RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 40, NUMBER 4 1 AUGUST 1989

Off-diagonal long-range order in Laughlin's states for particles obeying fractional statistics

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(Received 31 March 1989)

Laughlin has proposed mean-field wave functions for particles obeying $(1-1/J)$ fractional statistics where J is an integer. We demonstrate off-diagonal long-range order in these wave functions, which is closely related to the off-diagonal long-range order in the electronic wave functions corresponding to J completely filled Landau levels. The presence of off-diagonal long-range order signifies Bose condensation of bosonlike multiplets containing J particles. For $J=2$, which may be relevant to high-temperature superconductivity, our result implies Cooper pairing in Laughlin's wave function for half-fermions.

Exotic statistics are allowed in two dimensions.[1] Interchange of two particles changes the many-body wave functions by a phase factor $e^{i\nu\pi}$, where ν defines the statistics of the particles, and in principle can assume any value in two dimensions. Fermions and bosons correspond, respectively, to $\nu=1$ and $\nu=0$, which are the only allowed statistics in three or higher dimensions. Examples of fractional statistics are found in the fractional quantum Hall effect[2] (FQHE) where the quasiparticle excitations in the Laughlin ground state for $1/m$ filling factor have been shown to obey $\nu=1/m$ statistics.[3] Laughlin has also conjectured[4] that the "holons" and "spinons"[5] in a spin-liquid state (if it exists) of a spin-$\frac{1}{2}$ antiferromagnet obey $\nu=\frac{1}{2}$ statistics.

Due to the complicated statistics, the many-body states of particles obeying fractional statistics do not, in general, subscribe to a single-particle-like interpretation as is available for bosons and fermions. The two-ν-particle problem has been studied by Arovas, Schrieffer, Wilczek, and Zee,[6] where they compute the second virial coefficient. Recently, Laughlin[4] has proposed a mean-field solution for a liquid of particles obeying $\nu=1-1/J$ statistics where J is an integer. Laughlin's wave function is given by

$$\Psi_J[\{z_i\}]=\prod_{i>j}\frac{|z_i-z_j|^{1/J}}{(z_i-z_j)^{1/J}}\Phi_J[\{z_i\}]\,, \tag{1}$$

where $\Phi_J[\{z_i\}]$ is the antisymmetric wave function for (spinless) fermions in the presence of a transverse magnetic field such that J lowest Landau levels (LL's) are fully occupied. Particle positions (x,y) are labeled by $z=x+iy$. In this paper we show that there is off-diagonal long-range order[7] (ODLRO) in the state described by this wave function. This ODLRO is, not surprisingly, very closely related to an ODLRO in $\Phi_J[\{z_i\}]$. For $J=1$ the ODLRO in $\Phi_1[\{z_i\}]$ has been studied by Girvin and MacDonald[8] (GM). However, their method cannot be generalized straightforwardly to other values of J due to the lack of a simple form for $\Phi_J[\{z_i\}]$ such as the one available for $J=1$. We instead use second-quantized many-body techniques which allow us to study the ODLRO for arbitrary J. Of course, for $J=1$ we recover the result of GM.

An exchange of two composites of n ν-particles produces a statistical phase factor of $\exp(i\pi n^2\nu)$, implying that composites of J particles obeying $(1-1/J)$ statistics behave like bosons. Therefore, we consider as a candidate for ODLRO the J-body reduced density matrix defined as[7]

$$\rho_J[\eta_1,\ldots,\eta_J;\zeta_1,\ldots,\zeta_J]=\int dz_1\cdots dz_N\Psi_J^*[\eta_1,\ldots,\eta_J,z_1,\ldots,z_N]\times\Psi_J[\zeta_1,\ldots,\zeta_J,z_1,\ldots,z_N]\,. \tag{2}$$

The study of ODLRO requires an evaluation of ρ_J in the limit $|\eta_i-\eta|\le R_\eta$, $|\zeta_i-\zeta|\le R_\zeta$, and $|\eta-\zeta|\to\infty$. We will show that in this limit $\rho_J\sim|\eta-\zeta|^{-\alpha(J)}$, where $\alpha(J)=J/2$ is the exponent characterizing ODLRO in Ψ_J. For $J=1$ this result is in agreement with that of GM.

We first redefine the problem in a more amenable form in the fermion language. Substituting $\Psi_J[\{z_i\}]$ from Eq. (1) into Eq. (2) yields, in the second quantized notation,

$$\rho_J[\eta_1,\ldots,\eta_J;\zeta_1,\ldots,\zeta_J]=\langle\psi^\dagger(\eta_1)\cdots\psi^\dagger(\eta_J)S(\eta_1,\ldots,\eta_J)\times S^*(\zeta_1,\ldots,\zeta_J)\psi(\zeta_J)\cdots\psi(\zeta_1)\rangle\,, \tag{3}$$

where the expectation value refers to the fermion ground state Φ_J,

$$S(\eta_1,\ldots,\eta_J)=\prod_{k<k'}\frac{(\eta_k-\eta_{k'})^{1/J}}{|\eta_k-\eta_{k'}|^{1/J}}\prod_{k,i}\frac{(z_i-\eta_k)^{1/J}}{|z_i-\eta_k|^{1/J}}\,, \tag{4}$$

and $\psi(z)$ is the usual destruction field operator,

$$\psi(z)=\sum_{j,m}c_{j,m}\langle z\,|\,j,m\rangle\,. \tag{5}$$

Here the quantum numbers j and m are the LL and the angular momentum indices, respectively, and $c_{j,m}$ is the destruction operator for an electron in a single-particle state $\langle z\,|\,j,m\rangle$,

$$\langle r,\theta\,|\,j,m\rangle=\frac{e^{im\theta}}{(2\pi)^{1/2}}\left[\frac{j!}{(j+m)!}\right]^{1/2}e^{-t/2}t^{m/2}L_j^m(t)\,, \tag{6}$$

where $t = r^2/2$, and $m = -j, \ -j+1, \ldots, \ 0,1, \ldots,$ and $j = 0,1 \ldots J-1$. Clearly, $S(\eta_k)$ is a multivalued function and the off-diagonal density matrix defined in Eq. (2) depends upon the choice of the cuts as well as the branch on which the integral is evaluated. This fundamental difficulty with the very definition of the off-diagonal matrix elements arises in the case of fractional-statistics particles due to the multivalued nature of their wave functions. Even though we do not resolve the issue of an unambiguous definition of a generalized reduced density matrix, we note that the ambiguity is not so severe insofar as the exponent α describing the asymptotic power-law behavior of the ODLRO is concerned; a large class of cuts produces a unique value of the exponent even though the prefactor depends on the specific choice of the cuts. Consider a pair of nonoverlapping disks D_η and D_ζ of radii R_η and R_ζ with all points η_k lying in D_η and all points ζ_k lying in D_ζ. Then the exponent will be independent of R_η, R_ζ and the specific choice of cuts so long as one chooses branch cuts which are confined within one disk or other [in which case $S(\eta_i)S(\zeta_i)$ is single valued outside the disks] and R_η and R_ζ are held fixed as $|\eta-\zeta| \to \infty$. This results from the fact, which will become clear later in the paper, that the exponent is determined entirely by the integrand far from the points η and ζ. In light of this assertion, we approximate $S(\eta_k)$ by $S(\eta_k = \eta)$, and write, dropping the irrelevant phase factors,

$$S(\eta) = \prod_{i=1}^{N} \frac{(z_i - \eta)}{|z_i - \eta|} , \tag{7}$$

and a similar expression for $S(\zeta_k)$. In doing so we have completely eliminated the complications arising from the multivaluedness of S, without affecting the desired exponent. Notice that it is not allowed to identify the arguments of the creation or destruction field operators; ρ_J vanishes identically if any two of the η_k's or any two of the ζ_k's are equal.

Anticipating ODLRO, we write the following factorized asymptotic form for ρ_J:

$$\rho[\eta_k, \zeta_k] = g[\eta_k] g^\dagger[\zeta_k] , \tag{8}$$

$$g[\eta_k] \equiv \langle N+J | \psi^\dagger(\eta_J) \cdots \psi^\dagger(\eta_1) S(\eta) | N \rangle , \tag{9}$$

where $|N\rangle$ is the N-fermion state with filling factor J. The exponent α is closely related to the size dependence of the correlation function g, whose asymptotic form we now investigate. Without any loss of generality, we take η to be the origin. Application of the operator S on the N-particle state moves the particles away from the origin, thus leaving a charge deficiency near the origin. The correlation function g is the projection of the resulting state onto the N-particle state obtained by bodily removing J particles at $\eta_1, \ldots, \eta_J$ from the $(N+J)$-particle state. In second-quantized form

$$S = \frac{1}{N!} \sum_{\{l_i\}} \sum_{\{l_i'\}} \left(\prod_{i=1}^{N} T_{l_i, l_i'} c_{l_i}^\dagger c_{l_i'} \right) , \tag{10}$$

where the subscript l_i denotes a single-particle state (j,m), and

$$T_{jm,j'm'} \equiv \langle jm | \frac{z}{|z|} | j'm' \rangle . \tag{11}$$

Thus, under the operation of S on $|\{l_i'\}\rangle$, each single-particle state l_i' is mapped onto a single-particle state l_i

$$S | \{l_i'\} \rangle = \sum_{\{l_i\}} \left[\prod_i T_{l_i l_i'} \right] | \{l_i\} \rangle , \tag{12}$$

where the sum is over all distinct configurations $\{l_i\}$. Notice, (i) all l_i (also, all l_i') are distinct due to Fermi statistics, implying that the mapping is one-one. (ii) $T_{jm,j',m'}$ is nonzero only if $m = m'+1$. Thus, under the mapping the angular momentum quantum number of each particle is increased by one. It is useful to depict the single-particle states by points in the $j-m$ space and the mapping by arrows, as shown in Fig. 1. To each arrow connecting (j',m') to (j,m) is associated a matrix element $T_{jm,j'm'}$ and the product of all the matrix elements in a given mapping determines the amplitude of the resulting configuration. It is easy to see that S creates J holes near the origin, which makes it clear why one must consider the J-body reduced matrix, ρ_J, as the possible candidate for ODLRO; there is no possibility of ODLRO in $\rho_{J'}$ with $J' < J$, which is not surprising since a composite of J particles [each obeying $(1-1/J)$ statistics] is the smallest group that behaves like a boson.

Let us start with an initial fermion state $|\{l_i'\}, M\rangle$ in which all single-particle states $|j,m\rangle$ with $-j \le m \le M$ and $0 \le j \le J-1$ are occupied. S maps it onto states $|\{l_i\}\rangle$ with all angular momentum states $-j \le m \le M+1$ in the J LL's are occupied except for J holes near the origin, such that there is one hole for each angular momentum state $m \le 0$ (which can be in any available LL). Thus, there are $J!$ hole configurations, and all states with a given hole configuration λ can be obtained from a given state $|\Phi_\lambda\rangle$ with the same hole configuration by permutation of particles. Thus, we can write

$$g = \sum_{\lambda=1}^{J!} a_M^\lambda \langle \{l_i\}, M+1 | \psi^\dagger(\eta_J) \cdots \psi^\dagger(\eta_1) | \Phi_\lambda \rangle , \tag{13}$$

$$a_M^\lambda = \sum_{\{l_i\}} (-1)^P \prod_i T_{l_i l_i'} , \tag{14}$$

where the sum is over all distinct states $\{l_i\}$ that can be obtained from Φ_λ after P permutations. The asymptotic size dependence of g is the same as that of the coefficient a's. Evaluation of these coefficients is, in general, rather for-

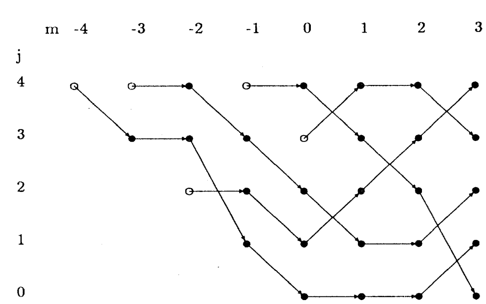

FIG. 1. A typical process illustrating the effect of the operator S (defined in the text) which maps each single-particle state (j,m) to another single-particle state $(j',m+1)$ as indicated by the arrows. J holes are created near the origin, one for each $m \le 0$.

midable, but we have been able to accomplish it in the limit $M \rightarrow \infty$ to order $(1/M)$, which is sufficient to yield the exponent characterizing the ODLRO. The simplifying feature that enables us to determine the exponent is the observation that S connects angular momentum m only to $m+1$, which allows us to relate a_M and a_{M+1} and obtain an asymptotic solution. The procedure is still rather involved[9] and here we give only a brief outline of the details. First, it can be shown that the leading order behavior of $T_{jM+1,j'M}$ is given by

$$T_{jM+1,j'M} \sim O(M^{-|j-j'|/2}) \,. \tag{15}$$

This implies that of the $J!$ possible terms, which correspond to $J!$ ways of connecting J points with angular momentum M to J points with angular momentum $M+1$, only the J terms of the type shown in Fig. 2 make a contribution to order $(1/M)$. The relative sign between the first and the rest of the terms takes care of the fact that they differ by an odd number of permutations. Thus, we get

$$a_{M+1} = a_M \prod_{j=0}^{J-1} T_{jM+1,jM} \times \left[1 - \sum_{i=0}^{J-2} \frac{T_{i+1M+1,iM} T_{iM+1,i+1M}}{T_{iM+1,iM} T_{i+1M+1,i+1M}} \right] . \tag{16}$$

We need to know all the matrix elements to order $(1/M)$. Here, we quote the results

$$T_{jM+1,jM} = 1 - \frac{2j+1}{8M} + O\left[\frac{1}{M^2}\right] , \tag{17}$$

$$T_{j+1M+1,jM} = -T_{jM+1,j+1M} = \left[\frac{j+1}{4M}\right]^{1/2} + O\left[\frac{1}{M^{3/2}}\right] , \tag{18}$$

with the help of which we obtain to $O(1/M)$

$$a_{M+1} = a_M \left[1 - \frac{J}{8M} \right] . \tag{19}$$

This implies that asymptotically $g \sim a_M \sim M^{-J/8} \sim R^{-J/4}$, since the size (radius) R of the system $R \sim M^{1/2}$. In Eq. (8), the distance $|\eta - \zeta|$ provides the natural scale for the distance from η and ζ where the integrals can be cut off. This finally leads to

$$\rho_J(\eta_i,\zeta_i) \sim |\eta - \zeta|^{-J/2} , \tag{20}$$

proving the existence of algebraic ODLRO in a gas of particles obeying $(1-1/J)$ statistics.

In the limit $M \rightarrow \infty$ the matrix elements $T_{jM+1,j'M}$ get exponentially vanishing contribution from z close to the origin, as the states $|j,M\rangle$ with large M have exponentially vanishing amplitude at the origin. Since the exponent is completely determined by the matrix elements in this limit, it is unaffected by the specific choice of the cuts so long as they are confined within a finite region near the origin, as indicated earlier. It is also worth mentioning that a proper treatment of the interference between the LL's (i.e., of the inter-LL coupling) is necessary to obtain the correct power-law behavior.

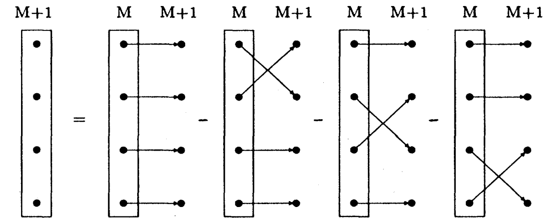

FIG. 2. The coefficient a_{M+1} is obtained from a_M. Only the terms that make a contribution to $O(1/M)$ are shown.

It is interesting to note the analogies between our study, where we investigate ODLRO in Laughlin's wave functions for particles obeying fractional statistics, and the work of GM, where they find ODLRO in Laughlin's wave functions for FQHE. GM find that for a partially filled LL with filling factor $1/J$, condensation occurs of composite objects consisting of one electron and J gauge flux tubes. We find that for J filled LL's, the condensing objects are comprised of J electrons and one gauge flux tube. This is not surprising since, in view of previous work,[8,10] one should expect the ratio of the number of electrons to the number of gauge flux tubes to be just the filling factor. Also, we find the same power-law behavior for filling factor J as GM do for filling factor $1/J$. Due to the appearance of gauge flux tubes, which introduce long-range gauge forces, the ODLRO in the *fermion* states is of a peculiar and unusual nature. However, in the case of filling factor J, a composite of J fermions and one flux tube is analogous to a composite of J fermions each carrying a $(1/J)$-flux tube, which is nothing but a composite of J particles obeying $(1-1/J)$-fractional statistics. Therefore, the peculiar ODLRO in the fermion system at filling factor J implies a *usual* ODLRO for a liquid of particles obeying $(1-1/J)$ statistics, in which case condensation occurs of J particles alone.

The asymptotic analysis in this paper implicitly assumes a finite J, and the results are not valid in the limit $J \rightarrow \infty$, which corresponds to fermions themselves.

In conclusion, we have identified the order parameter for an electron system containing J filled LL's and obtained the form of the algebraic ODLRO. This result translates into an algebraic ODLRO in Laughlin's liquid state for particles obeying $(1-1/J)$ statistics. The order parameter consists of J fractional statistics particles. In particular, for bosons ($J=1$), the ODLRO is analogous to Bose condensation, and for $\frac{1}{2}$-fermions ($J=2$), pairs of particles condense, which is analogous to superconductivity. These results strongly suggest that Laughlin's states *do* exhibit superfluidity.

N.R. thanks D. J. Thouless and S. M. Girvin for helpful comments.

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