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# Crossover theory for disordered ferromagnets

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We show that the critical behavior of disordered ferromagnets can be understood in terms of an effective, translational invariant Hamiltonian with renormalized coupling parameters which exhibit a tricritical as well as a critical point. The stable critical fixed point describes the critical behavior of weakly disordered systems, whereas the tricritical fixed point is responsible for the critical behavior of strongly disordered systems at least in the experimentally accessible range of temperature. This necessitates a crossover theory to describe the crossover between these two fixed points and to calculate critical properties as a function of concentration and temperature. In this paper we study the symmetric phase. Susceptibility and specific heat are calculated in scaling form using nonlinear scaling fields to verify that disordered ferromagnets exhibit universal critical behavior. The corresponding effective exponents exhibit nonmonotonous concentration- and temperature-dependent behavior. According to our theory, Fisher-renormalized tricritical behavior dominates in the experimentally accessible temperature range which leads to extreme values of the effective exponents $\gamma_{\text{eff}}=2$ and $\alpha_{\text{eff}}=-1$ in the limit of strong dilution. However, in accord with the Harris criterion we verify that the asymptotic behavior ($T \rightarrow T_c$) is not changed by dilution if the critical exponent of the specific heat of the pure system $\alpha_{\text{pure}}<0$, independent of the value of the concentration.

## I. INTRODUCTION

The critical behavior of ferromagnets has attracted considerable experimental and theoretical interest over the years. Related topics such as random-field systems, random-axis models, and spin glasses have appeared and are still of current interest. However, the original problem is not solved so far. The existing theories are partly contradictory and do not sufficiently explain the experiments.

Early investigations of disordered ferromagnets were performed on the basis of mean-field theories. High-temperature and concentration expansions[1-4] led to sensible results only for nonuniversal properties such as phase diagrams but failed in the critical region. A breakthrough to an understanding of the critical properties was achieved by the renormalization-group (RG) method. This method was first applied to systems with weak quenched disorder.[5-10] An important result was the confirmation of Harris's heuristic argument[11] that the critical behavior is unchanged by dilution if $\alpha_{\text{pure}}<0$. For systems with $\alpha_{\text{pure}}>0$, the critical exponents should change due to a new random fixed point. The common feature of these renormalization-group treatments is their restriction to weak quenched disorder so that their results are hardly applicable to strongly disordered ferromagnets.

A conceptionally different approach, which may be termed the quasiequilibrium method, has been proposed by Sobotta and Wagner.[12-14]

The basic idea is to describe the state of quenched disorder in terms of forces of constraints in analogy to the method of forces of constraints in classical mechanics. This idea has been introduced by Morita.[15] In this approach the quenched random ferromagnet is described in an equilibrium ensemble characterized by temperature, magnetic field, and generalized chemical potentials which are conjugate to products of the occupation numbers, which describe the quenched disordered state. In an information theoretical approach to statistical mechanics, it has been shown[13] that this method is equivalent to the commonly applied proposal of Brout,[9] who defines the true free energy of quenched disordered systems by averaging the free energy over all configurations. The quasiequilibrium method has been applied to dilute magnets with short-ranged interactions up to $O(\epsilon^2)$ (Ref. 14) as well as to dipolar and long-ranged isotropic interactions to $O(\epsilon)$.[16,17] In all cases a new random tricritical fixed point appears which shows distinctly different exponents compared to pure systems. A search for a random tricritical point by other methods was inconclusive so far.[18] The tricritical fixed point found in Refs. 12, 14, 16, and 17 is unstable but his exponents should be relevant for strongly diluted ferromagnets, because the RG flow drives these systems into the vicinity of this fixed point. Weakly diluted systems are driven directly to the stable random fixed point which governs the asymptotic behavior of ferromagnets with arbitrary concentration. A new stable fixed point appears only if $\alpha_{\text{pure}}>0$. It is characterized by Fisher-renormalized critical exponents.[12,14] Systems with $\alpha_{\text{pure}}<0$, such as dipolar magnets, do not show up a new stable random fixed point. The fixed point of the pure system remains stable so that the asymptotic behavior is not changed.[16] To summarize, the quasiequilibrium method supports the Harris criterion even in the limit of low concentration of magnetic atoms, but the values of the asymptotic critical exponents are different from other works on the same subject. Even more important is the occurrence of the tricritical fixed point which describes strongly diluted ferromagnets independent of the type of interaction.

In this paper we perform crossover calculations to investigate the crossover between the fixed points obtained by the quasiequilibrium method. We discuss the range of temperature and concentration, where one should observe the exponents of the tricritical fixed point and investigate the asymptotic temperature range, where the exponents of the stable critical fixed point apply. For dilute ferromagnets of arbitrary concentration, we calculate the crossover functions and effective critical exponents,[19] as well as the equation of state in the critical region. For the sake of clarity we split up the subject into two parts. The first covers the symmetric phase above $T_c$ without a magnetic field. This can be done within the frame of a $S^4$ theory. The symmetry-broken phase is treated in a forthcoming paper,[20] since one has to include a $S^6$ operator to calculate the equation of state of strongly diluted magnets. This is necessary because the effective $S^4$ coupling vanishes near the limiting concentration where ferromagnetism breaks down.

The outline of this paper is as follows. In Sec. II we briefly introduce our model and the quasiequilibrium method. In Sec. III we show that the disordered ferromagnet can be described by an effective spin Hamiltonian, which contains the same operators as the Hamiltonian of the corresponding pure ferromagnet. However, the coupling parameters are renormalized by the conditions of quenched randomness. Specifically, we point out that the renormalized temperature is a nonanalytic function of the reduced temperature $t=(T-T_c)/T_c$. This leads to a Fisher renormalization of the critical behavior of disordered ferromagnets. We show that these results are not changed in $O(\epsilon^2)$. In Sec. IV we work out in detail the crossover of the effective spin Hamiltonian on the basis of the trajectory integral method of Rudnick and Nelson.[23,24] Combining this with our results from Sec. III we obtain in Sec. V the complete crossover of the magnetic and caloric properties of disordered ferromagnets. The crossover functions of the susceptibility and specific heat are calculated using nonlinear scaling fields. We also evaluate the effective exponents in order to compare our results with experiments.

## II. THE MODEL AND METHOD

We study a quenched spin system with site disorder. Starting from a regular lattice, we occupy each site $i$ of the lattice independently by a magnetic atom with probability $p$ and by a nonmagnetic atom with probability $1-p$. We introduce occupation numbers $\tilde{K}_i$ which take the value $\tilde{K}_i=1$ if a magnetic atom is at site $i$ and $\tilde{K}_i=0$ otherwise. The Hamiltonian of the diluted spin system is then given by[12,14]

$$\mathcal{H}(K,\mathbf{S})=\sum_{i,j} J_{ij}\tilde{K}_i\mathbf{S}_i\tilde{K}_j\mathbf{S}_j+H_m\sum_i \tilde{K}_i S_i^1 \;. \tag{2.1}$$

In this paper, $J_{ij}$ is a short-ranged interaction of spins $\mathbf{S}_i$ and $\mathbf{S}_j$ of arbitrary dimension $n$. $H_m$ is the magnetic field in 1 direction. The factor $\beta=(kT)^{-1}$ is included in $\mathcal{H}$ as usual. A statistical description of a quenched spin system has to take into account the fundamental physical difference between spins and occupation numbers. The spins $\{\mathbf{S}_i\}$ are normal degrees of freedom. The occupation numbers $\{\tilde{K}_i\}$ are fixed by preparation and do not fluctuate at all. In order to apply convenient statistical mechanics to the complete system, we formally treat the occupation numbers as normal degrees of freedom. However, the conjugate generalized chemical potentials

$$\lambda_{i_1,\ldots,i_n}(T,H,\{\overline{\tilde{K}_{j_1},\ldots,\tilde{K}_{j_n}}\})$$

are determined in such a way that the configuration $\{\tilde{K}_i\}$ and its moments $\{\overline{\tilde{K}_{j_1}\cdots\tilde{K}_{j_n}}\}$, respectively, are kept fixed.[12-14] The Hamiltonian is then given by

$$\mathcal{H}=\mathcal{H}[\tilde{K},\mathbf{S}]+\beta^{-1}\sum_{\{i_1,\ldots,i_n\}}\lambda_{i_1,\ldots,i_n}\tilde{K}_{i_1},\ldots,\tilde{K}_{i_n}\;, \tag{2.2}$$

and the partition function is

$$Z(T,H,\{\lambda_{i_1\ldots i_n}\})=\mathrm{Tr}_{\mathbf{S},\tilde{K}}\,e^{\mathcal{H}}\;. \tag{2.3}$$

The functions

$$\lambda_{i_1\ldots i_n}(T,H,\{\overline{\tilde{K}_{j_1}\ldots\tilde{K}_{j_n}}\})$$

are calculated from the conditions of quenched randomness:

$$\langle\tilde{K}_{i_1}\ldots\tilde{K}_{i_n}\rangle=\overline{\tilde{K}_{i_1}\ldots\tilde{K}_{i_n}}\;, \tag{2.4}$$

where

$$\langle\cdots\rangle=\frac{1}{Z}\mathrm{Tr}_{\mathbf{S},\tilde{K}}\,e^{\mathcal{H}}\;. \tag{2.5}$$

It has been proven in a previous paper[13] that the thermodynamic properties calculated by this method should be the same as by Brout's configurational average.[9]

The advantage of our method is that one avoids averaging the logarithm of the partition function. However, in principle, the Hamiltonian (2.2) contains infinitely many chemical potentials. We introduce deviation numbers $K_i=\tilde{K}_i-p$ and Fourier-transformed variables $\{K_{\mathbf{q}}\}$ and $\{\mathbf{S}_{\mathbf{q}}\}$ to obtain a field-theoretic formulation.[12,14] The Hamiltonian, including the factor $(-\beta)$, then reads:[12,14]

$$\begin{aligned}\mathcal{H}=gK_0&-\frac{B}{2}\int_{\mathbf{q}}K_{\mathbf{q}}K_{-\mathbf{q}}-\frac{1}{2}\int_{\mathbf{q}}\mathbf{S}_{\mathbf{q}}\cdot\mathbf{S}_{-\mathbf{q}}(r+q^2)\\&-u\int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3}\mathbf{S}_{\mathbf{q}_1}\cdot\mathbf{S}_{\mathbf{q}_2}\mathbf{S}_{\mathbf{q}_1}\cdot\mathbf{S}_{-(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)}\\&+R\int_{\mathbf{q}_1}\int_{\mathbf{q}_2}K_{\mathbf{q}_1}\mathbf{S}_{\mathbf{q}_2}\cdot\mathbf{S}_{-(\mathbf{q}_1+\mathbf{q}_2)}+HS_0^1+h\int_{\mathbf{q}}K_{\mathbf{q}}\mathbf{S}_{-\mathbf{q}}^1\,,\end{aligned} \tag{2.6}$$

which can be written in short notation as

$$\mathcal{H}=gK_0-\frac{B}{2}K^2-\frac{r}{2}S^2+RKS^2-uS^4+HS_0^1+hKS_0^1\;. \tag{2.7}$$

In Eq. (2.6) we have used the abbreviation

$$\int_{\mathbf{q}} := \frac{1}{(2\pi)^d} \int_0^{\Lambda} d^d q \ ,$$

where $\Lambda$ is the cutoff. Equation (2.6) contains the usual spin operators $S^2$, $S_0^1$, and $S^4$ known from the pure spin system. The coupling parameters are $r \sim t = (T-T_c)/T_c$, $H \sim H_m$, and $u$. There are two operators $KS^2$ and $KS$, which couple spins and occupation numbers. Their coupling parameters $R$ and $h$ depend on the concentration. They obviously vanish for the pure system and increase with dilution. As usual, the exact dependence of the field-theoretic coupling parameters on temperature, concentration, and on the nonuniversal properties of the system is not known.

Only the first- and second-order terms in $K$, with the coupling parameters $g$ and $B$, are included in the Hamiltonian (2.6). $K$ operators of higher order in $K$ $(K^3, K^4, \ldots)$ or $\mathbf{q}$ dependence have been shown to be irrelevant in an $\epsilon$ expansion to $O(\epsilon)$ using the same arguments as in pure spin systems based on their scaling dimension and leading contributions in the RG equations.[12,14] The RG is applied to the Hamiltonian (2.6) similar to pure spin systems. In addition to the spin rescaling $\mathbf{S}_{\mathbf{q}}^{<} = \zeta \mathbf{S}'_{\mathbf{q}'}$, we introduce the two-parameter transformation $K_{\mathbf{q}}^{<} = \kappa K'_{\mathbf{q}'} + \rho \delta(\mathbf{q})$, where $\rho$ and $\kappa$ are determined by the condition of fixed moments:

$$\begin{aligned} &\langle K_0 \rangle = 0 \ , \\ &\langle K_{\mathbf{q}} K_{-\mathbf{q}} \rangle = p(1-p) \ , \end{aligned} \tag{2.8}$$

which leads to $1/B = p(1-p)$ in $O(\epsilon)$. We refer to Refs. 12 and 14 for details of the calculation. Besides the Gaussian and Heisenberg fixed point, one obtains two new random fixed points: an unstable tricritical fixed point and a new stable random fixed point. Both fixed points are characterized by $X = R^2/B = O(\epsilon)$. The RG flow shows that strongly disordered systems are characterized by $u \gtrsim X/2$. These systems are driven first to the tricritical fixed point before they approach the asymptotically stable fixed point. Weakly diluted systems are driven directly to this fixed point.

Before starting crossover calculations, we want to point out that the Hamiltonian (2.6) contains a redundant operator which gave rise to some ambiguities in previous papers.[14] In analogy to the redundant operator $\delta\mathcal{H}^*/\delta S_0$ of pure systems,[21] it is easy to show that the operator

$$\frac{\delta \mathcal{H}^*}{\delta K_0} = -B \left[ K_0 - \frac{R^*}{B} S^2 \right]$$

is redundant, where $\mathcal{H}$ is given by (2.6) (Ref. 22). Of course, the RG is simplified if the redundant operator is eliminated. The resulting recursion relations show that only the coupling parameters $\tilde{r} = r - 2gR/B$ and $\tilde{u} = u - X/2$ are physically relevant. The coupling parameter $X = R^2/B$ enters the physically significant RG transformation only via the effective coupling parameter $\tilde{u} = u - X/2$ so that only two physically different fixed points exist.[22]

### III. THE EFFECTIVE SPIN HAMILTONIAN

The reduction of the Hamiltonian $\mathcal{H}(K,\mathbf{S})$ [Eq. (2.6)] to an effective spin Hamiltonian can be readily derived by direct integration over the occupation numbers $\{K_{\mathbf{q}}\}$ in Eq. (2.3). This procedure is always applicable if additional nonordering degrees of freedom (phonons, impurities, etc.) are present which do not show critical fluctuations. Since we study quenched disorder, this condition is naturally fulfilled by the occupation numbers in our quasiequilibrium method $[B = O(1)]$.[12,14] As the RG transformation in $K-\mathbf{S}$ space in $O(\epsilon)$ has shown, the Hamiltonian (2.6) contains only $K-\mathbf{S}$ operators linear in $K$ in our $O(\epsilon)$ calculation. Therefore, the integrations over $\{K_{\mathbf{q}}\}$ in

$$Z = \int_{\mathbf{S}} \int_K \exp \mathcal{H}(K, \mathbf{S}) \tag{3.1}$$

can be carried out exactly leading to

$$Z = \int_{\mathbf{S}} \exp \mathcal{H}_{\text{eff}}(\mathbf{S}) \cdot \exp(-\Delta F) \tag{3.2}$$

with the effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = -\tfrac{1}{2} \tilde{r} S^2 - \tilde{w} S^3 - \tilde{u} S^4 + \tilde{H} S_0^1 \ . \tag{3.3}$$

The effective coupling parameters are related to the original coupling parameters by

$$\begin{aligned} \tilde{r} &= r - 2\frac{gR}{B} + \frac{h^2}{B} \ , \\ \tilde{w} &= -\frac{Rh}{B} \ , \\ \tilde{u} &= u - \frac{R^2}{2B} \ , \\ \tilde{H} &= H + \frac{gh}{B} \ . \end{aligned} \tag{3.4}$$

The free energy is then given by

$$F = \tilde{F} + \Delta F \tag{3.5}$$

with the singular part

$$\tilde{F} = -\ln \int_{\mathbf{S}} \exp \mathcal{H}_{\text{eff}}(\mathbf{S}) \tag{3.6}$$

and a regular part which results from the integrations over $K$:

$$\Delta F = -\frac{g^2}{2B} + \frac{1}{2} \int_{\mathbf{q}} \ln B \ . \tag{3.7}$$

The effective coupling parameters $\tilde{r}$ and $\tilde{u}$ (3.4) are exactly those combinations of the original coupling parameters which have been shown to be the only physically relevant coupling parameters in $\mathcal{H}(K,\mathbf{S})$.[22] $\tilde{r}$ and $\tilde{H}$ contain additional quadratic terms $h^2/B$ and $gh/B$, respectively. Since $g \sim r$ (see below) these terms are small in the critical region $(t, r \ll 1,\ H, h \ll 1)$ compared to terms linear in $r$ and $H$. The integrations over $\{K_{\mathbf{q}}\}$ have produced an operator $\tilde{w} \cdot \mathbf{S} \cdot \mathbf{S} \cdot S^1$ of $O(Rh/B)$. This operator can be neglected because of the fixed point value $\tilde{w}^* = 0$ and because it leads to terms of

$$O\left[ \tilde{w}^2 = \frac{R^2}{B} \frac{h^2}{B} \right]$$

and

$$O(\tilde{u}\tilde{w}^2,\tilde{w}^4)\,,$$

respectively, in the recursion relations for $\tilde{r}$ and $\tilde{u}$. The recursion relation for $\tilde{H}$,

$$\tilde{H}'=\zeta(\tilde{H}-3\tilde{w}A_1)\,, \tag{3.8}$$

is affected linearly by $\tilde{w}$, but this does not change the magnetic exponent $y_{\tilde{H}}$. One therefore recovers the well-known recursion relations of the $S^4$- model:

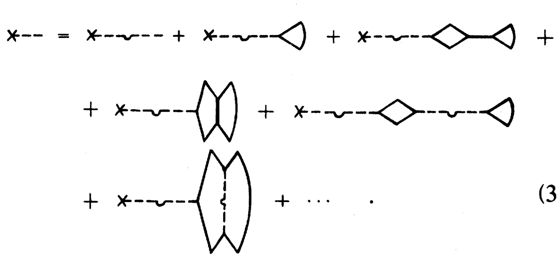
$$\begin{aligned}\tilde{r}'&=\zeta^2 b^{-d}[\tilde{r}+4(n+2)K_4\tilde{u}A_1]\,,\\ \tilde{u}'&=\zeta^4 b^{-3d}[\tilde{u}+4(n+8)K_4\tilde{u}^2A_2]\,,\\ \tilde{H}'&=\zeta\tilde{H}\,,\end{aligned} \tag{3.9}$$

with

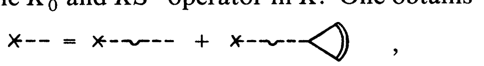
$$\begin{aligned}\tilde{r}&=r-2g\frac{R}{B}\,,\\ \tilde{u}&=u-\frac{R^2}{2B}\,,\\ \tilde{H}&=H\,.\end{aligned} \tag{3.10}$$

Thus, the renormalization of $\mathcal{H}(K,\mathbf{S})$ (2.6) (Refs. 12 and 22) and of $\mathcal{H}_{\text{eff}}(\mathbf{S})$ (3.8) lead to identical results in terms of the renormalized parameters $\tilde{r}$, $\tilde{u}$, and $\tilde{H}$.

The conditions of quenched disorder (2.8) relate the effective spin Hamiltonian (3.9) to the diluted ferromagnet we want to describe. The concentration $\langle K_0\rangle$ or more precisely the deviation from concentration $p$ can be calculated by a Feynman-graph expansion of $\mathcal{H}(K,\mathbf{S})$ with respect to the $K_0$, $KS^2$, and $S^4$ operator:

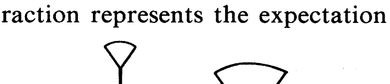

(3.11)

This series can be summed up exactly due to the linearity of the $K_0$ and $KS^2$ operator in $K$. One obtains


(3.12)

where the double contraction represents the expectation value

$$\int_q\langle\mathbf{S}_q\cdot\mathbf{S}_{-q}\rangle_c=$$


(3.13)

This can be written as the energy in terms of the effective spin system $\mathcal{H}(\mathbf{S})$ (3.3):

$$\tilde{E}=\left.\frac{\partial\tilde{F}}{\partial\tilde{t}}\right|_{\tilde{H}}=\left.\frac{\partial\tilde{F}}{\partial\tilde{r}}\right|_{\tilde{H}}=-\tfrac{1}{2}\int_q\langle\mathbf{S}_q\cdot\mathbf{S}_{-q}\rangle_c\,, \tag{3.14}$$

where $\tilde{F}$ is given by (3.6). We have used the scaling field[19] $\tilde{t}=\tilde{r}+2K_4(n+2)\tilde{u}$ in (3.14), which is the temperature of the effective spin system. Inserting (3.14) into (3.12) gives

$$\langle K_0\rangle=\frac{g}{B}+2\frac{R}{B}\tilde{E}(\tilde{t},\tilde{u},\tilde{H})\,, \tag{3.15}$$

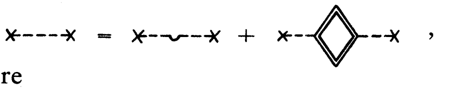
which is the exact equation of state for the concentration. Starting from the Hamiltonian (2.6) one can alternatively derive this equation using $g$ as a source term [see (2.6)].

$$\langle K_0\rangle=-\left.\frac{\partial F}{\partial g}\right|_{r,\ldots}\,, \tag{3.16}$$

where $F$ is the free energy of $\mathcal{H}(K,\mathbf{S})$ in terms of $r,g,u,\ldots$ given by $F=-\ln Z$ (3.1). Applying the relations (3.5)–(3.7) between $F$ and $\tilde{F}$ and the dependence of $\tilde{r}$ on $g$ (3.10), leads to

$$\langle K_0\rangle=\frac{g}{B}-\left.\frac{\partial\tilde{F}}{\partial\tilde{r}}\right|_{\tilde{H},\tilde{u}}\left.\frac{\partial\tilde{r}}{\partial g}\right|_{r} \tag{3.17}$$

and again to (3.15). Similarly, the equation of state for the variance $\langle K_{\mathbf{q}}K_{-\mathbf{q}}\rangle$ can be calculated. The Feynman-graph expansion can be summed up to give


(3.18)

where

$$=\int_q\int_{q'}\langle\mathbf{S}_q\cdot\mathbf{S}_{-q}\mathbf{S}_{q'}\cdot\mathbf{S}_{-q'}\rangle_c \tag{3.19}$$

is the four-point function $\tilde{\Gamma}^{(4)}(\tilde{r},\tilde{u},\mathbf{q})$. Equation (3.19) represents the energy density fluctuation, which for $\mathbf{q}=0$ is the specific heat

$$\tilde{C}(\tilde{t},\tilde{u},\tilde{H})=\left.\frac{\partial^2\tilde{F}}{\partial\tilde{t}^2}\right|_{\tilde{H},\tilde{u}} \tag{3.20}$$

of the effective spin system (3.3). Equations (3.18)–(3.20) lead to the equation of state of the variance

$$\langle K_{\mathbf{q}}K_{-\mathbf{q}}\rangle=\frac{1}{B}+\frac{R^2}{2B^2}\tilde{C}(\tilde{t},\tilde{u},\tilde{H})\,, \tag{3.21}$$

where we neglected the $\mathbf{q}$ dependence of the right-hand side of (3.21) because the $\mathbf{q}$ dependence of $\tilde{\Gamma}^{(4)}$ and $B$, respectively, was shown to be irrelevant in $O(\epsilon)$.[12] Equation (3.21) as well as the equations of state for higher moments (if necessary) could be derived starting from $\mathcal{H}(K,\mathbf{S})$ (2.6) supplemented by some source terms in the way described above [Eqs. (3.16) and (3.17)].

Throughout this paper a tilde indicates that physical quantities are expressed in terms of the effective spin system (3.3) with the coupling parameters $\tilde{r},\tilde{u}$ (3.4). We reserve the original notation for the $K-\mathbf{S}$ system (2.6) with the coupling parameters $r$, $u$, and $X$. Equations (3.15) and (3.21) yield the necessary relations between the chemical potentials $g$ and $B$ and their corresponding moments which are•fixed due to quenched randomness [Eq. (2.8)]. Using the constraint $\langle K_0\rangle=0$ and Eq. (3.15), $g$ is eliminated in the relation between $\tilde{r}$ and $r$ (3.10) leading to

$$r=\tilde{r}-4X\tilde{E}(\tilde{r},\tilde{u},\tilde{H}) \tag{3.22}$$

with $X=R^2/B$. The consequences of this relation are easily recognized if one looks at the critical behavior near the fixed points. The energy $\tilde{E}$ behaves as $\tilde{t}^{\,1-\tilde{\alpha}}$, where $\tilde{\alpha}$ is the exponent of the effective spin system (3.3). Equation (3.22) then leads to

$$t=\tilde{t}+q(X,u)\tilde{t}^{\,1-\tilde{\alpha}}\,, \tag{3.23}$$

where we neglect the $O(u)$ difference between $r$ and $t$. $q(X,u)$ depends on the concentration $p$ of the diluted ferromagnet. Without going into details, it is easy to see from (3.23) that the physical quantities of the quenched disordered system are Fisher renormalized with respect to those of the corresponding pure system if $\tilde{\alpha} > 0$:

$$\alpha = -\frac{\tilde{\alpha}}{1-\tilde{\alpha}}, \quad \beta = \frac{\tilde{\beta}}{1-\tilde{\alpha}}, \quad \gamma = \frac{\tilde{\gamma}}{1-\tilde{\alpha}}, \quad \nu = \frac{\tilde{\nu}}{1-\tilde{\alpha}} .$$

If $\tilde{\alpha} < 0$ the critical behavior near the fixed points is not changed by dilution.

Before we start the crossover calculations, we have to discuss the equation for the variance, Eq. (3.21). Besides the first term, which is of $O(1)$ (3.21) contains a correction of $O(R^2/B)$ proportional to the specific heat $\tilde{C}(\tilde{t},\tilde{u})$ of the effective spin Hamiltonian. The specific heat $\tilde{C}(\tilde{t},\tilde{u})$ diverges as $\tilde{t}^{-\tilde{\alpha}}$ and the $\mathbf{q}$ dependence of the four-point function $\tilde{\Gamma}^{(4)}$ of the effective spin system which occurred originally in (3.21) is proportional to $q^{-\tilde{\alpha}/\tilde{\nu}}$. Both divergences can be ignored in this $O(\epsilon)$ calculation because they are of $O(\epsilon^2)$ due to the $O(\epsilon)$ prefactor in (3.21). This fact was already used from the start in the Hamiltonian assuming a $\mathbf{q}$ independent chemical potential $B$. Actually, the same considerations were applied to the rescaling factor $\kappa$ of $K_{\mathbf{q}}$ in the renormalization of $\mathcal{H}(K,\mathbf{S})$,[12] which led to marginality of $B$ in $O(\epsilon)$. However, even in $O(\epsilon^2)$ these divergences present no severe problem since the effective temperature $\tilde{t}$ is related to the real reduced temperature $t$ by the Fisher-renormalization (3.23): The specific heat as a function of $t$ does not diverge but shows a finite cusp

$$\tilde{C} \sim \tilde{t}^{-\tilde{\alpha}} \sim t^{\alpha/(1-\tilde{\alpha})} \tag{3.24}$$

and the $\mathbf{q}$ dependence of the four-point function $\Gamma^{(4)}$ in terms of $\mathcal{H}(K,\mathbf{S})$ is given by

$$\Gamma^{(4)} \sim t^{\tilde{\alpha}/(1-\tilde{\alpha})} f(qt^{-\tilde{\nu}/(1-\tilde{\alpha})}) \xrightarrow[t\to 0]{} q^{\tilde{\alpha}/\tilde{\nu}} . \tag{3.25}$$

The temperature and momentum-dependence of $B$ resulting from (3.21) should be taken into account in $O(\epsilon^2)$. This will lead to different quantitative critical behavior compared to annealed magnets where the variance and higher moments are not constrained.

We also checked that the exponent renormalization is not changed in an $O(\epsilon^2)$ calculation. Three additional operators appear in $\mathcal{H}(K,\mathbf{S})$ in $O(\epsilon^2)$:

$$\begin{aligned} \mathcal{H} = {} & gK_0 - \tfrac{1}{2}BK^2 - \tfrac{1}{2}rS^2 + RKS^2 - uS^4 + HS_0^1 \\ & + hKS_0^1 + \sigma_1 K^2 S^2 + \sigma_2 K^3 + \sigma_3 K^4 . \end{aligned} \tag{3.26}$$

$\sigma_1$ is of $O(uX) = O(\epsilon^2)$, $\sigma_2$ is of $O(R^3) = O(\epsilon^{3/2})$, and $\sigma_3$ is of $O(X^2) = O(\epsilon^2)$. The integrations over $\{K_{\mathbf{q}}\}$ in $Z$ then lead to an effective $S^4$ Hamiltonian with the coupling parameters

$$\begin{aligned} \tilde{r} &= r - 2\frac{gR}{B} + c_1 \frac{\sigma_1}{B} + \cdots , \\ \tilde{u} &= u - \frac{R^2}{2B^2} + c_2 \frac{\sigma_1^2}{B^2} + c_3 \frac{gR\sigma_1}{B^2} + c_4 \frac{g^2\sigma_1}{B^2} + \cdots , \end{aligned} \tag{3.27}$$

where the $c_i$ contain combinatorial factors and integrals. The term $\sigma_1^2/B^2$ is of $O(\epsilon^4)$ and $gR\sigma_1/B^2$ and $g^2\sigma_1/B^2$ are of $O(\epsilon^3)$. They do not enter the $O(\epsilon^2)$ calculation. The term $\sigma_1/B$ is of $O(\epsilon^2)$, but it does not depend on $g$. Therefore, it cannot modify the $O(\epsilon)$ results via (3.15). The equation for the concentration (3.15) is modified by $O(\epsilon^{5/2})$ terms:

$$\begin{aligned} \langle K_0 \rangle = {} & \frac{g}{B} + \frac{R}{B}\int_{\mathbf{q}} \langle \mathbf{S}_{\mathbf{q}} \cdot \mathbf{S}_{-\mathbf{q}} \rangle_c + d_1 \frac{\sigma_1 g}{B^2} \int_{\mathbf{q}} \langle \mathbf{S}_{\mathbf{q}} \cdot \mathbf{S}_{-\mathbf{q}} \rangle_c \\ & + d_2 \frac{\sigma_1 R}{B^2} \int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3} \langle \mathbf{S}_{\mathbf{q}_1} \cdot \mathbf{S}_{\mathbf{q}_2} \mathbf{S}_{\mathbf{q}_3} \cdot \mathbf{S}_{-(\mathbf{q}_1+\mathbf{q}_2+\mathbf{q}_3)} \rangle_c \\ & + \cdots . \end{aligned} \tag{3.28}$$

Inserting this equation into (3.27) leads only to terms of $O(\epsilon^3)$. Qualitative changes of the $O(\epsilon)$ results would only occur if there were terms of $O(\epsilon^{3/2})$ in the equation for the variance (3.21) with a singularity stronger than that of the energy $\tilde{E}$.

## IV. CROSSOVER THEORY

We have seen in Sec. III that the disordered ferromagnet can be described by the effective spin Hamiltonian (3.3) with modified coupling parameters. Combining Eqs. (3.10), which relate the effective coupling parameters $\tilde{r}, \tilde{u}, \tilde{H}$ to the original ones $r$, $u$, $X$, $H$, and $h$, with the conditions of quenched randomness [Eqs. (3.15) and (3.21)], we are led to a Fisher renormalization of the critical exponents. However, the full treatment of the problem aiming at a complete description makes a double crossover calculation necessary. The first results from the crossover between the tricritical and the critical fixed point of the effective spin Hamiltonian (3.3). The second crossover results from the renormalization of the coupling parameters.

We use the well-known trajectory integral method proposed by Nelson and Rudnick,[23,24] who first calculated the crossover in the $S^4$ model. The method has been applied to various other problems, like the compressible magnet,[25] to bicritical points[26] and to the coexistence behavior in isotropic ferromagnets.[27,28]

The basic idea of the trajectory integral method is to renormalize the critical system until the fluctuations become small. Then the renormalized temperature $t(l^*)$ is of $O(1)$ and Landau theory with first-order fluctuation corrections can be applied. According to this idea, the free energy and the susceptibility are given by[24]

$$\begin{aligned} F(r,u,H) = {} & \int_0^{l^*} G_0(r(l),u(l)) e^{-dl} dl \\ & + e^{-dl^*} F[r(l^*),u(l^*),H(l^*)] , \end{aligned} \tag{4.1}$$

$$\chi = e^{2l^*} \chi(l^*) , \tag{4.2}$$

where $F(l^*)$ and $\chi(l^*)$ have to be calculated by Landau theory. We use the differential notation of the RG method which results from the discrete formulation as the infinitesimal limit $b = e^{\delta}$, $\delta \to 0$. $G_0(l)$ in [Eq. (4.1)] is the integration kernel which is the integrated free energy of an infinitesimal RG transformation:

$$G_0(l) = \tfrac{1}{2} n K_4 \{ \ln[1 + r(l)] - \tfrac{1}{2} \} , \tag{4.3}$$

where $K_4$ is the integral over the four-dimensional momentum sphere. In fact (4.3) is only the zero-order term in the $O[\epsilon, u(l)]$ expansion of the integrated free energy. The perturbation corrections, which are present in the discrete formulation of the RG ($b \neq 1$), are lost in the infinitesimal limit ($b \rightarrow 1$), which is an unsatisfactory feature of this formulation.[23,24,29] In a rather pragmatic way, Nelson and Rudnick[24] replaced (4.3) by the correct kernel

$$G_0(l)=\tfrac{1}{2}nK_4\{\ln[1+t(l)]-\tfrac{1}{2}\} , \tag{4.4}$$

where

$$t(l)=r(l)+\frac{A}{2}u(l)$$

is the $l$-dependent linear temperature scaling field.[20,24] We have shown in Ref. 30 that the kernel (4.4) can be derived in a more systematic way.

In order to calculate the free energy $\widetilde{F}(\widetilde{r},\widetilde{u})$ and the susceptibility $\widetilde{\chi}(\widetilde{r},\widetilde{u})$ of the effective spin Hamiltonian (3.3), one has to integrate the differential recursion relations[24] of the well-known $S^4$ model, but with the renormalized coupling parameters of the effective Hamiltonian

$$\frac{d\widetilde{r}}{dl}=2\widetilde{r}+A\frac{\widetilde{u}}{1+\widetilde{r}} , \tag{4.5}$$

$$\frac{d\widetilde{u}}{dl}=\epsilon\widetilde{u}-B\frac{\widetilde{u}^{\,2}}{(1+\widetilde{r})^2} , \tag{4.6}$$

with $A=4(n+2)K_4$ and $B=4(n+2)K_4$. In the critical region $[r(l) \ll 1]$, Eqs. (4.5) and (4.6) reduce to the crossover equations first studied by Riedel and Wegner.[19] We discuss the solutions of (4.5) and (4.6) given by Rudnick and Nelson[24] in the Appendix, where we point out that the matching procedure leads to cutoff-dependent terms. These nonuniversal terms are expected to occur naturally in the crossover regime to mean-field behavior. However, in the trajectory integral method they are an artifact of the imperfect matching of the critical theory valid for $r(1) \ll 1$ with the Landau theory valid for $r(l)=O(1)$. These cutoff-dependent terms must either be discarded or the solutions of (4.5) and (4.6) must be chosen appropriately in the way Nelson and Rudnick did.

The free energy $F(l^*)$ in (4.1) and the susceptibility $\chi(l^*)$ in (4.2) in the Landau regime have been evaluated by Rudnick and Nelson.[24] They used the matching condition $t(l^*)=1$, which simplifies the results since various $\log t(l^*)$ terms vanish. However, it has been proved that the results do not depend on this specific choice of the matching condition.

Using the results of Rudnick and Nelson[24] and the matching condition $\widetilde{t}(l^*)=1$ for the effective spin Hamiltonian, the susceptibility $\widetilde{\chi}(\widetilde{r},\widetilde{u})$ and the free energy $\widetilde{F}(\widetilde{r},\widetilde{u})$ of the effective Hamiltonian are given by

$$\widetilde{\chi}=e^{2l^*} \tag{4.7}$$

and

$$\widetilde{F}=-\frac{nK_4}{8\alpha_c}\frac{\widetilde{\mu}_1^{\,2}}{\widetilde{\mu}_2}\{[1+\widetilde{\mu}_2(e^{\epsilon l^*}-1)]^{\alpha_c/\alpha_t}-1\} , \tag{4.8}$$

where various regular terms of $O(1)$ have been neglected.[24] $\widetilde{\mu}_1=\widetilde{t}=\widetilde{r}+(A/2)\widetilde{u}$ and $\widetilde{\mu}_2=B\widetilde{u}/\epsilon$ are the linear tricritical scaling fields,[19] which transform like $\widetilde{\mu}_1(l)=\widetilde{\mu}_1 e^{y_{1t}l}$ with $y_{1t}=2$ and $\widetilde{\mu}_2(l)=\widetilde{\mu}_2 e^{y_{2t}l}$ with $y_{2t}=\epsilon$ near the tricritical fixed point. The transformation of the critical scaling fields is characterized by $y_{1c}=2-A/B\epsilon$ and $y_{2c}=-\epsilon$. $\alpha_c$ and $\alpha_t$ are the exponents of the specific heat at the critical and the tricritical fixed point, respectively, given by

$$\begin{aligned}\alpha_c&=\frac{2y_{1c}-d}{y_{1c}}=\left[1-2\frac{A}{B}\right]\frac{\epsilon}{2}=\frac{4-n}{n+8}\frac{\epsilon}{2}+O(\epsilon^2) ,\\ \alpha_t&=\frac{2y_{1t}-d}{y_{1t}}=\frac{\epsilon}{2} .\end{aligned} \tag{4.9}$$

For the complete description of the crossover regime it is sensible to introduce nonlinear scaling fields, which reduce to the linear scaling fields near the fixed points. They are given by[19]

$$\begin{aligned}\widetilde{g}_{1c}(l)&=\widetilde{g}_{1c}e^{y_{1c}l}=\widetilde{\mu}_1(l)\widetilde{\mu}_2(l)^{-\sigma_c/\phi_c} ,\\ \widetilde{g}_{2c}(l)&=\widetilde{g}_{2c}e^{y_{2c}l}=[1-\widetilde{\mu}_2(l)]\widetilde{\mu}_2(l)^{-1} ,\\ \widetilde{g}_{1t}(l)&=\widetilde{g}_{1t}e^{y_{1t}l}=\widetilde{\mu}_1(l)[1-\widetilde{\mu}_2(l)]^{-\sigma_t/\phi_t} ,\\ \widetilde{g}_{2t}(l)&=\widetilde{g}_{2t}e^{y_{2t}l}=[1-\widetilde{\mu}_2(l)]^{-1}\widetilde{\mu}_2(l) ,\end{aligned} \tag{4.10}$$

with the usual abbreviations

$$\begin{aligned}\phi_c&=\frac{y_{2c}}{y_{1c}}, \quad \sigma_c=\frac{y_{1c}-y_{1t}}{y_{1c}} ,\\ \phi_t&=\frac{y_{2t}}{y_{1t}}, \quad \sigma_t=\frac{y_{1t}-y_{1c}}{y_{1t}} .\end{aligned} \tag{4.11}$$

The matching parameter $e^{l^*}$ in Eq. (4.8) is obtained from the matching condition $\widetilde{\mu}_1(l^*)=1$. This condition may be evaluated using the solution of the recursion relations (4.5) and (4.6) (Refs. 19 and 24)

$$\widetilde{\mu}_1(l^*)=\widetilde{\mu}_1 e^{y_{1t}l^*}\left[1+\frac{\widetilde{\mu}_2}{1-\widetilde{\mu}_2}e^{y_{2t}l^*}\right]^{-\sigma_t/\phi_t}=1 . \tag{4.12}$$

To obtain results in explicit scaling form for the whole crossover region, it is preferable to express $\widetilde{\mu}_1(l^*)=1$ in terms of the nonlinear scaling fields (4.10) as

$$\widetilde{g}_{1c}e^{y_{1c}l^*}=(1+\widetilde{g}_{2c}e^{y_{2c}l^*})^{\sigma_c/\phi_c} . \tag{4.13}$$

One easily verifies that $\hat{L}(\widetilde{g}_{1c},\widetilde{g}_{2c})=e^{l^*}$ shows the scaling behavior

$$\hat{L}(\widetilde{g}_{1c},\widetilde{g}_{2c})=b\hat{L}(b^{y_{1c}}\widetilde{g}_{1c},b^{y_{2c}}\widetilde{g}_{2c}) . \tag{4.14}$$

We choose $b=\widetilde{g}_{1c}^{\;-1/y_{1c}}$ in (4.14) in order to extract the leading temperature dependence:

$$\hat{L}(\widetilde{g}_{1c},\widetilde{g}_{2c})=\widetilde{g}_{1c}^{\;-1/y_{1c}}L(\widetilde{g}_{2c}\widetilde{g}_{1c}^{\;-\phi_c}) . \tag{4.15}$$

This leads to the implicit equation for the critical scaling function $L(\widetilde{c}_c)$

$$L^{y_{1c}} = (1+\tilde{c}_c L^{y_{2c}})^{\sigma_c/\phi_c} \ . \tag{4.16}$$

$L$ depends only on the scaling variable $\tilde{c}_c = \tilde{g}_{2c}\tilde{g}_{1c}^{-\phi_c}$. Alternatively, it is possible to express $\tilde{\mu}_1(l^*)=1$ in terms of the tricritical scaling fields $\tilde{g}_{1t}(l)$ and $\tilde{g}_{2t}(l)$. With

$$\hat{L}(\tilde{g}_{1t},\tilde{g}_{2t}) = \tilde{g}_{1t}^{-1/y_{1t}} L(\tilde{g}_{2t}\tilde{g}_{1t}^{-\phi_t}) \tag{4.17}$$

one obtains the equation for the tricritical scaling function $L(\tilde{c}_t)$,

$$L^{y_{1t}} = (1+\tilde{c}_t L^{y_{2t}})^{\sigma_t/\phi_t} \ . \tag{4.18}$$

We use the same notation $L$ for both scaling functions indicating the critical and the tricritical case by means of the argument if ambiguities could arise. Inserting the solution for $e^{l^*}$, Eqs. (4.15) and (4.16) into Eq. (4.7), gives the susceptibility in scaling form

$$\tilde{\chi} = \tilde{g}_{1c}^{-\gamma_c}\tilde{\chi}_0(\tilde{c}_c) \tag{4.19}$$

with

$$\gamma_c = \frac{2}{y_{1c}} = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2) \ .$$

The crossover function of the susceptibility

$$\tilde{\chi}_0(\tilde{c}_c) = (1+\tilde{c}_c L^{y_{2c}})^{(\sigma_c/\phi_c)\gamma_c} = (1+\tilde{c}_c L^{y_{2c}})^{n+2/(n+8)} \tag{4.20}$$

depends only on the critical scaling variable $\tilde{c}_c$. It describes the variation of the singular behavior of $\tilde{\chi}$ in the whole $\tilde{\mu}_1-\tilde{\mu}_2$ space. Alternatively, one may choose tricritical scaling fields to obtain

$$\tilde{\chi} = \tilde{g}_{1t}^{-\gamma_t}\tilde{\chi}_0(\tilde{c}_t) \tag{4.21}$$

with the crossover function

$$\tilde{\chi}_0(\tilde{c}_t) = (1+\tilde{c}_t L^{y_{2t}})^{(\sigma_t/\phi_t)\gamma_t} = (1+\tilde{c}_t L^{y_{2t}})^{n+2/(n+8)} \ . \tag{4.22}$$

Equation (4.19) with Eq. (4.20), and Eq. (4.21) with Eq. (4.22), are equivalent descriptions of the susceptibility since both scaling functions $L(\tilde{c}_c)$ (4.16) and $L(\tilde{c}_t)$ (4.18), respectively, describe the crossover including both asymptotic regimes. However, the implicit equations for $L$ necessitate a numerical solution if one is interested in the complete crossover.

An analytic discussion of the asymptotic behavior is possible in terms of the local crossover variable $\tilde{c}^* = \tilde{c}L^{y_2}$. From (4.16) and (4.18), respectively, we obtain

$$\tilde{c}^* = \tilde{c}(1+\tilde{c}^*)^{\sigma} \tag{4.23}$$

with the corresponding $\sigma_c$ and $\sigma_t$. The asymptotic behavior in both regimes is easily obtained since

$$\begin{aligned} &\tilde{c}_c^* \rightarrow \tilde{c}_c \quad (\tilde{c}_c \ll 1) \ , \\ &\tilde{c}_c^* \rightarrow \tilde{c}_c^{\phi_t/\phi_c} = \tilde{c}_t^{-1} \quad (\tilde{c}_c \gg 1) \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} &\tilde{c}_t^* \rightarrow \tilde{c}_t \quad (\tilde{c}_t \ll 1) \ , \\ &\tilde{c}_t^* \rightarrow \tilde{c}_t^{\phi_c/\phi_t} = \tilde{c}_c^{-1} \quad (\tilde{c}_t \gg 1) \ , \end{aligned} \tag{4.25}$$

where we have used

$$1-\sigma_t = (1-\sigma_c)^{-1} = \frac{\phi_t}{\phi_c} \ .$$

The local critical scaling variable $\tilde{c}_c^*$ reduces to $\tilde{c}_c$ in the critical limit ($\tilde{c}_c \ll 1$) and to the inverse tricritical scaling variable $\tilde{c}_t^{-1}$ in the tricritical limit. The same applies to the local tricritical scaling variable $\tilde{c}_t^*$.

The critical crossover function $\tilde{\chi}_0$ (4.20) can be written in terms of the local scaling variable as

$$\tilde{\chi}_0 = (1+\tilde{c}_c^*)^{(\sigma_c/\phi_c)\gamma_c} \ , \tag{4.26}$$

which reduces to

$$\tilde{\chi}_0 = (1+\tilde{c}_c)^{(\sigma_c/\phi_c)\gamma_c} \tag{4.27}$$

for $\tilde{c}_c \ll 1$. The tricritical regime ($\tilde{c}_c \gg 1$) follows from Eqs. (4.19), (4.24), and (4.26) as

$$\tilde{\chi} = \tilde{g}_{1t}^{-\gamma_t}(1+\tilde{c}_t)^{(\sigma_t/\phi_t)\gamma_t} \ , \tag{4.28}$$

where we have used the relation

$$\tilde{g}_{1c}^{-1/y_{1c}} = \tilde{g}_{1t}^{-1/y_{1t}}\tilde{c}_t^{\sigma_c/y_{2c}} \ .$$

Equation (4.28) is the asymptotic form of the susceptibility in tricritical scaling fields:

$$\tilde{\chi} = \tilde{g}_{1t}^{-\gamma_t}(1+\tilde{c}_t^*)^{(\sigma_t/\phi_t)\gamma_t} \ . \tag{4.29}$$

In an analogous way one verifies that (4.21) and (4.22) include both types of critical behavior.

Using (4.15) and (4.16) or (4.17) and (4.18), the free energy can now be written in scaling form with nonlinear scaling fields. However, the free energy given by (4.8) still contains a regular part $\Delta\tilde{F}_r$ which has to be separated from the singular part $\tilde{F}_s$: $\tilde{F} = \tilde{F}_s + \Delta\tilde{F}_r$. $\Delta\tilde{F}_r$ is identified from the condition that the residual singular part $\tilde{F}_s$ can be written in scaling form. Splitting $\tilde{F}$ into

$$\tilde{F}_s = -\frac{nK_4}{8\alpha_c}\frac{\tilde{\mu}_1^2}{\tilde{\mu}_2}(1-\tilde{\mu}_2)^{\alpha_c/\alpha_t} \times \left[ \left[ 1 + \frac{\tilde{\mu}_2}{1-\tilde{\mu}_2}e^{\epsilon l^*} \right]^{\alpha_c/\alpha_t} - 1 \right] \tag{4.30}$$

and

$$\Delta\tilde{F}_r = \frac{nK_4}{8\alpha_c}\frac{\tilde{\mu}_1^2}{\tilde{\mu}_2}[1-(1-\tilde{\mu}_2)^{\alpha_c/\alpha_t}] \tag{4.31}$$

leads after some algebra to the free energy in scaling form

$$\begin{aligned} \tilde{F}_s &= \frac{nK_4}{8\alpha_c}\frac{\tilde{\mu}_1^2}{\tilde{\mu}_2}(1-\tilde{\mu}_2)^{\alpha_c/\alpha_t}[(1+\tilde{g}_{2c}^{-1}\tilde{g}_{1c}^{\phi_c}L^{\epsilon})^{\alpha_c/\alpha_t}-1] \\ &= -\frac{nK_4}{8\alpha_c}\tilde{g}_{1c}^{2-\alpha_c}\cdot\tilde{F}_0(\tilde{c}_c^*) \end{aligned} \tag{4.32}$$

with the crossover function

$$\tilde{F}_0(\tilde{c}_c)=L^{(\alpha_c/\alpha_t)y_{2t}}[(1+\tilde{c}_cL^{y_{2c}})^{\alpha_c/\alpha_t}-(\tilde{c}_cL^{y_{2c}})^{\alpha_c/\alpha_t}] \tag{4.33}$$

This result explicitly shows the scale invariance of the free energy $\tilde{F}_s$. In terms of the local scaling variable (4.23) the crossover function of the free energy reads

$$\tilde{F}_0(\tilde{c}_c^{\,*})=[(1+\tilde{c}_c^{\,*})^{\alpha_c/\alpha_t}-\tilde{c}_c^{\,*(\alpha_c/\alpha_t)}]\ , \tag{4.34}$$

where we neglected the prefactor in (4.33) since its exponent is an $O(\epsilon)$ smaller than the terms retained in (4.34). In terms of tricritical scaling fields, the calculation leads in $O(\epsilon)$ to

$$\tilde{F}_s=-\frac{nK_4}{8\alpha_t}\tilde{g}_{1t}^{\,2-\alpha_t}\tilde{F}_0(\tilde{c}_t^{\,*}) \tag{4.35}$$

with the tricritical crossover function

$$\tilde{F}_0(\tilde{c}_t^{\,*})=\frac{(1+\tilde{c}_t^{\,*})^{\alpha_c/\alpha_t}-1}{(\alpha_c/\alpha_t)\tilde{c}_t^{\,*}}\ , \tag{4.36}$$

where we have used (4.17) and (4.18) and the equality $\phi_t=\alpha_t$. The asymptotic behavior is recovered again for both descriptions using (4.24) and (4.25). Both representations, critical as well as tricritical, describe the crossover in an equivalent way. Moreover, it is the same scaling function $L(\tilde{c}_c)$ and $L(\tilde{c}_t)$, respectively, which appears in both physical quantities $\tilde{\chi}$ and $\tilde{F}$. This is the reason why one experimentally observes scaling behavior at all, even if the system is not at its fixed point.

The energy (3.14) and the specific heat (3.20) can be calculated either as derivatives of (4.1) or directly from the results (4.32) and (4.34) or (4.35) and (4.36). Ignoring terms of $O(\epsilon)$ compared to $O(1)$, one obtains

$$\tilde{E}_s=\left.\frac{\partial\tilde{F}_s}{\partial\tilde{\mu}_1}\right|_{\tilde{\mu}_2}=A_t^{+}(2-\alpha_t)\frac{\tilde{g}_{1t}^{\,2-\alpha_t}}{\tilde{\mu}_1}\tilde{F}_0(\tilde{c}_t)+A_t^{+}\tilde{g}_{1t}^{\,2-\alpha_t}\frac{d\tilde{F}_0}{d\tilde{c}_t^{\,*}}\frac{d\tilde{c}_t^{\,*}}{d\tilde{c}_t}(-\phi_t)\frac{\tilde{c}_t}{\tilde{\mu}_1}=2\frac{\tilde{F}_s}{\tilde{\mu}_1}\ , \tag{4.37}$$

where $A_t^{+}=-nK_4/8\alpha_t$ is the tricritical amplitude. We have used that $\tilde{g}_{1t}$ is linear in $\tilde{\mu}_1$ and that $\tilde{c}_t\sim\tilde{\mu}_1^{\,-\phi_t}$ with the crossover exponent $\phi_t$ being of $O(\epsilon)$. The specific heat is easily calculated from (4.37) as

$$\tilde{C}_s=\frac{\partial\tilde{E}_s}{\partial\tilde{\mu}_1}\bigg|\mu_2=-2\frac{\tilde{F}_s}{\tilde{\mu}_1^{\,2}}+2\frac{\tilde{E}_s}{\tilde{\mu}_1}=2\frac{\tilde{F}_s}{\tilde{\mu}_1^{\,2}}=\frac{\tilde{E}_s}{\tilde{\mu}_1}\ . \tag{4.38}$$

Thus, the energy and the specific heat are characterized by the same crossover functions (4.34) and (4.36), respectively, as the free energy. One gets an impression of the crossover between the tricritical and the critical point of the effective spin Hamiltonian, if one studies the effective exponents introduced by Riedel and Wegner.[20] The universal effective exponents are defined as

$$\tilde{\gamma}_{\text{eff}}=-\left.\frac{\partial\ln\tilde{\chi}}{\partial\ln\tilde{\mu}_1}\right|_{\tilde{\mu}_2}\ , \tag{4.39}$$

$$\tilde{\alpha}_{\text{eff}}=-\left.\frac{\partial\ln\tilde{C}}{\partial\ln\tilde{\mu}_1}\right|_{\tilde{\mu}_2}\ . \tag{4.40}$$

We use the notation $\tilde{\alpha}_{\text{eff}}$ and $\tilde{\gamma}_{\text{eff}}$ for the derivatives with respect to $\tilde{\mu}_1$ in order to preserve $\alpha_{\text{eff}}$ and $\gamma_{\text{eff}}$ for the derivative with respect to the real reduced temperature $\mu_1\sim t$ of the disordered ferromagnet. Inserting the result for the susceptibility in critical scaling fields (4.19) and (4.26) we obtain

$$\begin{aligned}\tilde{\gamma}_{\text{eff}}&=\gamma_c-\left.\frac{\partial\ln\tilde{\chi}_0}{\partial\ln\tilde{c}_c}\frac{\partial\ln\tilde{c}_c}{\partial\ln\tilde{\mu}_1}\right|_{\tilde{\mu}_2}\\&=\gamma_c+\phi_c\gamma_c\frac{\sigma_c}{\phi_c}\frac{\tilde{c}_c^{\,*}}{1+\tilde{c}_c^{\,*}}\frac{1}{1-\sigma_c(\tilde{c}_c^{\,*}/1+\tilde{c}_c^{\,*})}\\&=\frac{\gamma_c}{1-\sigma_c(\tilde{c}_c^{\,*}/1+\tilde{c}_c^{\,*})}\ ,\end{aligned} \tag{4.41}$$

where we used Eq. (4.23). $\tilde{\gamma}_{\text{eff}}$ (4.41) reduces to $\gamma_c$ in the critical regime ($\tilde{c}_c\ll 1$) and to $\gamma_t$ in the tricritical region ($\tilde{c}_c\gg 1$) since $\gamma_c/(1-\sigma_c)=\gamma_t$. The equivalent tricritical representation of $\tilde{\chi}$ [Eqs. (4.21) and (4.22)] leads to

$$\tilde{\gamma}_{\text{eff}}=\frac{\gamma_t}{1-\sigma_t(\tilde{c}_t^{\,*}/1+\tilde{c}_t^{\,*})}\ . \tag{4.42}$$

From (4.34) and (4.38) we calculate $\tilde{\alpha}_{\text{eff}}$ according to (4.40). In critical scaling fields we obtain in $O(\epsilon)$ after some algebra

$$\tilde{\alpha}_{\text{eff}}=\alpha_c(1+\tilde{c}_c^{\,*})^{-(1-\alpha_c/\alpha_t)}\tilde{F}_0^{-1}(\tilde{c}_c^{\,*})\ , \tag{4.43}$$

which is equivalent to the tricritical form, resulting from (4.36):

$$\tilde{\alpha}_{\text{eff}}=\alpha_t(1+\tilde{c}_t^{\,*})^{-(1-\alpha_c/\alpha_t)}\tilde{F}_0^{-1}(\tilde{c}_t^{\,*})\ . \tag{4.44}$$

## V. FISHER RENORMALIZATION

We now turn to the second crossover phenomenon in disordered ferromagnets, the Fisher renormalization. In Sec. III we have already discussed the Fisher renormalization near the fixed points in a qualitative way. In this section we analyze this additional crossover in detail.

The relation (3.22) between $r$ and $\tilde{r}$ leads to the relation

$$\mu_1=\tilde{\mu}_1-4X\tilde{E}(\tilde{\mu}_1,\tilde{\mu}_2) \tag{5.1}$$

between the real reduced temperature $t=\mu_1=r+(A/2)u$ of the disordered ferromagnet and the temperature scaling field $\tilde{\mu}_1=\tilde{r}+(A/2)\tilde{u}$ of the effective spin Hamiltonian. We neglected the difference $\tilde{u}-u=-X/2$ in (5.1) because it is only a temperature-independent constant of $O(\epsilon)$ which represents the difference between the critical temperatures of the disordered and the pure system. Inserting the energy $\tilde{E}(\tilde{\mu}_1,\tilde{\mu}_2)$ [Eq. (4.37)] of the effective spin system including the leading regular term arising from $\Delta\widetilde{F}_r$, (4.31) into (5.1), gives

$$\mu_1=\widetilde{\mu}_1\left[1-\frac{nK_4X}{\alpha_t}\frac{1-(1-\widetilde{\mu}_2)^{\alpha_c/\alpha_t}}{\frac{\alpha_c}{\alpha_t}\widetilde{\mu}_2}\right]+\frac{nK_4X}{4\alpha_t}\frac{\widetilde{g}_{1t}^{\,2-\alpha_t}}{\widetilde{\mu}_1}\widetilde{F}_0(\widetilde{c}_t) \tag{5.2}$$

in tricritical and

$$\mu_1=\widetilde{\mu}_1\left[1-\frac{nK_4X}{\alpha_c}\frac{1-(1-\widetilde{\mu}_2)^{\alpha_c/\alpha_t}}{\widetilde{\mu}_2}\right]+\frac{nK_4X}{\alpha_c}\frac{\widetilde{g}_{1c}^{\,2-\alpha_c}}{\widetilde{\mu}_1}\widetilde{F}_0(\widetilde{c}_c) \tag{5.3}$$

in critical scaling fields with the crossover functions (4.36) and (4.34), respectively. This temperature transformation is the important link between the critical properties of the effective spin system and the disordered ferromagnet. Given the reduced temperature $t\equiv\mu_1$ and the concentration-dependent coupling parameters $u$ and $X$, (5.2) and (5.3), respectively, give the effective temperature $\widetilde{\mu}_1$ which enters the crossover results for the susceptibility $\widetilde{\chi}(\widetilde{\mu}_1,\widetilde{\mu}_2)$ [Eqs. (4.21) and (4.22) or (4.19) and (4.20)] and the specific heat $\widetilde{C}(\widetilde{\mu}_1,\widetilde{\mu}_2)$ [Eq. (4.38)]. The effective coupling parameter $\widetilde{\mu}_2$ depends on the coupling parameters $u$ and $X$ of the original Hamiltonian (3.9) as

$$\widetilde{\mu}_2=\frac{B}{\epsilon}\left[u-\frac{X}{2}\right] . \tag{5.4}$$

which follows from (3.10). The concentration dependence of $\widetilde{\mu}_2$ via $u$ and $X$ is not known in our field-theoretic Hamiltonian. However, we know that the pure ferromagnet is characterized by $X=0$ and $\widetilde{\mu}_2=\mu_2=Bu/\epsilon$ [Eq. (5.4)]. In this case, the temperature transformation (5.2) reduces to $\widetilde{\mu}_1=\mu_1$. Extremely diluted ferromagnets near the limiting concentration of ferromagnetism $p_c$ are characterized by $\widetilde{\mu}_2\approx 0$ since for $\widetilde{\mu}_2<0$ there is obviously no second-order phase transition in our effective spin system. It is therefore sensible to assume that $\widetilde{\mu}_2$ and $X$ are monotonous functions of the concentration $p$ and that $\widetilde{\mu}_2$ vanishes at the limiting concentration $p_c$ below which no ferromagnetism occurs.

The crossover between the temperature scales $\mu_1$ and $\widetilde{\mu}_1$ is governed by the coupling parameter $X$. Its role in the complete crossover is more easily discussed if one studies the tricritical behavior ($\widetilde{c}_t \ll 1$) separately from the critical behavior ($\widetilde{c}_c \ll 1$) of the effective spin system.

Tricritical behavior mainly occurs in the experimentally accessible temperature range. In this case (5.2) reduces to

$$\begin{aligned}\mu_1&=\widetilde{\mu}_1\left[1-\frac{nK_4X}{\alpha_t}\right]+\frac{nK_4X}{\alpha_t}\widetilde{\mu}_1^{\,1-\alpha_t}\widetilde{F}_0(\widetilde{c}_t)\\&=\widetilde{\mu}_1[1-\mu_x+\widetilde{p}_t\widetilde{F}_0(\widetilde{c}_t)] ,\end{aligned} \tag{5.5}$$

where we have introduced the scaling field $\mu_x=nK_4X/\alpha_t$ and the scaling variable

$$\widetilde{p}_t=\mu_x\widetilde{\mu}_1^{\,-\alpha_t} .$$

Equation (5.5) shows that the tricritical regime ($\widetilde{c}_t \ll 1$), which is a large $\widetilde{\mu}_1$ interval in the case of strongly diluted systems [$\widetilde{\mu}_2 \ll 1, \mu_X=O(1)$], is described by renormalized behavior since $\widetilde{p}_t \gg 1$. Only a small part of the tricritical regime far away from the critical temperature is governed by unrenormalized behavior ($\widetilde{p}_t \ll 1$). The actual width of the unrenormalized interval obviously depends on $\mu_x$.

Since the critical fixed point of the effective Hamiltonian is the stable one, the asymptotic temperature range is governed by the critical behavior ($\widetilde{c}_c \ll 1$) of the effective spin Hamiltonian. With $\mu_x=nK_4X/\alpha_c$ and

$$\widetilde{p}_c=\mu_x\widetilde{\mu}_1^{\,-\alpha_c} ,$$

(5.3) simplifies to

$$\begin{aligned}\mu_1&=\widetilde{\mu}_1\left[1-\frac{nK_4X}{\alpha_c}\right]+\frac{nK_4x}{\alpha_c}\widetilde{\mu}_1^{\,1-\alpha_c}\widetilde{F}_0(\widetilde{c}_c)\\&=\widetilde{\mu}_1[1-\mu_x+\widetilde{p}_c\widetilde{F}_0(\widetilde{c}_c)] .\end{aligned} \tag{5.6}$$

Equation (5.6) implies that the asymptotic behavior ($\widetilde{\mu}_1\rightarrow 0$, $\widetilde{p}_c \gg 1$), which is critical for weakly ($\widetilde{\mu}_2=\mu_c$) as well as strongly diluted ($\widetilde{\mu}_2 \gtrsim 0$) systems, is governed by Fisher-renormalized exponents.

The actual behavior of a dilute magnet with given $u$ and $X$ at some temperature $t$ depends on the values of the scaling variables $\widetilde{c}_t$ or $\widetilde{c}_c$, and $\widetilde{p}_t$ or $\widetilde{p}_c$. A numerical solution of the temperature transformation [Eqs. (5.2) and (5.3)] together with the iteration [Eq. (4.23)] gives the precise variation of the critical behavior as function of temperature and concentration.

The concept of effective exponents[19] is seemingly the only possible way to get an impression about the crossover in disordered ferromagnets. The effective exponent $\gamma_{\text{eff}}$ which is defined as the derivative with respect to the real reduced temperature $\mu_1$ follows from $\chi(\mu_1,\mu_2,X)=\chi(\widetilde{\mu}_1,\widetilde{\mu}_2,X)$ and (4.39) as

$$\begin{aligned}\gamma_{\text{eff}}&=-\left.\frac{\partial\ln\chi}{\partial\ln\mu_1}\right|_{u,X}=-\left.\frac{\partial\ln\widetilde{\chi}}{\partial\ln\widetilde{\mu}_1}\right|_{\widetilde{\mu}_2}\left.\frac{\partial\ln\widetilde{\mu}_1}{\partial\ln\mu_1}\right|_{u,X}\\&=\frac{\widetilde{\gamma}_{\text{eff}}}{1-\tau} .\end{aligned} \tag{5.7}$$

$\tau$ is defined as the logarithmic derivative of the real temperature $\mu_1$ with respect to the temperature $\widetilde{\mu}_1$ of the effective spin system:

$$\left.\frac{\partial\ln\mu_1}{\partial\ln\widetilde{\mu}_1}\right|_{\widetilde{\mu}_2,\mu_x}=:1-\tau(\widetilde{\mu}_1,\widetilde{\mu}_2,\mu_x) . \tag{5.8}$$

Straightforward calculation of $\tau$ using (5.2) and (4.23) for the derivative of the crossover function $\widetilde{F}_0(\widetilde{c}_t)$ [Eq. (4.35)] with respect to $\widetilde{c}_t$ leads to

$$\tau=\alpha_t\frac{\Delta_t p_t(1+\widetilde{c}_t^{\,*})^{(\alpha_c/\alpha_t)-1}}{1-\mu_x\Delta_t+\Delta_t p_t\widetilde{F}_0(\widetilde{c}_t^{\,*})} \tag{5.9}$$

with the abbreviations

$$q_t = \frac{1-(1-\tilde{\mu}_2)^{\alpha_c/\alpha_t}}{\dfrac{\alpha_c}{\alpha_t}\tilde{\mu}_2}$$

and

$$\Delta_t = (1-\tilde{\mu}_2)^{-\sigma_t/\phi_t(2-\alpha_t)} \; .$$

Alternatively, one may use critical scaling fields to calculate $\tau$. Comparison of $\tau$ [Eq. (5.9)] with $\tilde{\alpha}_{\text{eff}}$ [Eq. (4.44)] shows that in the disordered regime $\tilde{p}_t \gg 1$: $\tau \rightarrow \tilde{\alpha}_{\text{eff}}$. Figure 1 contains a plot of $\tau$ as a function of $\log_{10}(t)$ for diluted systems with different concentrations and coupling parameters $\tilde{\mu}_2, \mu_X$, respectively. Strongly diluted systems [$\tilde{\mu}_2 \ll 1$, $\mu_X = O(1)$] exhibit a broad temperature range with renormalized behavior [$\tau = O(\alpha_t)$]. Asymptotically, one observes $\tau \rightarrow \alpha_c$ for strongly as well as weakly diluted systems. This applies to all spin dimensions since $\alpha_c > 0$ in $O(\epsilon)$ for all $n$. In Fig. 2 we show a plot of $\log_{10}\tilde{\mu}_1$ versus $\log_{10}\mu_1$ according to (5.2) or (5.3). It shows that for strongly diluted systems there is a temperature region [$\tau = O(\alpha_t)$] where the effective temperature scale $\log\tilde{\mu}_1$ is contracted by a factor $\tau^{-1} = O(2)$ compared to the real reduced temperature $\log_{10}\mu_1$. Figure 3 shows the result for $\gamma_{\text{eff}}$ [Eq. (5.7)] with $\tilde{\gamma}_{\text{eff}}(\tilde{\mu}_1, \tilde{\mu}_2)$ given by (4.42) and $\tau$ [Eq. (5.9)]. $\gamma_{\text{eff}}$ is a nonmonotonous function of the temperature. In the experimentally accessible critical temperature range $10^{-4} < t < 10^{-1}$, $\gamma_{\text{eff}}$ rises up to concentration-dependent maximum values. Extremely diluted systems near the limiting concentration of ferromagnetism even reach values up to $\gamma_{\text{eff}} = 2$, which is the Fisher-renormalized (FR) tricritical exponent $\gamma_t^{\text{FR}} = \gamma_t/(1-\alpha_t)$. The behavior of $\gamma_{\text{eff}}(t)$ for $t > 10^{-1}$ should not be taken seriously since the RG method is not applicable in this temperature range. $\gamma_{\text{eff}}$ should approach the mean-field value $\gamma_{\text{MF}} = 1$ for $t > 10^{-1}$ in a nonuniversal manner. Actually, $\gamma_{\text{eff}}$ drops down to $\gamma_t = 1$, the unrenormalized tricritical exponent, which incidentally has the same value as $\gamma_{\text{MF}}$. In the asymptotic temperature region $t < 10^{-4}$ $\gamma_{\text{eff}}$ gradually approaches the Fisher-renormalized critical exponent $\gamma_c^{\text{FR}} = \gamma_c/(1-\alpha_c)$, independent of the degree of disorder.

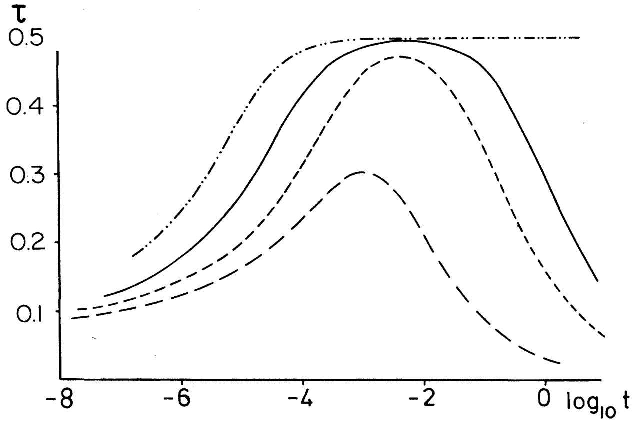


FIG. 1. $\tau$ (5.9) as function of the $\log_{10}$ of the reduced temperature $t = (T - T_c)/T_c$ for diluted ferromagnets with different concentrations and coupling parameters respectively ($\tilde{\mu}_2 = B\tilde{u}/\epsilon$, $\mu_X = nK_4X/\alpha_t$): —··— $\tilde{\mu}_2 = 10^{-5}$, $\mu_X = 1$; —— $\tilde{\mu}_2 = 10^{-4}$, $\mu_X = 0.5$; – – – $\tilde{\mu}_2 = 10^{-3}$, $\mu_X = 0.05$; — — $\tilde{\mu}_2 = 10^{-2}$, $\mu_X = 10^{-2}$; ---- $\tilde{\mu}_2 = 0.1$, $\mu_X = 0$. The last set of coupling parameters corresponds to a pure system with unrenormalized behavior ($\tau = 0$).

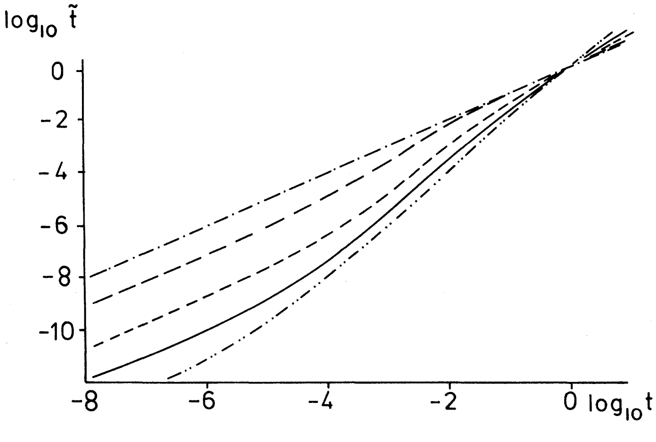


FIG. 2. $\text{Log}_{10}$ of the effective temperature $\tilde{t}$ [Eqs. (5.2) and (5.3)] vs the $\log_{10}$ of the reduced temperature $t$ for systems with the same concentrations as in Fig. 1.

The specific heat of the disordered ferromagnet is easily calculated from the free energy of the effective spin system using

$$F(\mu_1, \mu_2, X) = \tilde{F}[\tilde{\mu}_1(\mu_1, \mu_2, X), \tilde{\mu}_2(\mu_2, X)]$$

as

$$\begin{aligned} C(\mu_1, u, X) &= \left.\frac{\partial^2 F}{\partial \mu_1^2}\right|_{u,X} \\ &= \tilde{C}(\tilde{\mu}_1, \tilde{\mu}_2)\left.\frac{\partial \tilde{\mu}_1}{\partial \mu_1}\right|_{u,X}^2 + \tilde{E}(\tilde{\mu}_1, \tilde{\mu}_2)\left.\frac{\partial^2 \tilde{\mu}_1}{\partial \mu_1^2}\right|_{u,X} \end{aligned} \tag{5.10}$$

Using $\tilde{E} = \tilde{\mu}_1 \tilde{C}$ [Eq. (4.38)] and the definition of $\tau$ [Eq. (5.8)], one obtains

$$C(\mu_1, u, X) = \tilde{C}(\tilde{\mu}_1, \tilde{\mu}_2)\frac{\tilde{\mu}_1^2}{\mu_1^2}\frac{1+\tau}{(1-\tau)^2} \tag{5.11}$$

if one neglects terms which are small in the critical region $\mu_1, \tilde{\mu}_1 \ll 1$. $\tau$ is a smooth function of the temperature so that the singular behavior of $\tilde{C}$ dominates in (5.11). Therefore, the specific heat $C$ of the disordered ferromagnet simplifies to

$$C(\mu_1, u, X) = \tilde{C}(\tilde{\mu}_1, \tilde{\mu}_2)\frac{\tilde{\mu}_1^2}{\mu_1^2} \; , \tag{5.12}$$

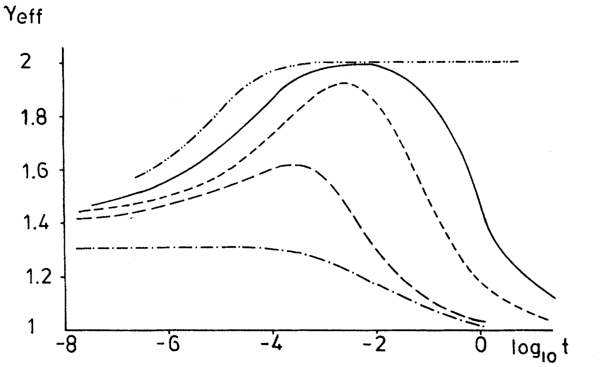


FIG. 3. Effective exponent $\gamma_{\rm eff}$ [Eq. (5.7)] given by (4.42) and (5.9) for diluted ferromagnets with the same concentrations as in Fig. 1.

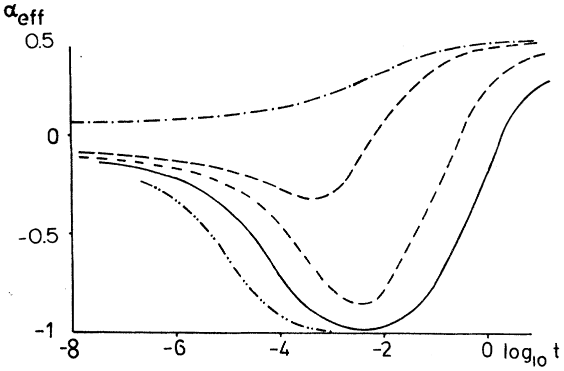


FIG. 4. Effective exponent $\alpha_{\rm eff}$ [Eq. (5.13)] given by (4.44) and (5.9) for systems with the same concentrations as in Fig. 1.

where one has to insert the temperature transformation (5.2) or (5.3). The effective exponent $\alpha_{\rm eff}$ defined with respect to the real reduced temperature $\mu_1$ is readily derived from (5.12) and (5.8) as

$$\alpha_{\rm eff} = -\left.\frac{\partial \ln C}{\partial \ln \mu_1}\right|_{u,X} = \frac{\tilde{\alpha}_{\rm eff}}{1-\tau} - \frac{2}{1-\tau} + 2 = \frac{\tilde{\alpha}_{\rm eff} - 2\tau}{1-\tau} \; . \tag{5.13}$$

Figure 4 contains plots of $\alpha_{\rm eff}$ for systems with the same coupling parameters as in Fig. 3 for the susceptibility. There is a broad temperature range where $\alpha_{\rm eff}$ changes drastically. For extremely diluted systems even $\alpha_{\rm eff}$ reaches the value of $-1$, which is the value of the Fisher-renormalized tricritical exponent $\alpha_t^{\rm FR} = -\alpha_t/(1-\alpha_t)$. This corresponds to systems with a rounded specific heat; it simply has a maximum value at $T_c$, but no sharp cusp or stronger singularity. Nevertheless, the phase transition is well defined and exhibits universal critical behavior as we have shown above. The asymptotic behavior of the specific heat is characterized by the Fisher-renormalized critical exponent $\alpha_c^{\rm FR} = -\alpha_c/(1-\alpha_c)$. This corresponds to the asymptotic critical behavior we found for $\gamma_{\rm eff}$ (Fig. 3).

The results obtained above for the susceptibility and the specific heat in the symmetric phase allow a direct comparison with experimental data. It goes without saying that one should not fit the data to the effective exponents which would necessitate error-increasing numerical derivatives; we have calculated the effective exponents to get a rough idea of the crossover. Instead, one should fit experimental data to the crossover scaling functions $\tilde{\chi}_0$ and $\tilde{F}_0$ and to the explicit temperature transformation (5.2) and (5.3). The subject of such an investigation would be first to verify the crossover scaling functions and second to identify the coupling parameters $u$ and $X$ as a function of the concentration. A comparison of experimental data to a power law such as

$$\chi = C t^{-\bar{\gamma}_{\rm eff}}$$

with some averaged effective exponent $\bar{\gamma}_{\rm eff}$, which characterizes the singularity in the observed temperature interval, is only a preliminary step towards the complete understanding of the subject.

There are many experimental investigations of the critical behavior of disordered ferromagnets. Reference 31 contains a summary of the experimental results for many different substances covering the whole range of concentrations between the pure and the extremely diluted ferromagnet. The measured critical exponents, which characterize the critical singularities in the experimental temperature range $10^{-4} < t < 10^{-1}$, are found to be in very good agreement with our theory. Moreover, the latest Monte Carlo simulations of the site-diluted Ising model also show concentration-dependent critical exponents.[32,33]

The calculations in this paper were done in $O(\epsilon)$. This may raise the question in what way the results are changed by terms of $O(\epsilon^2)$. It is well known that tricritical exponents are not changed by higher-order terms. We have shown that apart from very weakly diluted systems the experimental temperature range is characterized by tricritical behavior. Therefore, we do not expect significant changes of the crossover results in that temperature range. The asymptotic behavior will be changed in $O(\epsilon^2)$ because the numerical value of the exponents of the critical fixed point depend on $O(\epsilon^2)$ terms.

It is well known that one needs the equation of state in order to verify scaling behavior. We will publish the equation of state in exponentiated scaling form for disordered ferromagnets in a forthcoming paper. This will be done in the framework of $S^6$ theory since the effective $S^4$ coupling vanishes for extremely diluted systems.

## APPENDIX

Before applying the trajectory-integral method of Rudnick and Nelson[23,24] to a crossover problem, one has to solve the differential recursion relations. In their study of the crossover between the tricritical and the critical point of the $S^4$ model, Rudnick and Nelson[24] claim to have solved the equations (4.5) and (4.6) in the region $r(l) < O(1)$ and $u(l) = O(\epsilon)$.

They rewrote (4.5) and (4.6) as

$$\frac{dr}{dl}=2r(l)+Au(l)-Au(l)r(l)+A\frac{u(l)r^2(l)}{1+r(l)}\ , \qquad \text{(A1)}$$

$$\frac{du}{dl}=\epsilon u(l)-Bu^2(l)+B\frac{u^2(l)[2r(l)+r^2(l)]}{[1+r(l)]^2}\ . \qquad \text{(A2)}$$

They first studied the truncated set of equations without the last terms in (A1) and (A2) as an approximation to (4.5) and (4.6) valid in the critical region $r(l)\ll 1$. The resulting equations are identical to the model equations studied by Riedel and Wegner[19] if one uses $t(l)=r(l)+(A/2)u(l)$ instead of $r(l)$. Nelson and Rudnick[24] estimated the effect of the last terms in (A1, A2); they claim that the solution for $u(l)$ is not changed in $O(\epsilon)$ and that the scaling field

$$t(l)=r(l)+(A/2)u(l)$$

changes to

$$t(l)=r(l)+\frac{A}{2}u(l)-\frac{A}{2}u(l)r(l)\ln[1+r(l)]\ , \qquad \text{(A3)}$$

which leads to a marginal change for $r(l)$. Our numerical analysis of (4.5) and (4.6) does not support this conjecture. We find that the solutions given by Rudnick and Nelson[24] deviate strongly from our numerical solutions of the correct recursion relations (4.5) and (4.6) for $r(l)>0.1$. The deviations are more pronounced in the critical region than in the tricritical region. This is easily understood since for $u(l)\ll u_c=\epsilon/B$ the first terms in (A1) and (A2) dominate.

The effort of Rudnick and Nelson[24] to solve the correct recursion relations up to $r(l)=O(1)$ was motivated by the matching procedure of the trajectory integral method. Matching the critical theory with Landau theory at $r(l^*)=O(1)$ seemed to make it necessary to work out the correct solutions in this region. However, the approximate solutions valid for $r(l)\ll 1$ are absolutely sufficient for the calculation if one is interested in the critical behavior only. This becomes clear from the following argument: The last terms in (A1) and (A2) lead to deviations only outside the critical region $r(l)\ll 1$. Actually, the same kind of deviations would occur if one would supplement the $S^4$ model by additional irrelevant operators. These operators would not change the $l$ dependence of the scaling fields $\mu_1(l)$ and $\mu_2(l)$ in the critical region $\mu_1(l)\ll 1$; deviations would only occur in the region $\mu_1(l)=O(1)$. That is, the last terms in (A1) and (A2) are unimportant for the critical behavior in very much the same way as irrelevant operators are. The corrections induced by the last terms in (A1) and (A2) as well as irrelevant operators become important if one is interested in the true crossover from critical behavior to nonuniversal mean-field behavior.

These considerations lead to a second aspect of the trajectory integral method which has to be taken care of. The $S^4$ model is designed to give a sensible description of universal properties only. In contrast, Landau theory naturally includes nonuniversal properties like cutoff dependencies, etc. It is for that reason one cannot expect a perfect match of the $S^4$ model with Landau theory. To exemplify this, we calculate the susceptibility with an arbitrary cutoff $\Lambda$. The Feynman-graph expansion for $\chi^{-1}(l^*)$ with the Hamiltonian

$$\mathcal{H}(l^*)=-\tfrac{1}{2}r(l^*)S^2-u(l^*)S^4 \qquad \text{(A4)}$$

leads to

$$\chi^{-1}(l^*)=r(l^*)+Au(l^*)\int_0^\Lambda\frac{q^3}{r(l^*)+q^2}dq$$
$$+O(u^2(l^*),\epsilon u(l^*)) \qquad \text{(A5)}$$

in $O(\epsilon,u(l^*))$. Evaluating the integral, gives

$$\chi^{-1}(l^*)=r(l^*)+\frac{A}{2}u(l^*)$$
$$\times\{\Lambda^2+r(l^*)\ln r(l^*)-\ln[\Lambda^2+r(l^*)]\}. \qquad \text{(A6)}$$

In terms of the scaling field $t(l)=r(1)+(A/2)\Lambda^2u(l)$ with arbitrary cutoff, one obtains

$$\chi^{-1}(l^*)=t(l^*)+\frac{A}{2}u(l^*)t(l^*)$$
$$\times\{\ln t(l^*)-\ln[\Lambda^2+t(l^*)]\}$$
$$+O(\epsilon u(l^*),u^2(l^*))\ . \qquad \text{(A7)}$$

Using the matching condition $t(l^*)=1$, the susceptibility (4.2) follows as

$$\chi=e^{2l^*}\left[1+\frac{A}{2}u(l^*)\ln(1+\Lambda^2)\right]\ . \qquad \text{(A8)}$$

Equation (A8) shows that the imperfect matching of the $S^4$ model with Landau theory leads to artificial cutoff-dependent terms in the critical theory. However, if one inserts the scaling field $\tilde{t}(l)$ [Eq. (A3)] with general $\Lambda$ into (A6), then the cutoff-dependent term is canceled and one recovers $\chi=e^{2l^*}$ with $l^*$ given by $\tilde{t}(l^*)=1$. Thus, one may either use the approximate solutions (4.10) given by Riedel and Wegner[19] and neglect these artificial terms or one may use the solutions of Rudnick and Nelson,[24] where these terms are absorbed into the scaling fields.

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