsilon_{p\sigma}^0 \delta n_{p\sigma} + \frac{1}{2} \sum_{p,\sigma,p',\sigma'} \tilde{f}_{pp'}^{\sigma\sigma'} \delta n_{p\sigma} \delta n_{p'\sigma'}. \quad (3.7)$$

$$\begin{aligned}\delta n_{p\sigma} &= \delta\varepsilon_{k_F^\sigma} \frac{\partial}{\partial\varepsilon_{k_F^\sigma}} \Theta(\varepsilon_{k_F^\sigma} - \varepsilon_{p\sigma}^0) + \frac{1}{2} (\delta\varepsilon_{k_F^\sigma})^2 \frac{\partial^2}{\partial\varepsilon_{k_F^\sigma}^2} \Theta(\varepsilon_{k_F^\sigma} - \varepsilon_{p\sigma}^0) \\ &= \delta\varepsilon_{k_F^\sigma} \delta(\varepsilon_{k_F^\sigma} - \varepsilon_{p\sigma}^0) + \frac{1}{2} (\delta\varepsilon_{k_F^\sigma})^2 \frac{\partial}{\partial\varepsilon_{k_F^\sigma}} \delta(\varepsilon_{k_F^\sigma} - \varepsilon_{p\sigma}^0).\end{aligned} \quad (3.9)$$

The corresponding δn_σ is obtained by integrating Eq. (3.9) and gives

$$\delta n_\sigma = \delta\varepsilon_{k_F^\sigma} N_\sigma(0) + \frac{1}{2} (\delta\varepsilon_{k_F^\sigma})^2 \frac{\partial}{\partial\varepsilon_{k_F^\sigma}} N_\sigma(0), \quad (3.10)$$

where

$$N_\sigma(0) = \frac{k_F^\sigma m_\sigma^*}{2\pi^2}$$

is the density of states of spin σ at the Fermi surface. Substituting Eq. (3.9) into Eq. (3.7) and making use of Eq. (3.10), we obtain $\delta\varepsilon$ up to quadratic order in $\delta\varepsilon_{k_F^\sigma}$. For the kinetic energy term we have that

$$\sum_{p\sigma} \varepsilon_{p\sigma}^0 \delta n_{p\sigma} = \sum_\sigma \left[\varepsilon_{k_F^\sigma}^0 N_\sigma(0) \delta\varepsilon_{k_F^\sigma} + \frac{1}{2} \delta\varepsilon_{k_F^\sigma}^2 \frac{\partial}{\partial\varepsilon_{k_F^\sigma}} [\varepsilon_{k_F^\sigma}^0 N_\sigma(0)] \right] = \sum_\sigma [\varepsilon_{k_F^\sigma}^0 \delta n_\sigma + \frac{1}{2} \delta\varepsilon_{k_F^\sigma}^2 N_\sigma(0)], \quad (3.11a)$$

and for the interaction term we get

$$\frac{1}{2} \sum_{p,\sigma,p',\sigma'} \tilde{f}_{pp'}^{\sigma\sigma'} \delta n_{p\sigma} \delta n_{p'\sigma'} = \frac{1}{2} \sum_{\sigma,\sigma'} \tilde{f}_0^{\sigma\sigma'} N_\sigma(0) \delta\varepsilon_{k_F^\sigma} N_{\sigma'}(0) \delta\varepsilon_{k_F^{\sigma'}}. \quad (3.11b)$$

Here, we have used the standard expansion of $\tilde{f}_{pp'}^{\sigma\sigma'}$ in terms of Legendre polynomials¹⁴ and $\tilde{f}_0^{\sigma\sigma'}$ is just the $l=0$ term. For the change in the energy we get, upon combining Eqs. (3.11a) and (3.11b), that

$$\begin{aligned}\delta\varepsilon &= \sum_\sigma \left[\varepsilon_{k_F^\sigma}^0 \delta n_\sigma + \frac{1}{2} \left(\frac{1}{N_\sigma(0)} + \tilde{f}_0^{\sigma\sigma} \right) \delta n_\sigma^2 \right] \\ &\quad + \tilde{f}_0^{\uparrow\downarrow} \delta n_\uparrow \delta n_\downarrow.\end{aligned} \quad (3.11c)$$

The energy density change, Eq. (3.11c) can be rewritten using $n_\sigma = \frac{1}{2}(n + \sigma m)$ as

$$\begin{aligned}\delta\varepsilon &= \frac{1}{2} (\varepsilon_{k_F^\uparrow}^0 + \varepsilon_{k_F^\downarrow}^0) \delta n + \frac{1}{2} (\varepsilon_{k_F^\uparrow}^0 - \varepsilon_{k_F^\downarrow}^0) \delta m \\ &\quad + \frac{1}{8} (C^{\uparrow\uparrow} + C^{\downarrow\downarrow} + 2\tilde{f}_0^{\uparrow\downarrow}) \delta n^2 \\ &\quad + \frac{1}{8} (C^{\uparrow\uparrow} + C^{\downarrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow}) \delta m^2 \\ &\quad + \frac{1}{4} (C^{\uparrow\uparrow} - C^{\downarrow\downarrow}) \delta n \delta m,\end{aligned} \quad (3.12)$$

It is evident from this analysis that the transverse component of the quasiparticle interaction

$$\tilde{g}_{pp'}^{\uparrow\downarrow} = [\tilde{f}_{pp'}]_{\uparrow\downarrow, \downarrow\uparrow} = 2\phi_4 \quad (3.8)$$

is not needed in the description of longitudinal fluctuations.

In order to make contact with experiment, we need to compute the static response functions of the system: the compressibility and the magnetic susceptibility. To do this, we must obtain the change in the energy density due to a longitudinal, uniform distortion of the Fermi surfaces. The change in the distribution function due to this distortion is^{14,18}

where

$$C^{\sigma\sigma} = \frac{1}{N_\sigma(0)} + \tilde{f}_0^{\sigma\sigma}.$$

We can now obtain the thermodynamical properties of the system at $T=0$ K and the response functions by differentiation of Eq. (3.12). The magnetic field, obtained from Eqs. (2.4) and (3.12), equals¹⁴

$$H = H_0 + \delta H, \quad (3.13)$$

where

$$H_0 = \frac{1}{2} (\varepsilon_{k_F^\uparrow}^0 - \varepsilon_{k_F^\downarrow}^0)$$

is the equilibrium magnetic field and

$$\delta H = \frac{1}{4} (C^{\uparrow\uparrow} + C^{\downarrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow}) \delta m + \frac{1}{4} (C^{\uparrow\uparrow} - C^{\downarrow\downarrow}) \delta n$$

is the change in the magnetic field. From this expression

and Eq. (2.10), we get that the susceptibility is²

$$X_n = [\frac{1}{4}(C^{\uparrow\uparrow} + C^{\downarrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow})]^{-1}. \quad (3.14)$$

The pressure, evaluated using Eqs. (2.3) and (3.12), is given by¹⁴

$$P = P_0 + \delta P, \quad (3.15)$$

where

$$P_0 = -\varepsilon_0 + n_{\uparrow} \varepsilon_{k_F}^0 + n_{\downarrow} \varepsilon_{k_F}^0$$

is the equilibrium pressure and

$$\begin{aligned} \delta P = & \frac{1}{4}[n(C^{\uparrow\uparrow} + C^{\downarrow\downarrow} + 2\tilde{f}_0^{\uparrow\downarrow}) + m(C^{\uparrow\uparrow} - C^{\downarrow\downarrow})]\delta n \\ & + \frac{1}{4}[n(C^{\uparrow\uparrow} - C^{\downarrow\downarrow}) + m(C^{\uparrow\uparrow} + C^{\downarrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow})]\delta m \end{aligned}$$

is the pressure change. From this expression and Eq. (2.13), we obtain that the compressibility $K_{M,N,T=0K}$ equals¹³

$$K_{M,N,T=0} = (n_{\uparrow}^2 C^{\uparrow\uparrow} + n_{\downarrow}^2 C^{\downarrow\downarrow} + 2n_{\uparrow} n_{\downarrow} \tilde{f}_0^{\uparrow\downarrow})^{-1}. \quad (3.16)$$

To calculate $K_{H,N,T=0}$, we note that

$$\begin{aligned} \frac{1}{K_{H,N,T=0}} &= n \frac{\partial P}{\partial n}(T=0, m, n)|_{N,H} \\ &= n \frac{\partial P}{\partial n}(T=0, m, n)|_m + \left[\frac{\partial P}{\partial m} \right]_n \left[\frac{\partial m}{\partial n} \right]_{N,H} \end{aligned} \quad (3.17)$$

The derivative

$$\left[\frac{\partial m}{\partial n} \right]_{N,H},$$

evaluated from Eq. (3.13), is given by

$$\left[\frac{\partial m}{\partial n} \right]_{N,H} = - \frac{(C^{\uparrow\uparrow} - C^{\downarrow\downarrow})}{(C^{\uparrow\uparrow} + C^{\downarrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow})}. \quad (3.18)$$

Using Eqs. (3.17), (3.18), and (3.15), we obtain after some algebra

$$K_{H,N,T=0} = \frac{1}{n^2} \frac{C^{\uparrow\uparrow} + C^{\downarrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow}}{C^{\uparrow\uparrow} C^{\downarrow\downarrow} - \tilde{f}_0^{\uparrow\downarrow 2}}. \quad (3.19)$$

For completeness, we note that the chemical potential, obtained using Eqs. (2.4) and (3.12), is

$$\mu = \mu_0 + \delta\mu, \quad (3.20)$$

where

$$\mu_0 = \frac{1}{2}(\varepsilon_{k_F}^0 + \varepsilon_{k_F}^0)$$

is the equilibrium chemical potential and

$$\delta\mu = \frac{1}{4}(C^{\uparrow\uparrow} + C^{\downarrow\downarrow} + 2\tilde{f}_0^{\uparrow\downarrow})\delta n + \frac{1}{4}(C^{\uparrow\uparrow} - C^{\downarrow\downarrow})\delta m$$

is the change in the chemical potential.

IV. THE KINETIC EQUATION

We are interested in studying nonequilibrium and inhomogeneous processes characterized by a distribution function

$$[n_p(\mathbf{r}, t)] = [n_p^0] + [\delta n_p(\mathbf{r}, t)] \quad (4.1)$$

that differs slightly from the homogeneous equilibrium distribution [Eq. (3.2)]. This distribution function obeys the quasiparticle kinetic equation^{18,19}

$$\frac{\partial}{\partial t} [n_p] + \frac{1}{2}(\nabla_p[\varepsilon_p] \cdot \nabla_r [n_p] + \nabla_r [n_p] \cdot \nabla_p [\varepsilon_p]) - \frac{1}{2}(\nabla_r [\varepsilon_p] \cdot \nabla_p [n_p] + \nabla_p [n_p] \cdot \nabla_r [\varepsilon_p]) = I([n_p]) - i[[\varepsilon_p], [n_p]], \quad (4.2)$$

where $I([n_p])$ is the collision integral and the commutator $-i[[\varepsilon_p], [n_p]]$ takes into account the spin precession in a magnetic field. Dropping the collision integral ($T=0$ K) and linearizing with respect to $[\delta n_p]$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} [\delta n_p] + \frac{1}{2}(\nabla_p[\varepsilon_p^0] \cdot \nabla_r [\delta n_p] + \nabla_r [\delta n_p] \cdot \nabla_p [\varepsilon_p^0]) - \frac{1}{2}(\nabla_r [\delta \varepsilon_p] \cdot \nabla_p [n_p^0] + \nabla_p [n_p^0] \cdot \nabla_r [\delta \varepsilon_p]) \\ = -i[[\delta \varepsilon_p], [n_p^0]] - i[[\varepsilon_p^0], [\delta n_p]], \end{aligned} \quad (4.3)$$

where

$$[\varepsilon_p(\mathbf{r}, t)] = [\varepsilon_p^0] + [\delta \varepsilon_p(\mathbf{r}, t)],$$

and $[\varepsilon_p^0]$ is given by Eq. (3.3). As expected, Eq. (4.3) simplifies for a longitudinal fluctuation $[\delta n_p]_{\sigma\sigma'} = \delta n_{p\sigma} \delta_{\sigma\sigma'}$. In that case, all matrices are diagonal in spin space [see Eq. (3.6)] and the kinetic equation reduces to

$$\frac{\partial}{\partial t} \delta n_{p\sigma} + \mathbf{v}_{p\sigma} \cdot \nabla_r \left[\delta n_{p\sigma} - \frac{\partial n_{p\sigma}^0}{\partial \varepsilon_{p\sigma}^0} \delta \varepsilon_{p\sigma} \right] = 0, \quad (4.4)$$

where

$$\mathbf{v}_{p\sigma} = \nabla_p \varepsilon_{p\sigma}^0$$

and

$$\delta\epsilon_{p\sigma} = U_{\sigma}(\mathbf{r}, t) + \sum_{p'\sigma'} \tilde{f}_{pp'}^{\sigma\sigma'} \delta n_{p'\sigma'}(\mathbf{r}, t).$$

Here $U_{\sigma}(\mathbf{r}, t)$ is the longitudinal external potential driving the system. The Fourier transform of this equation is given by¹³

$$(\omega - \mathbf{v}_{p\sigma} \cdot \mathbf{q}) \delta n_{p\sigma}(\mathbf{q}, \omega) + \mathbf{q} \cdot \mathbf{v}_{p\sigma} \frac{\partial n_{p\sigma}^0}{\partial \epsilon_{p\sigma}^0} \sum_{p'\sigma'} \tilde{f}_{pp'}^{\sigma\sigma'} \delta n_{p'\sigma'}(\mathbf{q}, \omega) = - \frac{\partial n_{p\sigma}^0}{\partial \epsilon_{p\sigma}^0} \mathbf{q} \cdot \mathbf{v}_{p\sigma} U_{\sigma}(\mathbf{q}, \omega). \quad (4.5)$$

The form of the kinetic equation for a transversal disturbance^{18,19} is of considerable in the study of spin waves. Here, we consider that the magnetic field acting on the system is made up of a static longitudinal component $\mathbf{H}_0 = H_0 \hat{\mathbf{e}}_z$ and a small (compared to H_0), time dependent transversal part $\delta \mathbf{H}_{\perp}(\mathbf{x}, t) = \delta H_x \hat{\mathbf{e}}_x + \delta H_y \hat{\mathbf{e}}_y$. The effect of the longitudinal field has been analyzed before [see Eq. (3.2)]. The transverse field produces a fluctuation in the quasiparticle distribution given by

$$[\delta n_{\mathbf{p}}] = [n_{\mathbf{p}}] - [n_{\mathbf{p}}^0] = \delta \mathbf{m}_{\mathbf{p}\perp}(\mathbf{x}, t) \cdot \boldsymbol{\tau} = \begin{bmatrix} 0 & \delta m_{\mathbf{p}}^{(-)} \\ \delta m_{\mathbf{p}}^{(+)} & 0 \end{bmatrix}, \quad (4.6)$$

where $[n_{\mathbf{p}}^0]$ is the quasiparticle distribution in the presence of only \mathbf{H}_0 ,

$$\delta \mathbf{m}_{\mathbf{p}\perp} = (\delta m_{\mathbf{p}})_x \hat{\mathbf{e}}_x + (\delta m_{\mathbf{p}})_y \hat{\mathbf{e}}_y,$$

and

$$\delta m_{\mathbf{p}}^{(\pm)} = (\delta m_{\mathbf{p}})_x \pm i (\delta m_{\mathbf{p}})_y.$$

This fluctuation induces a change in the quasiparticle energy operator given by

$$[\delta \epsilon_{\mathbf{p}}] = [\epsilon_{\mathbf{p}}] - [\epsilon_{\mathbf{p}}^0] = \begin{bmatrix} 0 & -\delta H^{(-)} + \sum_{p'} \tilde{g}_{pp'}^{\uparrow\downarrow} \delta m_{p'}^{(-)} \\ -\delta H^{(+)} + \sum_{p'} \tilde{g}_{pp'}^{\uparrow\downarrow} \delta m_{p'}^{(+)} & 0 \end{bmatrix}, \quad (4.7)$$

where

$$\delta H^{(\pm)} = \delta H_x \pm i \delta H_y.$$

In this case, the linearized spin precession term in the kinetic equation becomes

$$[[\delta \epsilon_{\mathbf{p}}], [n_{\mathbf{p}}^0]] + [[\epsilon_{\mathbf{p}}^0], [\delta n_{\mathbf{p}}]] = \begin{bmatrix} 0 & -2m_{\mathbf{p}}^{(0)} \delta \epsilon_{\mathbf{p}}^{(-)} + (\epsilon_{p\uparrow}^0 - \epsilon_{p\downarrow}^0) \delta m_{\mathbf{p}}^{(-)} \\ 2m_{\mathbf{p}}^{(0)} \delta \epsilon_{\mathbf{p}}^{(+)} - (\epsilon_{p\uparrow}^0 - \epsilon_{p\downarrow}^0) \delta m_{\mathbf{p}}^{(+)} & 0 \end{bmatrix}, \quad (4.8)$$

where

$$\delta \epsilon_{\mathbf{p}}^{(\pm)} = -\delta H^{(\pm)} + \sum_{p'} \tilde{g}_{pp'}^{\uparrow\downarrow} \delta m_{p'}^{(\pm)}$$

and

$$m_{\mathbf{p}}^{(0)} = \frac{1}{2} (n_{p\uparrow}^0 - n_{p\downarrow}^0).$$

Substituting Eqs. (4.6), (4.7), and (4.8) into Eq. (4.3) and taking the Fourier transform of the resultant equation, we obtain the kinetic equation for a transverse disturbance at $T=0$ K,

$$\left[\omega - \mathbf{q} \cdot \frac{(\mathbf{v}_{p\uparrow} + \mathbf{v}_{p\downarrow})}{2} \right] \delta m_{\mathbf{p}}^{(+)} + \mathbf{q} \cdot \frac{1}{2} \left[\frac{\partial n_{p\uparrow}^0}{\partial \epsilon_{p\uparrow}^0} \mathbf{v}_{p\uparrow} + \frac{\partial n_{p\downarrow}^0}{\partial \epsilon_{p\downarrow}^0} \mathbf{v}_{p\downarrow} \right] \delta \epsilon_{\mathbf{p}}^{(+)} = 2m_{\mathbf{p}}^{(0)} \delta \epsilon_{\mathbf{p}}^{(+)} - (\epsilon_{p\uparrow}^0 - \epsilon_{p\downarrow}^0) \delta m_{\mathbf{p}}^{(+)} \quad (4.9a)$$

and

$$\left[\omega - \mathbf{q} \cdot \frac{(\mathbf{v}_{p\uparrow} + \mathbf{v}_{p\downarrow})}{2} \right] \delta m_{\mathbf{p}}^{(-)} + \mathbf{q} \cdot \frac{1}{2} \left[\frac{\partial n_{p\uparrow}^0}{\partial \epsilon_{p\uparrow}^0} \mathbf{v}_{p\uparrow} + \frac{\partial n_{p\downarrow}^0}{\partial \epsilon_{p\downarrow}^0} \mathbf{v}_{p\downarrow} \right] \delta \epsilon_{\mathbf{p}}^{(-)} = -[2m_{\mathbf{p}}^{(0)} \delta \epsilon_{\mathbf{p}}^{(-)} - (\epsilon_{p\uparrow}^0 - \epsilon_{p\downarrow}^0) \delta m_{\mathbf{p}}^{(-)}], \quad (4.9b)$$

where

$$\mathbf{v}_{p\sigma} = \nabla_{\mathbf{p}} \epsilon_{p\sigma}^0.$$

V. COLLECTIVE MODES

The eigenvectors of the longitudinal kinetic equation, Eq. (4.5), correspond to the collective modes, e.g., zero sound in the polarized liquid.²⁰ Thus, we now turn to the task of finding the dispersion relation of these modes.¹⁸ It is convenient to write the fluctuation $\delta n_{p\sigma}$ as

$$\delta n_{p\sigma} = - \frac{\partial n_{p\sigma}^0}{\partial \epsilon_{p\sigma}^0} v_{p\sigma}, \quad (5.1)$$

where the $v_{p\sigma}$ are the normal displacements of the Fermi surfaces due to the collective mode. In terms of the $v_{p\sigma}$, the longitudinal kinetic equation is

$$(\omega - \mathbf{q} \cdot \mathbf{v}_{p\sigma}) v_{p\sigma} + \mathbf{q} \cdot \mathbf{v}_{p\sigma} \sum_{p'\sigma'} \tilde{f}_{pp'}^{\sigma\sigma'} \frac{\partial n_{p'\sigma'}^0}{\partial \epsilon_{p'\sigma'}^0} v_{p'\sigma'} = 0, \quad (5.2)$$

where we have set the external potential U_σ in Eq. (4.5) equal to zero since we are interested in the free modes of oscillation.

In the most general case, $v_{p\sigma}$ is a function of the spherical angles (Θ, ϕ) specifying the direction of \mathbf{p} with respect to \mathbf{q} . For the azimuthally symmetric solution, $v_{p\sigma}$ will only depend on the angle Θ between \mathbf{p} and \mathbf{q} , and we can expand it in a Legendre series as

$$v_{p\sigma} = \sum_{l=0}^{\infty} v_{l\sigma} P_l(\cos\Theta). \quad (5.3)$$

Substituting Eq. (5.3) and the expansion

$$\tilde{f}_{pp'}^{\sigma\sigma'} = \sum_{l=0}^{\infty} \tilde{f}_l^{\sigma\sigma'} P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')$$

into Eq. (5.2), and then using the addition theorem for the spherical harmonics, we get

$$(s_\sigma - \cos\Theta) \sum_l v_{l\sigma} P_l(\cos\Theta) - \cos\Theta \sum_{\sigma', l} N_{\sigma', l}(0) \frac{\tilde{f}_l^{\sigma\sigma'}}{2l+1} v_{l\sigma'} P_l(\cos\Theta) = 0, \quad (5.4)$$

where

$$s_\sigma = \frac{\omega}{qv_{F\sigma}}.$$

If we now project out the moments of this equation, we find the following set of coupled equations

$$s_\sigma v_{l\sigma} - \sum_{\sigma'} \left[\frac{l+1}{2l+3} v_{l+1, \sigma'} \left[\delta_{\sigma\sigma'} + \frac{N_{\sigma'}(0)}{2l+3} \tilde{f}_{l+1}^{\sigma\sigma'} \right] + \frac{l}{2l-1} \left[\delta_{\sigma\sigma'} + \frac{N_{\sigma'}(0)}{2l-1} \tilde{f}_{l-1}^{\sigma\sigma'} \right] v_{l-1, \sigma'} \right] = 0. \quad (5.5)$$

In order to solve this set of coupled equations, we assume that

$$\tilde{f}_l^{\sigma\sigma'} = 0 \quad l \geq 2$$

and

$$v_{l\sigma} = 0 \quad l > 2.$$

We will come back to these assumptions at the end of this section. Making use of Eq. (5.6), Eq. (5.5) reduces to

$$s_\sigma v_{0\sigma} - \frac{1}{3} \sum_{\sigma'} \left[\delta_{\sigma\sigma'} + \frac{N_{\sigma'}(0)}{3} \tilde{f}_1^{\sigma\sigma'} \right] v_{1\sigma'} = 0, \quad (5.7a)$$

$$s_\sigma v_{1\sigma} - \sum_{\sigma'} \{ [\delta_{\sigma\sigma'} + N_{\sigma'}(0) \tilde{f}_0^{\sigma\sigma'}] v_{0\sigma'} + \frac{2}{3} \delta_{\sigma\sigma'} v_{2\sigma'} \} = 0, \quad (5.7b)$$

$$s_\sigma v_{2\sigma} - \frac{2}{3} \sum_{\sigma'} \left[\delta_{\sigma\sigma'} + N_{\sigma'}(0) \frac{\tilde{f}_1^{\sigma\sigma'}}{3} \right] v_{1\sigma'} = 0. \quad (5.7c)$$

Now, we proceed as follows: from Eqs. (5.7a) and (5.7c), we get

$$v_{2\sigma} = 2v_{0\sigma}. \quad (5.8)$$

Solving Eq. (5.7b) for $v_{1\sigma}$ and substituting it in Eq. (5.7a), we obtain

$$s_\sigma v_{0\sigma} - \frac{1}{3} \sum_{\sigma', \sigma''} \left[\delta_{\sigma, \sigma'} + \frac{N_{\sigma'}(0)}{3} \tilde{f}_1^{\sigma\sigma'} \right] \times \left[\frac{2}{3} \delta_{\sigma', \sigma''} + N_{\sigma''}(0) \tilde{f}_0^{\sigma'\sigma''} \right] \frac{v_{0\sigma''}}{s_{\sigma'}} = 0. \quad (5.9)$$

This equation can be rewritten, by making the substitution

$$v_{0\sigma} = s_\sigma \eta_{0\sigma} = (c_{0m}/v_{F\sigma}) \eta_{0\sigma},$$

as

$$\sum_{\sigma''} \left[\frac{c_{0m}^2}{v_{F\sigma}^2} \delta_{\sigma\sigma''} - A_{\sigma\sigma''} \right] \eta_{0\sigma''} = 0, \quad (5.10)$$

where

$$c_{0m} = \frac{\omega}{q}$$

is the velocity of propagation of the collective mode and

$$A_{\sigma\sigma''} = \frac{1}{3} \sum_{\sigma'} \left[\delta_{\sigma\sigma'} + \frac{N_{\sigma'}(0)}{3} \tilde{f}_1^{\sigma\sigma'} \right] \\ \times \frac{v_{F\sigma'}}{v_{F\sigma''}} \left[\frac{2}{3} \delta_{\sigma'\sigma''} + N_{\sigma''}(0) \tilde{f}_0^{\sigma'\sigma''} \right].$$

The collective mode velocity is obtained by requiring that

$$\det \left[\frac{c_{0m}^2}{v_{F\sigma}^2} \delta_{\sigma\sigma''} - A_{\sigma\sigma''} \right] = 0$$

and is given by

$$\bar{a}_{pp'}^{\sigma\sigma'}(\mathbf{q}, \omega) = \langle \mathbf{p} - \mathbf{q}/2, \sigma; \mathbf{p}' + \mathbf{q}/2, \sigma' | t | \mathbf{p} + \mathbf{q}/2, \sigma; \mathbf{p}' - \mathbf{q}/2, \sigma' \rangle \quad (6.1a)$$

and the spin-flip scattering amplitude

$$\tilde{G}_{pp'}^{\uparrow\downarrow}(\mathbf{q}, \omega) = \langle \mathbf{p} - \mathbf{q}/2, \downarrow; \mathbf{p}' + \mathbf{q}/2, \uparrow | t | \mathbf{p} + \mathbf{q}/2, \uparrow; \mathbf{p}' - \mathbf{q}/2, \downarrow \rangle, \quad (6.1b)$$

where t is the transition operator, \mathbf{q} is the momentum transfer, and ω is the energy transfer. The similarity of this situation and that for the quasiparticle interaction is not coincidental; both functions correspond to different limits of the renormalized four-point vertex function.²² In the limit of small energy transfers and small \mathbf{q} (forward scattering), the kinetic equation can be used to construct the Landau version of the Bethe-Salpeter equation obeyed by these scattering amplitudes. To calculate the longitudinal scattering amplitudes in this limit, we must compute the change in the quasiparticle energy $\delta\varepsilon_{p\sigma}(\mathbf{q}, \omega)$ induced by putting a quasiparticle in the empty state \mathbf{p}', σ' allowing for rearrangements in the quasiparticle distribution. Thus, we have¹⁸

$$\bar{a}_{pp'}^{\sigma\sigma'}(\mathbf{q}, \omega) \equiv \frac{\delta\varepsilon_{p\sigma}(\mathbf{q}, \omega)}{\delta n_{\mathbf{p}'\sigma'}(\mathbf{q}, \omega)} = \tilde{f}_{pp'}^{\sigma\sigma'} \\ + \sum_{p'', \sigma'', p'' \neq p'} \tilde{f}_{pp''}^{\sigma\sigma''} \frac{\delta n_{\mathbf{p}''\sigma''}}{\delta n_{\mathbf{p}'\sigma'}}. \quad (6.2)$$

The $\delta n_{\mathbf{p}''\sigma''}$, obtained from the longitudinal kinetic equation [Eq. (4.5)], is given by

$$c_{0m}^2 = \frac{A_{\uparrow\uparrow} v_{F\uparrow}^2 + A_{\downarrow\downarrow} v_{F\downarrow}^2}{2} \\ \pm \left[\left(\frac{A_{\uparrow\uparrow} v_{F\uparrow}^2 + A_{\downarrow\downarrow} v_{F\downarrow}^2}{2} \right)^2 - v_{F\uparrow}^2 v_{F\downarrow}^2 (A_{\uparrow\uparrow} A_{\downarrow\downarrow} - A_{\uparrow\downarrow} A_{\downarrow\uparrow}) \right]^{1/2}. \quad (5.11)$$

The two solutions given in Eq. (5.11) correspond to the two possible zero-temperature oscillations; zero sound c_{0m}^+ (the solution corresponding to the positive square root) and spin-zero sound c_{0m}^- . In liquid ³He, only zero sound propagates; spin-zero sound, i.e., the paramagnon, is not a propagating mode. The spin-zero sound appears as a propagating mode due to the approximation of Eq. (5.6), where $v_{l\sigma} = 0$ for $l > 2$. At $T = 0$, this approximation is good only when the collective mode velocity is much larger than the Fermi velocity.²¹ In liquid ³He, we have for small polarizations that $c_{0m}^+ \gg v_F$ and $c_{0m}^- \leq v_F$, thus, only the zero sound solution is consistent with Eq. (5.6).

VI. SCATTERING AMPLITUDES AND SUM RULES

There are four distinct scattering amplitudes in polarized ³He. These are defined as follows: the three longitudinal scattering amplitudes

$$\delta n_{\mathbf{p}''\sigma''}(\mathbf{q}, \omega) = - \left[\frac{\partial n_{\mathbf{p}''\sigma''}^0}{\partial \varepsilon_{\mathbf{p}''\sigma''}^0} \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}''\sigma''} \delta \varepsilon_{\mathbf{p}''\sigma''} \right] / (\omega - \mathbf{v}_{\mathbf{p}''\sigma''} \cdot \mathbf{q}). \quad (6.3)$$

Substituting this result into Eq. (6.2), we get that, in the limit of small (\mathbf{q}, ω) , $\bar{a}_{pp'}^{\sigma\sigma'}$, satisfies the equation,

$$\bar{a}_{pp'}^{\sigma\sigma'}(\mathbf{q}, \omega) = \tilde{f}_{pp'}^{\sigma\sigma'} - \sum_{p'', \sigma''} \tilde{f}_{pp''}^{\sigma\sigma''} \frac{\partial n_{\mathbf{p}''\sigma''}^0}{\partial \varepsilon_{\mathbf{p}''\sigma''}^0} \frac{\mathbf{q} \cdot \mathbf{v}_{\mathbf{p}''\sigma''}}{\omega - \mathbf{v}_{\mathbf{p}''\sigma''} \cdot \mathbf{q}} \bar{a}_{\mathbf{p}'\mathbf{p}'}^{\sigma''\sigma'}(\mathbf{q}, \omega). \quad (6.4)$$

The derivation of the Bethe-Salpeter equation obeyed by $\tilde{G}_{pp'}^{\uparrow\downarrow}$ is more complicated. If we require that the quasiparticles be near their respective Fermi surfaces, this will need a finite four momentum transfer for the spin-flip process at arbitrary polarizations.²³ In order to obtain the equation obeyed by $\tilde{G}_{pp'}^{\uparrow\downarrow}$ from the transverse kinetic equation, we are forced to relax this requirement. To proceed, we define $\tilde{G}_{pp'}^{\uparrow\downarrow}$ by

$$\tilde{G}_{pp'}^{\uparrow\downarrow} = \frac{\delta \varepsilon_{\mathbf{p}'}^{(-)}(\mathbf{q}, \omega)}{\delta m_{\mathbf{p}'}^{(-)}(\mathbf{q}, \omega)} = \bar{g}_{pp'}^{\uparrow\downarrow} + \sum_{p'', p'' \neq p'} \bar{g}_{pp''}^{\uparrow\downarrow} \frac{\delta m_{\mathbf{p}''}^{(-)}}{\delta m_{\mathbf{p}'}^{(-)}}. \quad (6.5)$$

The $\delta m_{\mathbf{p}'}^{(-)}$, obtained from the transverse kinetic equation [Eq. (4.9b)], is given by

$$\delta m_{\mathbf{p}''}^{(-)} = - \left[\left[n_{\mathbf{p}''\uparrow}^0 - n_{\mathbf{p}''\downarrow}^0 + \frac{1}{2} \frac{\partial n_{\mathbf{p}''\uparrow}^0}{\partial \epsilon_{\mathbf{p}''\uparrow}^0} \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}''\uparrow} + \frac{1}{2} \frac{\partial n_{\mathbf{p}''\downarrow}^0}{\partial \epsilon_{\mathbf{p}''\downarrow}^0} \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}''\downarrow} \right] / \left[\omega - \mathbf{q} \cdot \frac{(\mathbf{v}_{\mathbf{p}''\uparrow} + \mathbf{v}_{\mathbf{p}''\downarrow})}{2} - \epsilon_{\mathbf{p}''\uparrow}^0 + \epsilon_{\mathbf{p}''\downarrow}^0 \right] \right] \delta \epsilon_{\mathbf{p}''}^{(-)}. \quad (6.6)$$

Substituting this equation into Eq. (6.5) gives $\tilde{G}_{\mathbf{p}\mathbf{p}'}^{\uparrow\downarrow}$ in the limit of small four momentum transfer

$$\tilde{G}_{\mathbf{p}\mathbf{p}'}^{\uparrow\downarrow} = \tilde{g}_{\mathbf{p}\mathbf{p}'}^{\uparrow\downarrow} - \sum_{\mathbf{p}''} \tilde{g}_{\mathbf{p}\mathbf{p}''}^{\uparrow\downarrow} \left\{ \left[n_{\mathbf{p}''\uparrow}^0 - n_{\mathbf{p}''\downarrow}^0 + \frac{1}{2} \frac{\partial n_{\mathbf{p}''\uparrow}^0}{\partial \epsilon_{\mathbf{p}''\uparrow}^0} \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}''\uparrow} + \frac{1}{2} \frac{\partial n_{\mathbf{p}''\downarrow}^0}{\partial \epsilon_{\mathbf{p}''\downarrow}^0} \mathbf{q} \cdot \mathbf{v}_{\mathbf{p}''\downarrow} \right] / \left[\omega - \mathbf{q} \cdot \left[\frac{\mathbf{v}_{\mathbf{p}''\uparrow} + \mathbf{v}_{\mathbf{p}''\downarrow}}{2} \right] - \epsilon_{\mathbf{p}''\uparrow}^0 + \epsilon_{\mathbf{p}''\downarrow}^0 \right] \right\} \tilde{G}_{\mathbf{p}''\mathbf{p}'}^{\uparrow\downarrow}. \quad (6.7)$$

This equation has been generalized to arbitrary four momentum transfer by Quader and Bedell,²³ however, this is beyond the scope of the current paper. The spin-flip scattering amplitude goes beyond the standard Landau quasiparticle picture since it involves states that are far from their respective Fermi surface when the momentum transfer is less than a minimum value²³ given by $|\mathbf{q}_{\min}| = k_F^\uparrow - k_F^\downarrow$. A detailed discussion of this situation involving quasiparticles far from their Fermi surface can be found in the papers by Stamp¹¹ and Quader and Bedell.²³

There are two important sum rules in polarized ³He. To obtain them, we need first to solve for $\tilde{a}_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'}(\mathbf{q}, \omega=0)$ in the limit of small \mathbf{q} . Substituting the expansion

$$\tilde{a}_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma'}(\mathbf{q} \rightarrow 0, \omega=0) = \sum_{l=0}^{\infty} \tilde{a}_l^{\sigma\sigma'} P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \quad (6.8)$$

into Eq. (6.4) and using the addition theorem of the spherical harmonics, we get

$$\tilde{a}_l^{\sigma\sigma'} = \tilde{f}_l^{\sigma\sigma'} - \sum_{\sigma''} \tilde{f}_l^{\sigma\sigma''} \frac{N^{\sigma''}(0)}{2l+1} \tilde{a}_l^{\sigma''\sigma'}. \quad (6.9)$$

This set of coupled algebraic equations is solved for the $\tilde{a}_l^{\sigma\sigma'}$. We get^{12,23}

$$\tilde{a}_l^{\sigma\sigma} = \frac{1}{D_l} \left[\tilde{f}_l^{\sigma\sigma} \left(1 + \frac{\tilde{f}_l^{-\sigma-\sigma} N^{-\sigma}(0)}{2l+1} \right) - \frac{\tilde{f}_l^{\uparrow\downarrow} N^{-\sigma}(0)}{2l+1} \right], \quad (6.10a)$$

$$\tilde{a}_l^{\uparrow\downarrow} = \frac{\tilde{f}_l^{\uparrow\downarrow}}{D_l}, \quad (6.10b)$$

where

$$D_l = \left[1 + \frac{\tilde{f}_l^{\uparrow\downarrow} N^\uparrow(0)}{2l+1} \right] \left[1 + \frac{\tilde{f}_l^{\uparrow\downarrow} N^\downarrow(0)}{2l+1} \right] - \left[\frac{\tilde{f}_l^{\uparrow\downarrow}}{2l+1} \right]^2 N^\uparrow(0) N^\downarrow(0).$$

The sum rules are obtained by noting that the antisymmetry of the wave function under exchange of identical particles requires that^{12,13}

$$\tilde{a}_{\mathbf{p}\mathbf{p}'}^{\sigma\sigma}(\mathbf{q}=0, \omega=0) = \sum_{l=0}^{\infty} \tilde{a}_l^{\sigma\sigma} = 0, \quad (6.11)$$

where the $\tilde{a}_l^{\sigma\sigma}$ are given by Eq. (6.10a). These sum rules serve as important constraints on the Landau parameters of the polarized liquid.

VII. A MODEL FOR THE LONGITUDINAL LANDAU PARAMETERS

The results derived in the previous sections can now be used to determine the parameters of a model for the polarization dependence of the Landau parameters. We will follow the steps of Bedell and Sanchez-Castro² whereby a power series expansion in the polarization, Δ , is proposed for the $\tilde{f}_l^{\sigma\sigma'}$'s. It was argued by Bedell and Sanchez-Castro² that a finite order expansion, up to Δ^2 in $\tilde{f}_0^{\sigma\sigma'}$ and to Δ^4 for $\tilde{f}_1^{\sigma\sigma'}$, would provide a reasonable model for the polarization dependence of these parameters over the whole polarization range. While no rigorous arguments exist for truncating this expansion, the physical arguments for keeping only terms to order Δ^2 in $\tilde{f}_0^{\sigma\sigma'}$ are plausible. Given the ansatz for the Δ dependence of the Landau parameters, the derived quantities, X_n , ϵ , etc., can be obtained to all orders in Δ . In the next section, the results for these quantities will be given. For now we will assume the truncated form for the expansion in Δ and refer the reader to Ref. 2 for the physical motivation.

We begin by noting that if one requires symmetry under a reversal of the external magnetic field, it is then evident that the expansion of $\tilde{f}_l^{\uparrow\uparrow}$ differs only in the sign of the odd terms from the expansion of $\tilde{f}_l^{\downarrow\downarrow}$ and that $\tilde{f}_l^{\uparrow\downarrow}$ has only even powers in its expansion. The $l=0$ moments are assumed to be weakly dependent on the polarization, namely,

$$\tilde{f}_0^{\sigma\sigma}(\Delta) = f_0^{\uparrow\uparrow} (1 - \sigma b_0 \Delta + b_1 \Delta^2), \quad (7.1a)$$

$$\tilde{f}_0^{\uparrow\downarrow}(\Delta) = f_0^{\uparrow\downarrow} (1 + c_1 \Delta^2). \quad (7.1b)$$

For the $l=1$ moments, the situation is more complicated. They must satisfy^{2,12}

$$\tilde{f}_1^{\uparrow\downarrow}(\Delta=1) = 0, \quad (7.2a)$$

$$\tilde{f}_1^{\downarrow\downarrow}(\Delta=1) = 0 \quad (7.2b)$$

as a consequence of the fact that only s wave scattering is possible for the down spins when $k_F^\downarrow = 0$. To account for the more detailed structure anticipated, the $l=1$ moments are expanded up to fourth order in Δ , namely,

$$\tilde{f}_1^{\sigma\sigma}(\Delta) = f_1^{\uparrow\uparrow} (1 - d_0 \sigma \Delta + d_1 \Delta^2 - d_2 \sigma \Delta^3 + d_3 \Delta^4), \quad (7.3a)$$

$$\tilde{f}_1^{\uparrow\downarrow}(\Delta) = f_1^{\uparrow\downarrow} (1 + g_1 \Delta^2 + g_3 \Delta^4). \quad (7.3b)$$

In addition, the $l \geq 2$ moments are assumed to be zero.

To determine these parameters, one makes use of sum rules and thermodynamics. The m_σ^* that appear through various formulas are related by Galilean invariance to the

$\tilde{f}_1^{\sigma\sigma'}$ as follows:^{2,12}

$$\frac{m_\sigma^*(\Delta)}{m} = \left\{ 1 - \frac{1}{3} N_\sigma^0(0) \left[\tilde{f}_1^{\sigma\sigma} + \left(\frac{k_F^{-\sigma}}{k_F^\sigma} \right)^2 \tilde{f}_1^{\uparrow\downarrow} \right] \right\}^{-1}, \quad (7.4)$$

where

$$N_\sigma^0(0) = \frac{k_F^\sigma m}{2\pi^2}$$

and m is the ^3He bare mass. It is convenient, for small Δ , to expand this relation up to order Δ^2 as

$$\frac{m_\sigma^*}{m} = 1 - a_0 \sigma \Delta + a_1 \Delta^2 + \dots, \quad (7.5)$$

where

$$a_0 = \frac{1}{6} (F_1^{\uparrow\uparrow} d_0 - \frac{1}{3} F_1^{\uparrow\uparrow} + F_1^{\uparrow\downarrow})$$

and

$$a_1 = a_0^2 - \frac{1}{6} [F_1^{\uparrow\uparrow} (\frac{1}{9} + \frac{1}{3} d_0 - d_1) - F_1^{\uparrow\downarrow} (g_1 + \frac{1}{3})].$$

The microscopic theory gives us six of the nine relations needed to solve for the parameters: four coming from the sum rules and two from Eqs. (7.2). The sum rules were enforced in the following approximate way: For small Δ , Eq. (6.11) was expanded up to Δ^2 and each coefficient required to vanish. The two additional sum rule relations came from a consideration of fully polarized ^3He . For fully polarized ^3He , the sum rule for $\bar{a}^{\uparrow\downarrow}$ terminates because of Eq. (7.2). This sum rule gives^{2,12}

$$\tilde{f}_0^{\uparrow\downarrow} [1 + N_\uparrow(0) \tilde{f}_0^{\uparrow\uparrow}] - N_\uparrow(0) \tilde{f}_0^{\uparrow\downarrow} = 0. \quad (7.6)$$

Using this result, we can express m_\uparrow^* for $\Delta=1$ as

$$\frac{m_\uparrow^*}{m^*} (\Delta=1) = \frac{2^{2/3}}{N(0)} \frac{\tilde{f}_0^{\uparrow\downarrow}}{\tilde{f}_0^{\uparrow\downarrow} - \tilde{f}_0^{\uparrow\uparrow} \tilde{f}_0^{\uparrow\downarrow}}, \quad (7.7)$$

where

$$N(0) = \frac{k_F m^*}{\pi^2}$$

is the total density of states of the unpolarized liquid. Solving simultaneously Eq. (7.7), the expression for m_\uparrow^* in terms of the $\tilde{f}_1^{\sigma\sigma'}$, Eq. (7.4),

$$\frac{m_\uparrow^*}{m} = 1 + \frac{\bar{F}_1^{\uparrow\uparrow}}{3} \quad (7.8)$$

and the sum rule for $\bar{a}^{\uparrow\uparrow}$

$$\frac{\bar{F}_0^{\uparrow\uparrow}}{1 + \bar{F}_0^{\uparrow\uparrow}} + \frac{\bar{F}_1^{\uparrow\uparrow}}{1 + \frac{\bar{F}_1^{\uparrow\uparrow}}{3}} = 0, \quad (7.9)$$

where

$$\bar{F}_1^{\uparrow\uparrow} = N^\uparrow(0) \tilde{f}_1^{\uparrow\uparrow} (\Delta=1),$$

gives the two desired relations.

The last three relations were obtained using thermo-

dynamics. Expanding $K_{M,N,T=0K}$, Eq. (3.16), up to Δ^2 and comparing it with the corresponding thermodynamical relation, Eq. (2.14), gives²

$$a_0^2 + \frac{5}{3} a_0 - a_1 + \frac{F_0^{\uparrow\uparrow}}{2} b_1 - F_0^{\uparrow\uparrow} b_0 + \frac{F_0^{\uparrow\downarrow}}{2} c_1 + F_0^a + \frac{5}{9} = \frac{N(0)}{3} \left[n \frac{d^2 T_{\text{SF}}}{dn^2} + 2 \frac{dT_{\text{SF}}}{dn} \right]. \quad (7.10)$$

Similarly, expanding the microscopic expression for

$$\left(\frac{\partial P}{\partial m} \right)_n,$$

Eq. (3.15), up to Δ^2 , and comparing it with Eq. (2.12) gives^{2,14}

$$a_0 - \frac{1}{2} F_0^{\uparrow\uparrow} b_0 + \frac{2}{3} + F_0^a = \frac{n}{\epsilon_F} \frac{dT_{\text{SF}}}{dn}. \quad (7.11)$$

Similarly, comparing Eq. (2.17) and the low-temperature expansion for the heat capacity,

$$C_{V,N,H} = \frac{\pi^2}{3} [N_\uparrow(0) + N_\downarrow(0)] T,$$

expanded up to Δ^2 , gives²

$$a_1 - \frac{a_0}{3} - \frac{1}{9} = \frac{-4}{3} \frac{a}{\pi^2} (1 + F_0^a) \epsilon_F^2. \quad (7.12)$$

Thus, we have all the relations needed to determine the coefficients. Table I shows the set of experimental inputs as well as the calculated coefficients for four different pressures. A discussion of the results and implications of the model will be done in Sec. VIII.

VIII. PREDICTIONS OF THE MODEL

Given the Landau parameters, $\tilde{f}_0^{\sigma\sigma'}$, and the effective masses, m_σ^* , one of the quantities we can calculate is the susceptibility at constant density, X_n . The most striking feature of the susceptibility, shown in Fig. 1, is that X_n initially increases with increasing Δ . At some finite polarization, $\Delta_{\text{max}} < 1$, X_n reaches a maximum and then drops to zero as $\Delta \rightarrow 1$. This rise in X_n is at first surprising given that the total density of states $[N_\uparrow(0) + N_\downarrow(0)]$ decreases with increasing Δ (see Fig. 2). The rise comes about since the parameter combination, $\tilde{f}_0^{\uparrow\uparrow} + \tilde{f}_0^{\uparrow\downarrow} - 2\tilde{f}_0^{\uparrow\downarrow}$, that appears in X_n , Eq. (3.14), becomes more attractive with increasing Δ . The fact that this parameter combination becomes more attractive in the model of Bedell and Sanchez-Castro indicates a tendency of liquid ^3He to come closer to a magnetic instability when a field is applied. At finite field, a magnetic transition for itinerant electrons was called a metamagnetic transition by Wohlfarth and Rhodes.⁸ Since, in the model of Bedell and Sanchez-Castro, ^3He becomes more magnetic with increasing field, without a real metamagnetic transition occurring, they introduced the notion of the "nearly metamagnetic" picture for the MEOS of ^3He .

From the susceptibility, it is possible to derive a number of important thermodynamic quantities. For exam-

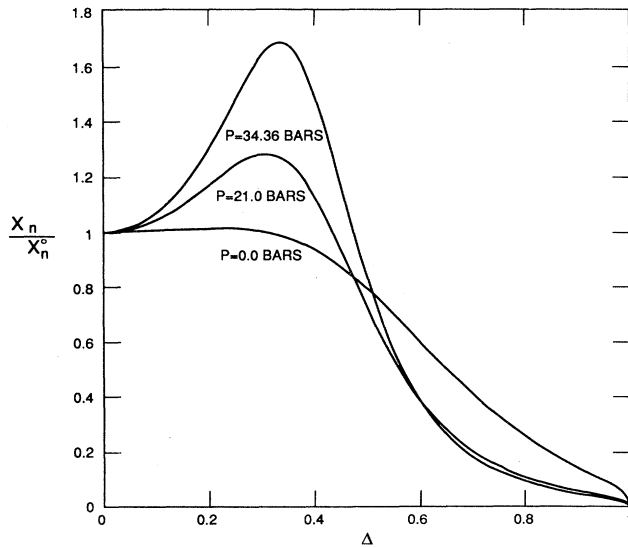


FIG. 1. The ratio of the magnetic susceptibility to the unpolarized magnetic susceptibility for three different pressures.

ple, the magnetic field, H at $T=0$ K, needed to produce a magnetization m is given by

$$H(m) = \int_0^m dm' X_n^{-1} = n \int_0^\Delta d\Delta' X_n^{-1}. \quad (8.1)$$

In Fig. 3, we plot the magnetic field versus polarization Δ . There are features of these curves that should be noted. The first point is that roughly 50% polarization is achieved at high pressure ($P \geq 21$ bars) for a field of approximately $100 \text{ T} \approx 0.1 H_{\max}$, where H_{\max} is the field needed for $\Delta=1$. Another feature of Fig. 3 worth noting is the size of H_{\max} which is of the order of 10^3 T . This is about six times larger than the spin-fluctuation field, H_{SF} , defined by the zero-field susceptibility

$$X_n^0 = \frac{3}{2} \frac{n}{(1+F_0^a)T_F} = \frac{n}{H_{\text{SF}}}, \quad (8.2)$$

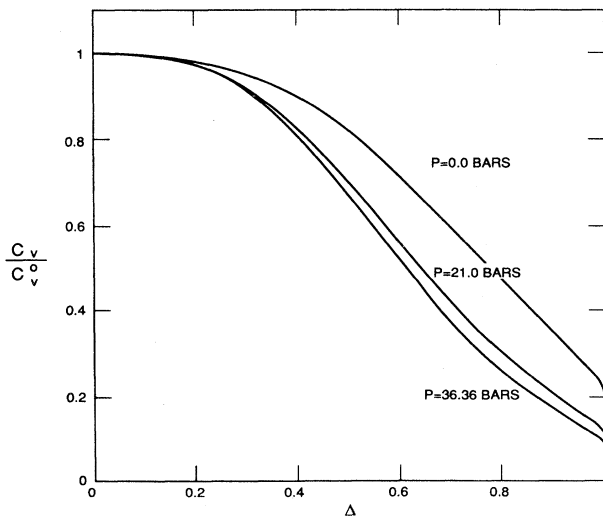


FIG. 2. The ratio of the specific heat to the unpolarized value for three different pressures.

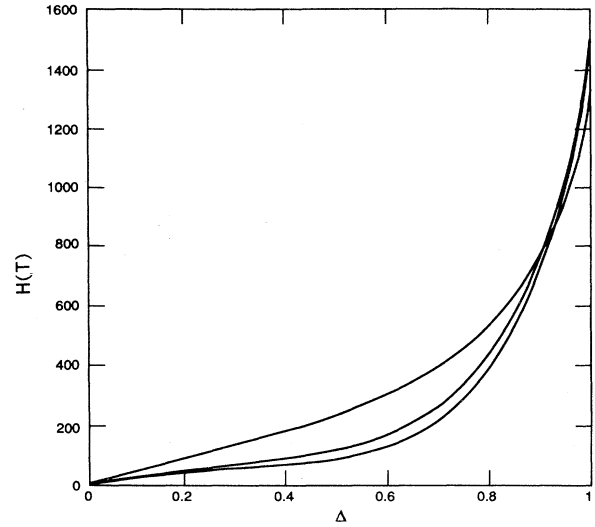


FIG. 3. The magnetic field necessary to produce a given polarization. The upper curve at $\Delta=0.5$ is at $P=0.0$ bars, the middle curve is at $P=21.0$ bars, and the lower curve is at $P=34.36$ bars.

where $H_{\text{SF}} = \frac{2}{3}(1+F_0^a)T_F$. Clearly, any simple estimates based on the assumption of constant X_n will underestimate H_{\max} and, as we will see, the energy density.

The large value of H_{\max} in the "nearly metamagnetic" model is due largely to the fact that X_n must vanish when $\Delta \rightarrow 1$, thus, requiring larger fields to turn over the last few spins. For example, the last 10% of polarization requires roughly a doubling of the field that produced the first 90% of the polarization. Another feature we should note is that the density dependence of H_{\max} is the opposite of that found for H_{SF} . We can understand the densi-

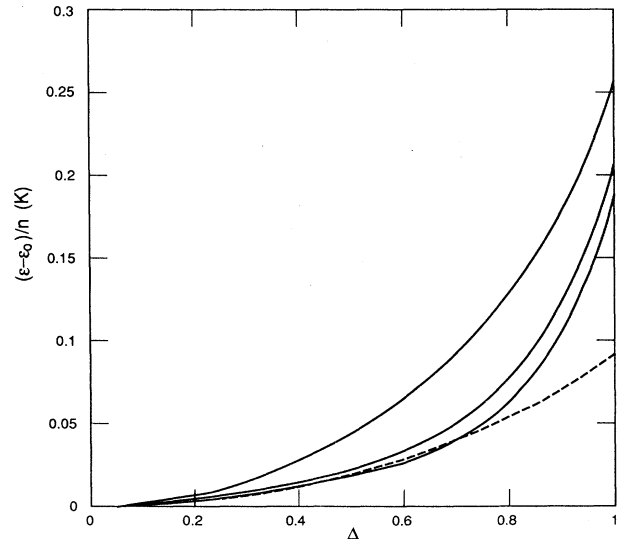


FIG. 4. The magnetic energy per atom as a function of polarization. The upper solid curve at $\Delta=0.5$ is at $P=0.0$ bars, the middle curve is at $P=21.0$ bars and the lower curve is at $P=34.36$ bars. The dashed curve is at $P=34.36$ bars and was calculated assuming a constant susceptibility.

ty dependence of H_{\max} as follows: In the fully polarized limit of the “nearly metamagnetic” model, ^3He is a weakly interacting Fermi gas. For this gas we might expect that

$$2H_{\max} \approx T_F^\dagger = k_F^{\dagger 2} / 2m_\dagger^*,$$

thus, since m_\dagger^* decreases with increasing density, we would expect H_{\max} to increase.

From the function $H(m)$ obtained using Eq. (8.1), we can determine the energy per particle, $\varepsilon(n, m)/n$. By fixing the density, we can calculate the change in energy from the experimentally determined value $\varepsilon_0 = \varepsilon(n, m=0)$ as follows:

$$(\varepsilon - \varepsilon_0)/n = \int_0^\Delta d\Delta' H(m'). \quad (8.3)$$

In Fig. 4, we have plotted $(\varepsilon - \varepsilon_0)/n$ as a function of Δ for three pressures. We have also plotted (dashed line) the $(\varepsilon - \varepsilon_0)/n$ for $n = 0.0236 \text{ \AA}^{-3}$, the density at the melting pressure in zero field, assuming a constant susceptibility, Eq. (8.2).

One of the more important lessons we can learn from Fig. 4 has to do with the comparison between the $(\varepsilon - \varepsilon_0)/n$ curves obtained from the “nearly metamagnetic” model and the constant susceptibility results. The feature in $(\varepsilon - \varepsilon_0)/n$ curve that gives rise to the initial increase in X_n with increasing m is the fact that the energy calculated in the “nearly metamagnetic” model initially drops below the constant susceptibility result. To be able to distinguish between a X_n that initially increases or one that decreases with increasing m from a microscopic calculation of the energy density would require an accuracy of a few percent in this quantity. Clearly, this is beyond the ability of the currently available microscopic approaches to ^3He .

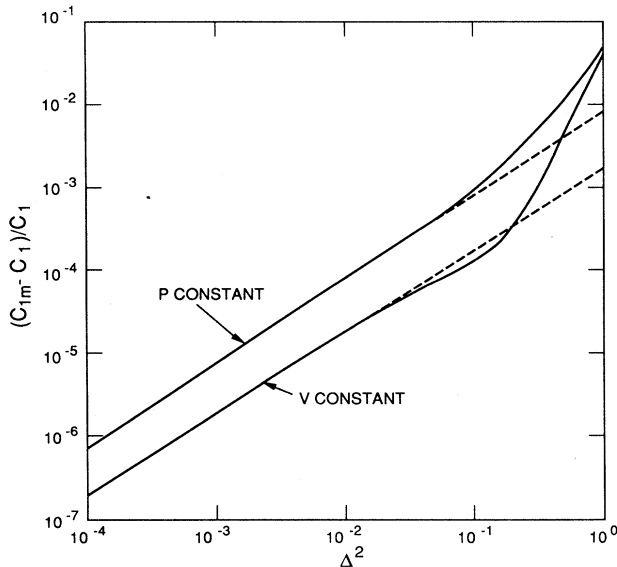


FIG. 5. A log-log plot of the relative change in the first sound velocity at constant magnetization (with constant pressure $P = 30.0$ bars and with constant density $n = 0.023 \text{ \AA}^{-3}$) vs Δ^2 .

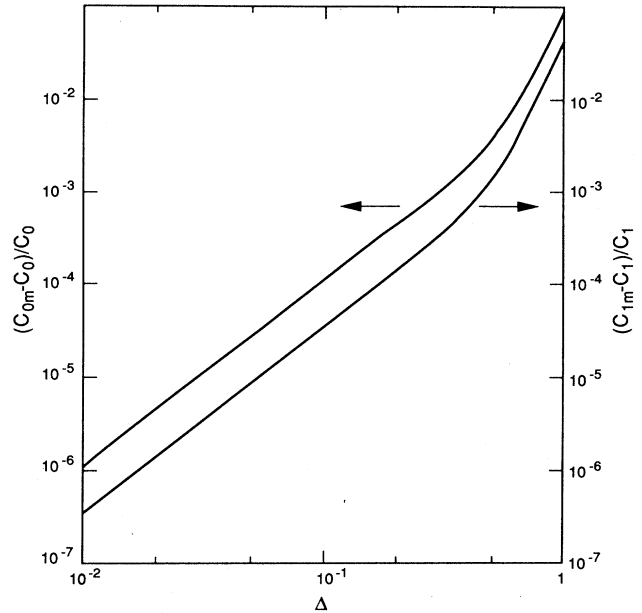


FIG. 6. The relative change in the zero sound velocity and the relative change in the first sound velocity vs Δ in a log-log scale. The density is fixed at $n = 0.023 \text{ \AA}^{-3}$.

In our discussion of the thermodynamics of polarized Fermi liquids, we noted that there are several compressibilities or equivalently several first sound velocities. The particular sound speed that propagates depends on which thermodynamic variables we keep fixed. For example, Eq. (2.8a) defines the isothermal sound speed at fixed magnetization and Eq. (2.8b) the isothermal sound speed at fixed magnetic field (or equivalently at a fixed value for the difference $\varepsilon_{k_F^\dagger}^0 - \varepsilon_{k_F^\dagger}^0$). The conditions that are needed to control which of these two modes of sound propagation will result have been discussed by Greywall and Paalanen.²⁴

Here we are interested in a feature that is common to

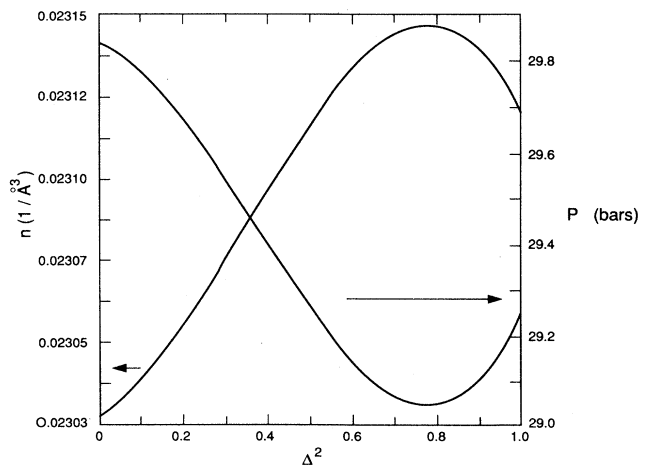


FIG. 7. A plot of the density vs Δ^2 at constant pressure $P \approx 30$ bars and a plot of the pressure vs Δ^2 at constant density $n = 0.023 03 \text{ \AA}^{-3}$.

both of these sound velocities in our model. This is the result that the change in the sound velocity up to approximately 50% polarization is determined to a good approximation by the term quadratic in Δ . For example, in Fig. 5, we plot on a log-log scale the relative change in the first sound velocity at constant $m, (c_{1m} - c_1)/c_1$, where $c_1 = c_{1m=0}$, versus Δ^2 with the additional constraint that the density is fixed. There is a small deviation downward from the Δ^2 term around 30% polarization, reflecting of course the peak in X_n . This is also the case for the fixed field. This fact that the Δ^2 feature holds over a wide range of polarization is consistent with the experiments of Bonfait *et al.*²⁵ That they did not observe the deviation downward could be due to the fact that they performed their experiments at constant pressure (or analyzed their data back to constant pressure) rather than at constant density (or as a function of density). A direct comparison between this experiment and the results given in Fig. 5 cannot be made since the experiment was carried out at a high temperature, $T \approx 200$ mK. As we will see later on, such high temperatures will have a pronounced effect on the susceptibility.

At very low temperatures we know that first sound cannot propagate; however, zero sound does propagate. In Fig. 6, we have plotted on a log-log scale $(c_{0m} - c_0)/c_0$, where $c_0 = c_{0m=0}$, and $(c_{1,m} - c_1)/c_1$, versus Δ . As we increase Δ , we see that the difference between c_{0m} and c_{1m} increases. To understand this trend, it is instructive to study the two end points of Eq. (5.11). We have that for $\Delta = 0$,

$$\frac{c_0^2 - c_1^2}{c_1^2} \approx \frac{4}{5} \frac{1}{1 + F_0^s}, \quad (8.4a)$$

and for $\Delta = 1$, i.e., $m = n$, we have

$$\frac{c_{0m}^2 - c_{1m}^2}{c_{1m}^2} \approx \frac{4}{5} \frac{1}{1 + \tilde{F}_0^{\uparrow\uparrow}}. \quad (8.4b)$$

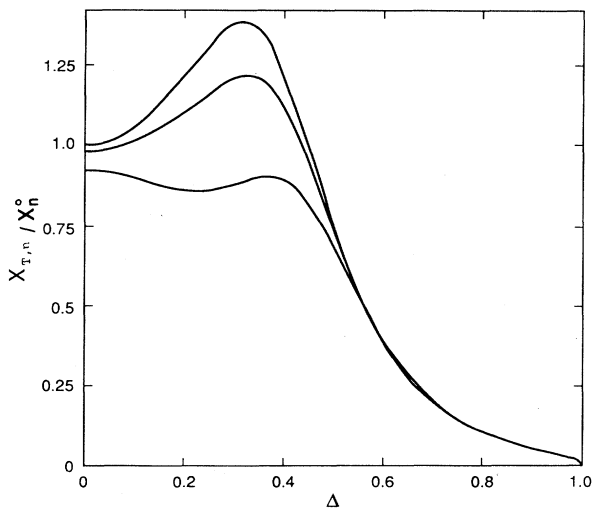


FIG. 8. The ratio of the magnetic susceptibility at $P=25$ bars and three different temperatures to the $T=0$ K unpolarized magnetic susceptibility. The upper curve at $\Delta=0.3$ is at $T=0$ K, the middle curve at $T=50$ mK and the lower curve at $T=100$ mK.

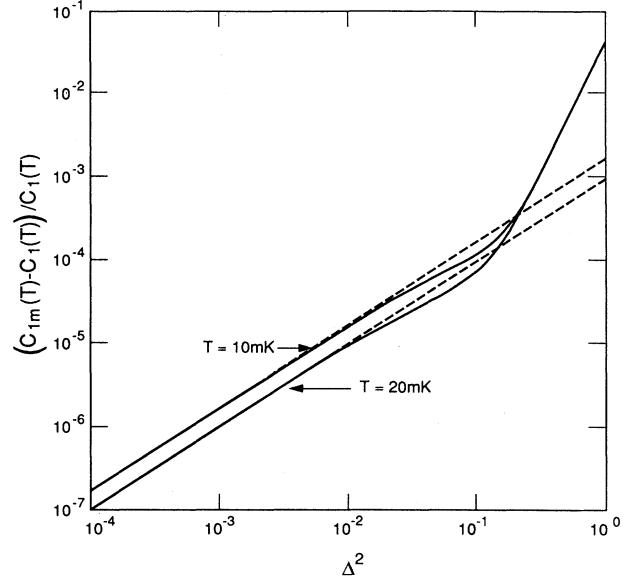


FIG. 9. A log-log plot of the relative change in the isothermal first sound velocity at constant magnetization vs Δ^2 at two different temperatures. The density is fixed at $n=0.023 \text{ \AA}^{-3}$.

From our calculated results for $\tilde{F}_0^{\uparrow\uparrow}$, we have that at high pressure

$$F_0^s \gg \tilde{F}_0^{\uparrow\uparrow}.$$

The fact that the sound velocity difference increases simply reflects the decrease in the dimensionless quasiparticle interaction strength. This decrease arising largely due to the decrease in the density of states, or equivalently the effective mass m^* .

In calculating some of the sound velocities shown in

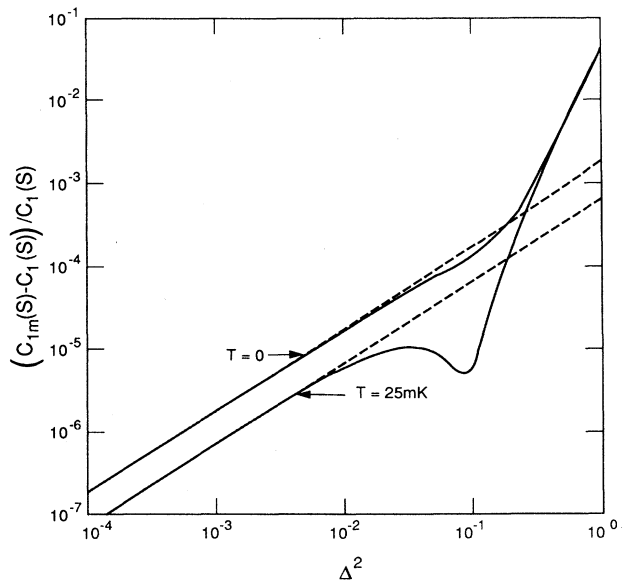


FIG. 10. A log-log plot of the relative change in the adiabatic first sound velocity at constant magnetization vs Δ^2 at two different temperatures. The density is fixed at $n=0.023 \text{ \AA}^{-3}$.

Figs. 5 and 6, we have assumed that the density or equivalently the volume is fixed as a function of Δ . This is important to take into account due to the magnetostriction effects in ^3He for finite polarization. As can be seen from Fig. 5, where the upper curve is the constant pressure plot, there are two key differences: First, the magnitude of the shift never drops below the Δ^2 term in the velocity. Second, the value of the shift is largest for the fixed P plot. At full polarization ($n_{\downarrow}=0$), the two velocities are very close with the value at fixed P just slightly higher. That the two sound velocities are close when $n_{\downarrow}=0$ follows from Eqs. (3.16) and (3.19), where the two compressibilities become equal with the one at fixed P evaluated at a higher density.

The larger value for the density at full polarization (with P fixed) is just the result of the magnetostriction effect in liquid ^3He , i.e., a decreasing volume with increasing polarization. We see from Fig. 7 that the density increases to a maximum value at $\Delta \simeq 0.9$, for $P \simeq 30$ bars, and then it decreases for $\Delta > 0.9$. This behavior is in fact consistent with the earlier discussion concerning the behavior of H_{max} . The point is that in the highly polarized phase of the "nearly metamagnetic" model of ^3He behaves like a weakly interacting Fermi gas. In the noninteracting limit, we have

$$P = \frac{2}{5}(n_{\uparrow}\epsilon_F^{\uparrow} + n_{\downarrow}\epsilon_F^{\downarrow}),$$

thus, n must decrease to keep P fixed as we increase Δ . This is also consistent with the feature seen in Fig. 7 in the plot of P versus Δ , for fixed n , for $\Delta > 0.9$. Clearly, P must increase for fixed n in the noninteracting limit.

Up to now our results have been concerned with the

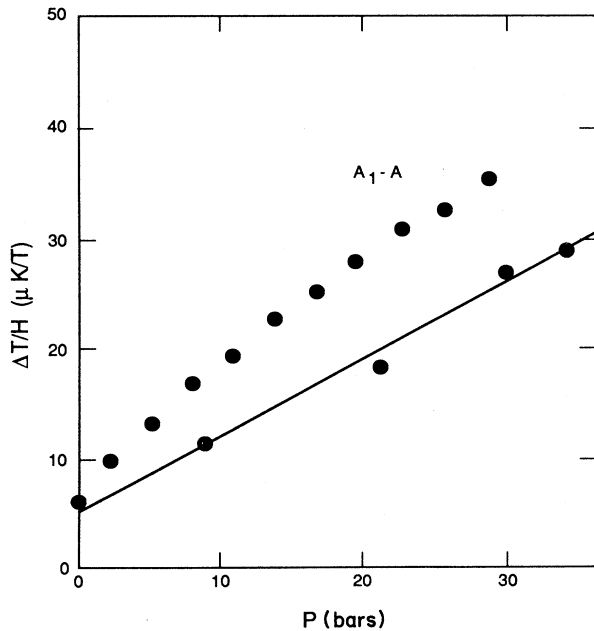


FIG. 11. The solid line joins our calculated t_{c_1} as a function of pressure. The other curve is the experimental values of Israelson *et al.* (Ref. 28).

zero-temperature properties of the model. To account for the leading order finite temperature corrections to our model, we make use of the various Maxwell relations developed in Sec. II. For example, Eq. (2.20) gives the leading temperature corrections to the susceptibility. For $T < 100$ mK, this term will give a reasonable estimate of the thermal effects on $X_{T,n}$ for several temperatures. In Fig. 8, we have plotted $X_{T,n}$ for several temperatures. The peak seen in Fig. 8 in $X_{T,n}$ at $T=0$ occurs for a magnetic field $H_{\text{peak}} \sim 100$ T ($\simeq 80$ mK), thus, it is not surprising that for $T \sim 100$ mK the structure is suppressed. Here we have a competition between the magnetic field, which is initially driving the system towards a magnetically ordered state, and the thermal fluctuations, which tend to suppress the ordering. As we get closer to the ordered state the thermal effects get larger, however, once we pass through the maximum in X_n , the finite temperature corrections get smaller. The trend we see with increasing temperature in Fig. 8 is as we would expect, however, for $T > 70$ mK, we must go beyond the leading order expansion in T given in Eq. (2.20) when the polarization is close to 35%. Although the 100 mK plot in Fig. 8 is pushing our results a bit too far, we believe it has the correct qualitative behavior, i.e., a suppression of the peak structure in $X_{T,n}$ with increasing temperature.

At finite temperature, we can have the sound propagate either at fixed temperature, i.e., isothermal first sound, or at fixed entropy, i.e., adiabatic first sound. In Fig. 9, we have plotted the relative change in the isothermal first sound velocity versus Δ^2 , on a log-log plot, at constant magnetization with the volume fixed for $T=10$ and 20 mK. Apart from the decrease in the shift of the sound velocity, compared with the $T=0$ value, Fig. 5, there is very little difference in the qualitative behavior of the $T=0$ and T finite curves. However, for the adiabatic sound there is a rather pronounced difference between the $T=0$ and 25 mK results, see Fig. 10. This is

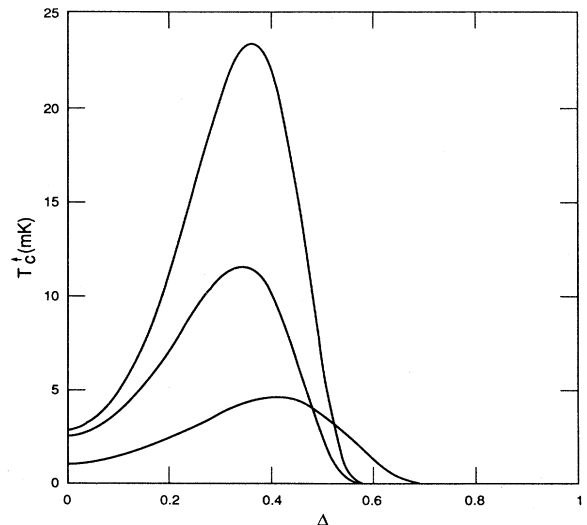


FIG. 12. Calculated T_c^{A1} vs polarization for three different pressures. Upper curve at $\Delta=0.4$ is at $P=34.36$ bars, middle curve is at $P=21.0$ bars and lower curve is at $P=0.0$ bars.

under the conditions of constant m as well as constant V . The local minimum in $[c_{1m}(S) - c_1(S)]/c_1(S)$ at $\Delta \simeq 0.35$ is approximately at the point where the maximum in X_n occurs. Clearly, the adiabatic first sound would be a more sensitive probe of the behavior of X_n than the isothermal sound.

One of the more exciting predictions of the "nearly metamagnetic" model has to do with the field dependence of the superfluid transition temperature. To determine this from theory, we need to use the generalization of the Patton-Zaringhulam formula²⁶ introduced by Bedell and Quader.^{2,20,27} Their formula for $T_c^{A_1}$ is given by

$$T_c^{A_1} = 1.13a \varepsilon_F^\dagger e^{(1/g_1^\dagger)}, \quad (8.5)$$

where a , ε_F^\dagger , and \tilde{g}_1^\dagger are a constant independent of the density and the polarization, the Fermi energy, and the pairing interaction for the up spin moments in the polarized Fermi liquid. The pairing interaction is approximated by the normal state, forward scattering amplitude in the limit of zero total momentum, that is

$$\tilde{g}_1^\dagger = \frac{N_1(0)}{6} \tilde{A}^{\uparrow\uparrow}(\Theta = \pi, \phi = 0), \quad (8.6)$$

where Θ is the angle between the momentum vectors of the incident quasiparticles and ϕ is the angle between the incident and outgoing planes. In the s - p approximation,¹⁸ $\tilde{A}^{\uparrow\uparrow}(\Theta, \phi)$ is given by

$$\tilde{A}^{\uparrow\uparrow}(\Theta, \phi) = \sum_{l=0}^{\infty} \tilde{A}_l^{\uparrow\uparrow} P_l(\cos\Theta) \cos\phi, \quad (8.7)$$

where the $\tilde{A}_l^{\uparrow\uparrow}$ are given by Eq. (6.10a). Thus, we have

$$\tilde{A}^{\uparrow\uparrow}(\Theta = \pi, \phi = 0) = \sum_{l=0}^1 (-1)^l \tilde{A}_l^{\uparrow\uparrow} = 2\tilde{A}_0^{\uparrow\uparrow}, \quad (8.8)$$

where we have kept the first two partial waves and used

the forward scattering sum rule, Eq. (6.11).

To make contact with experiment, we have expanded $T_c^{A_1}$ to linear order in the polarization, i.e.,

$$\frac{T_c^{A_1} - T_c}{H} \simeq t_{c_1}. \quad (8.9)$$

The results of this calculation are shown in Fig. 11. The agreement with experiment²⁸ is quite impressive considering the fact that we are calculating a quantity of the order of $\mu\text{K}/\text{T}$. In addition, we have included a calculation of $T_c^{A_1}$ to all orders in the polarization for three pressures in Fig. 12. The curves are qualitatively similar. It is seen that at melting pressure, $T_c^{A_1}$ initially increases with field and reaches a maximum of 23 mK at a polarization of 40%. Another striking feature of these curves is the reentrant behavior of the superfluid for temperatures larger than the zero field value of $T_c^{A_1}$. For example, at melting pressure and 5 mK, the superfluid phase reenters when $\Delta = 0.1$. The superfluid then disappears for $\Delta > 0.52$. We refer the reader to a paper by Frossati *et al.*²⁰ for a discussion of an experimental procedure for reaching this region of the phase diagram and a discussion of a number of experiments designed to explore this reentrant behavior.

The maximum value of T_c^\dagger of 23 mK at the melting pressure is achieved assuming that the melting pressure is unchanged with polarization. However, in the paper of Wiegiers *et al.*,²⁹ the melting pressure is lowered when Δ is increased. Using the "nearly metamagnetic" model, Wiegiers *et al.*²⁹ have approximately a 3-bar depression of the melting curve for $\Delta \simeq 0.35$. Thus, the maximum T_c^\dagger will be lower than 23 mK, lying somewhere between 15 and 20 mK. This is clearly a large and possibly observable increase in T_c^\dagger , and while T_c 's are notoriously difficult to predict, it is certainly worth the effort to try to observe this effect.

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