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Broken symmetry and $\frac{1}{2}$ statistics of spinons in $(2+1)$-dimensional antiferromagnets

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We consider the Gutzwiller-projected Hofstadter-Slater determinant as a resonating-valence-bond wave function in a $(2+1)$-dimensional antiferromagnet. A strong case is made that spinons in the spin liquid state obey $\frac{1}{2}$ statistics when a mass gap is present. We arrive at this conclusion from an effective continuum limit gauge theory. When a mass gap opens up, the parity symmetry is spontaneously broken resulting in twofold degeneracy of the ground state. The picture is consistent with that of the fractional quantum Hall state advocated by Laughlin.

I. INTRODUCTION

The two-dimensional quantum antiferromagnet is currently under intense investigations, largely because of its relevance to the high-temperature superconductivity. There is a growing amount of evidence that the resonating-valence-bond (RVB) state, a spin liquid state, proposed by Anderson,[1] is the correct context for discussing the high-T_c superconductivity. Although much work has been done[2] since Anderson's original work, many aspects of the theory are evidently controversial. The statistics of the spinons, among other things, are being hotly debated among those who are working on the problem.

The one-dimensional analog of the spinon is known to obey Fermi statistics. It seems natural to assume that the spinons in two dimensions are also fermions. This generalization is, however, less obvious, since in one dimension fermion and boson can be transformed back and forth according to the bosonization procedure. Laughlin[3] has argued that the spinon in two dimensions necessarily obeys $\frac{1}{2}$ statistics: In the absence of the spin-spin interactions, the elementary excitations of the spin system consist of spin flips, which have spin 1 and are bosons. The only way the system could acquire spin-$\frac{1}{2}$ excitation is by charge fractionalization which then gives rise to the fractional statistics. Laughlin's approach is closely related to the phenomenon of fractional quantum Hall (FQH) effect. The $\frac{1}{2}$-statistics spinon has the FQH as its paradigm.

The equivalence of RVB and FQH states was first proposed by Kalmeyer and Laughlin.[4] Zou, Doucot, and Shastry[5] have later shown that the FQH wave function is equivalent to a Gutzwiller-projected Slater determinant and that the wave function is manifestly a spin singlet.

Parallel to the development of FQH wave function approach, there are many other proposals to characterize the RVB state in terms of $(2+1)$-dimensional quantum field theory. In particular, Dzyaloshinky, Polyakov, and Wiegmann[6] have attempted to identify spinons with instantons of an O(3) nonlinear σ model, and argued that a Hopf invariant term could stablize the instanton state and hence gives rise to the fermion character of the spinon. Many field theory models have been studied.[7] Generally speaking, these models all contain a Chern-Simons term, though it is not clear how such a topological term might occur in real magnets. There have been many attempts to derive the Chern-Simons term from the original Heisenberg model, but most of them failed to find one in $2+1$ dimensions. On the other hand, it has been shown that the large U Hubbard model possesses a local SU(2) gauge symmetry.[8] Using this gauge symmetry of the Heisenberg model in the fermion representation, the present author has pointed out the possibility of adding a parity-breaking term in the effective action, namely, the source of the Chern-Simons term in the $(2+1)$-dimensional antiferromagnets lies in the so-called parity anomaly in $2+1$ dimensions.[9] This point of view was criticized because of the incorrect treatment of the well-known fermion doubling problem in Ref. 9. The conventional wisdom is that in taking the continuum limit, the anomalies of the two species of fermion will cancel each other.

In this paper, I shall show that there exists another possibility in the presence of the two fermion species, namely, the presence of two fermions can double the coefficient of the topological term. As a result of this, the quasiparticle excitations, spinons, will be $\frac{1}{2}$ fermions. We discuss these assertions within the context of the SU(2)-invariant representation of the flux RVB state.[10] Although the flux state was initially obtained as a saddle-point solution of the nearest-neighbor Heisenberg model (NNHM), it is here assumed that the Gutzwiller-projected flux state is an exact ground state of some unknown spin Hamiltonian whose ground state is a spin liquid. We shall argue later that the approach developed in this paper is valid for a class of RVB wave functions, namely, the Gutzwiller-projected Hofstadter-Slater determinants, independent of the way we obtain the flux state. The Hofstadter model[11] is essentially a tight-binding model with a strong uniform magnetic field. The flux per square measured in the flux quantum is rational: $\phi/\phi_0=p/q$ and p,q are mutual primes. A typical hopping matrix element is of the form

$$t_{ij}\exp\left[i\int_i^j \tilde{A}\cdot d\mathbf{s}\right], \tag{1}$$

where the "magnetic field" is specified in our case to correspond to $p/q=\frac{1}{2}$. The flux state discussed in Ref. 10 is a special case of the Hofstadter model in which t_{ij} is nonzero only for nearest-neighbor hopping. In general t_{ij} can be nonzero for long-ranged hopping as well. We can form a Slater determinant on the basis of the eigenfunctions of the model (1): $\{\psi(\mathbf{k},\mathbf{r}_j)\}$, where $\mathbf{r}_j$'s are coordinates of the jth particle,

$$\Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=\mathrm{Det}\{\psi(\mathbf{k}_\alpha,\mathbf{r}_j)\} \ . \tag{2}$$

The variational wave function of the spin Hamiltonian is taken to be the Gutzwiller projected Slater determinant

$$\Psi_G(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=P_G\Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N) = \prod_i (1-n_{i\downarrow}n_{i\uparrow})\Psi \ . \tag{3}$$

The important feature of this state is that the long-wavelength fluctuations consist of fermions coupled to a gauge field which implements the Gutzwiller projection. For the simplest flux state, namely, the nearest-neighbor hopping model, the long-wavelength excitations are massless fermions. We shall first use the flux state in our discussion for the purpose of motivation. After integrating out the long-wavelength fermions, one obtains an effective action for the gauge field with a Chern-Simons term[12] of abnormal "parity" $\theta W[A]$. It is well known that the action is periodic in $\theta\to\theta+2\pi$. However, in $2+1$ dimensions, the state with $\theta=2\pi$ and the state with $\theta=0$ correspond to two physically very different states. The reason is that the Chern-Simons term in $2+1$ dimensions has the dimension of mass for the gauge field. It is the only gauge covariant form for the mass of the gauge field. Thus $\theta=0$ describes a massless gauge field, and $\theta=2\pi$ a massive gauge field. Whether $\theta=0$ or 2π in our effective gauge field action should be determined dynamically. For $\theta=2\pi$ we obviously have confinement of the gauge field and the system is in the disordered liquid phase. The excitations in this state are vortices and they obey $\frac{1}{2}$ statistics as we shall discuss in Sec. IV.

In order to see how the fermion doubling can lead to the fractional statistics of the spinons, we must first address the important question of the symmetry, namely, is there any symmetry spontaneously broken in the RVB state? What is the degeneracy of the ground state? We find that the unit cell is doubled. Kivelson and Rakhsar[13] have pointed out that the time-reversal symmetry breaking seems necessary for the fractional statistics. Authors of Ref. 14 have recently considered a model with next-nearest-neighbor hopping, which breaks the time-reversal symmetry. In this case, the ground state is twofold degenerate from time reversal. That the spontaneous symmetry breaking of the ground state and that the ground state is disordered, namely, the gauge field acquires mass, mandate the presence of a Chern-Simons term which explicitly breaks parity and provides a mass term for the gauge field simultaneously. The fact that we have two fermion species merely means that $\theta=2\pi$ and the spinons obey $\frac{1}{2}$ statistics.

II. THE FLUX STATE AND ITS CONTINUUM LIMIT

We shall focus our discussion on the flux state which is obtained by a mean-field solution[10] of the nearest-neighbor Heisenberg model (NNHM). We do not expect this solution be the ground state of the model, but we assume it to be the exact ground state of some unknown Hamiltonian. It remains a challenge to find a spin Hamiltonian for which the Gutzwiller-projected Hofstadter state is the exact ground state. The only important feature relevant to our discussion is the existence of fermionic excitations before the Gutzwiller projection. We shall show in Sec. VI how the result obtained can be generalized to a whole class of RVB wave functions, the Gutzwiller-projected Hofstadter states.

The Heisenberg model can be written in terms of electron operators as

$$H=J\sum_{\langle i,j\rangle}\chi_{ij}^\dagger\chi_{ij}+\text{const} \ , \tag{4}$$

where

$$\chi_{ij}=\sum_\sigma c_{i\sigma}^\dagger c_{j\sigma} \ .$$

The fermion operators are subject to the constraint of one particle per site, $\sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}=1$. One can also add frustration terms into this model such as next-nearest-neighbor antiferromagnetic coupling.[14] The flux state is obtained by making a mean-field approximation with "order parameter" $\langle\chi_{ij}\rangle$. The phase of the order parameter is chosen as $\chi_x=i\chi_y$. The single-particle excitation spectrum in this mean-field theory is given by

$$E_k=\pm(\cos^2k_x+\cos^2k_y)^{1/2} \ . \tag{5}$$

This spectrum is gapped everywhere except at $\mathbf{k}=(\pm\pi/2,\pm\pi/2)$. But this spectrum is not gauge invariant. A U(1) gauge transformation[15] on the link variables χ_{ij} have the effect

$$E(k_x,k_y)\to E(k_x+p,k_y+q) \ .$$

This does not, however, lead to any inconsistence, since there is a local SU(2) gauge symmetry[8] and only particle-hole excitations are allowed by the gauge symmetry.

The order parameter of the flux state $\langle\chi_{ij}\rangle$ is not invariant against a U(1) gauge transformation $c_{i\sigma}\to e^{i\theta_i}c_{i\sigma}$. However, the elementary plaquette variable

$$P(i,j,k,l)=\langle\chi_{ij}\chi_{jk}\chi_{kl}\chi_{li}\rangle \tag{6}$$

is gauge invariant and observable. $P(i,j,k,l)$ is essentially the flux enclosed within the plaquette, which is equal to π in the flux state. Upon recognizing this, we see that the saddle-point effective Hamiltonian of the flux state is equivalent to the Hofstadter model[11] on the square lattice with $\frac{1}{2}$ flux quantum per square

$$H_{\text{eff}}=J\sum_{\langle i,j\rangle\sigma}|\chi_{ij}|\exp\left[i\frac{e^*}{\hbar c}\int_i^j\tilde{A}\cdot ds\right]c_{i\sigma}^\dagger c_{j\sigma} \ , \tag{7}$$

where e^* is a factitious charge. Because of this "magnetic" field, the unit cell is doubled or the Brillouin zone is halved. The eigenstates of (7) form a representation of the "magnetic" translation group.

Let us choose a Landau gauge for the magnetic field $\tilde{A}=(0,Bx,0)$. The usual translation operators do not commute with the Hamiltonian in the presence of the magnetic field. Instead one introduces the magnetic translation operators $\hat{T}(\mathbf{a}_j)$ $(j=1,2)$, $\mathbf{a}_j$'s are the basis vectors of the square lattice. The magnetic translation operators form a so-called magnetic translation group[16] satisfying

$$\hat{T}^n(\mathbf{a}_1)\hat{T}^m(\mathbf{a}_2)=(-)^{nm}\hat{T}(n\mathbf{a}_1+m\mathbf{a}_2) . \tag{8}$$

The phase factor $(-)^{nm}$ ensures the correct group multiplication. $\hat{T}(\mathbf{a}_1)$ and $\hat{T}(\mathbf{a}_2)$ both commute with the effective Hamiltonian

$$[\hat{T}(\mathbf{a}_j),H_{\text{eff}}]=0 \quad (j=1,2) \tag{9}$$

but not with each other since

$$\hat{T}(\mathbf{a}_1)\hat{T}(\mathbf{a}_2)=\hat{T}(\mathbf{a}_2)\hat{T}(\mathbf{a}_1)e^{i2\pi(\phi/\phi_0)} , \tag{10}$$

where ϕ/ϕ_0 is the flux per square measured in flux quantum ϕ_0. The magnetic translation group is well defined only if $\phi/\phi_0=p/q$, where p and q are mutual primes. For our flux phase $p/q=\frac{1}{2}$, $\hat{T}(\mathbf{a}_1)$ commutes with $\hat{T}(2\mathbf{a}_2)$. Thus we can define a simultaneous eigenstate and two wave vectors k_1 and k_2 by

$$\begin{aligned} &H_{\text{eff}}\psi(\mathbf{k},\mathbf{r})=E(\mathbf{k})\psi(\mathbf{k},\mathbf{r}) ,\\ &\hat{T}(\mathbf{a}_1)\psi(\mathbf{k},\mathbf{r})=e^{ik_1a}\psi(\mathbf{k},\mathbf{r}) ,\\ &\hat{T}(2\mathbf{a}_2)\psi(\mathbf{k},\mathbf{r})=e^{ik_2\cdot 2a_2}\psi(\mathbf{k},\mathbf{r}) . \end{aligned} \tag{11}$$

The wave vectors k_1 and k_2 are confined within the range

$$\begin{aligned} &0\le k_1\le\frac{2\pi}{a_1} ,\\ &0\le k_2\le\frac{\pi}{a_2} . \end{aligned} \tag{12}$$

It is easy to see that $\hat{T}(\mathbf{a}_2)\psi(\mathbf{k},\mathbf{r})$ is also an eigenstate of H, $\hat{T}(\mathbf{a}_1)$, and $\hat{T}(2\mathbf{a}_2)$, and that $\hat{T}(\mathbf{a}_2)\psi$ is orthorgonal to ψ. Thus each state is twofold degenerate and they are related by a magnetic translation along the $\mathbf{a}_2$ direction (in our particular gauge):

$$\psi(\mathbf{k},\mathbf{r}), \quad \hat{T}(\mathbf{a}_2)\psi(\mathbf{k},\mathbf{r}) . \tag{13}$$

Whether or not the projected many-body wave function (3) is also degenerate due to the magnetic translation is very subtle, and we simply do not understand it.

We note that the most important feature of this mean-field theory is that the low-energy effective theory is a (2+1)-dimensional relativistic quantum field theory.

We may introduce a two-component Dirac fermion $\psi=(\psi_e,\psi_0)$; here ψ_e and ψ_0 denote fermions on the even and odd sites, respectively. Consequently, the Brillouin zone is halved. Therefore, only two Fermi points are inequivalent, say$(\pm\pi/2,\pi/2)$. If there were no gauge field, the low-energy theory would be described by Lagrangian

$$\mathcal{L}=\sum_{a=1}^{2}\int d^3x\,\bar{\psi}_a\gamma_\mu\partial_\mu\psi_a , \tag{14}$$

where $a=1,2$ correspond to the two independent massless fermions,

$$\gamma_\mu=(\gamma_0,\gamma_1,\gamma_2)=(\sigma_3,\sigma_2,\sigma_1)$$

and $\bar{\psi}=\psi^\dagger\gamma_0$.

Since the lattice model possesses the local SU(2) gauge symmetry, the continuum limit should also preserve this symmetry. The only way to implement the gauge symmetry in the continuum limit is to introduce a vector SU(2) gauge potential A_μ $(\mu=0,1,2)$. The spatial components of A_μ live on the link $\langle ij\rangle$ through the line integral

$$\mathbf{A}\text{:}\quad \int_i^j \mathbf{A}\cdot d\mathbf{s} . \tag{15}$$

[The factitious magnetic field $\tilde{A}$ defined earlier is not to be confused with the SU(2) gauge field defined here]. Under an SU(2) gauge transformation, A_μ transforms as

$$A_\mu\to g[A_\mu-\partial_\mu]g^{-1} . \tag{16}$$

The SU(2) gauge group is acting on the SU(2) doublet

$$\psi_{e(0)}=\begin{bmatrix} c_\uparrow \\ c_\downarrow^\dagger \end{bmatrix} , \tag{17}$$

namely the SU(2) transformation mixes up-spin electron and down-spin hole. Thus we may write our Lagrangian as

$$\mathcal{L}=\sum_{a=1}^{2}\int d^3x\,\bar{\psi}_a\gamma_\mu(\partial_\mu+iA_\mu)\psi_a \tag{18}$$

and

$$\psi_a=\begin{bmatrix} \psi_e \\ \psi_0 \end{bmatrix}, \quad \psi_{e(0)}=\begin{bmatrix} c_\uparrow \\ c_\downarrow^\dagger \end{bmatrix} . \tag{19}$$

We note that the ψ_a contains SU(2) indices as well as "chiral" (even or odd) indices.

One may also derive this continuum action directly from the SU(2) gauge-invariant lattice model. This was done by the author in Ref. 9. In contrast to the Ref. 9, a more careful derivation gives rise to two fermion spices, which are, of course, consistent with our flux phase picture. This fermion doubling problem is well known in the lattice gauge theory, where when going to the continuum limit an extra fermion appears. Here in our case, the fermion doubling is necessary to account for the correct low-energy degrees of freedom. It is a consequence of the reduction of the Brillouin zone.

In summary, the generalized flux states, namely, the states belonging to the same representation of the magnetic translation group defined previously, the single-particle states are twofold degenerate as indicated in (13). This seems rather obvious from the way the flux is constructed. When next-nearest-neighbor hopping is included in the Hofstadter model,[14] the time-reversal symmetry is then broken. We note that in Euclidean space the time reversal does not reverse $x_3=-it=\tau$, but simply complex conjugates c numbers and transforms the fermion fields according to

$$T_E:\ \psi \to i\gamma_1\psi \quad (\gamma_1=\sigma_2)\ , \tag{20}$$

namely,

$$\psi=\begin{bmatrix}\psi_e\\ \psi_0\end{bmatrix}\to\begin{bmatrix}\psi_a\\ -\psi_e\end{bmatrix}\ . \tag{21}$$

So we see that T_E interchange two "chiral" components of ψ: $\psi_0 \leftrightarrow \psi_e$, i.e., T_E has the effect of permute even and odd lattice sites.

III. EFFECTIVE ACTON AND THE θ TERM

We now want to integrate out the fermion degrees of freedom. We have two fermions coupled to an SU(2) gauge field. In order to calculate the fermion determinant, we need to introduce regularization scheme. The conventional method is to introduce a parity-preserving mass term for the two fermions:

$$\sum_a \bar{\psi}_a\gamma_\mu(\partial_\mu+iA_\mu)\psi_a+m(\bar{\psi}_1\psi_1-\bar{\psi}_2\psi_2)\ . \tag{22}$$

As a result of this, no parity-violating topological term can arise in the effective gauge field action $I_{\text{eff}}[A]$. However, we argue that there is another possibility which seems more consistent with our preceding discussions.

As discussed in Ref. 14, the time-reversal symmetry can be spontaneously broken in the presence of sufficiently strong frustrations. Integrating out low-energy fermions in the background of the gauge field can not restore the broken symmetry. Therefore, it seems to us that the parity-preserving regulator in (22) is inconsistent with the fact the parity has already been broken in the mean-field theory. Thus, the physically sensible way to regulate the fermion determinant is by the Pauli-Villars regulator with the same sign for the two fermion mass terms. These fermion mass terms can be thought of as being dynamically generated due to frustrations. These mass terms can also stablize the flux phase against the dimerization due to the amplitude ($|\chi_{ij}|$) fluctuations as suggested by Dombre and Kotliar.[17]

Therefore, using the Pauli-Villars regulator, the effective action for the gauge field becomes[12]

$$\begin{aligned}S_{\text{eff}}[A]&=2\ln \mathrm{Det}[\gamma_\mu(\partial_\mu+iA_\mu+m]\\ &=S_0[A]+\frac{m}{|m|}2\pi W[A]\\ &=S_0[A]\pm 2\pi W[A]\ ,\end{aligned} \tag{23}$$

where

$$W[A]=-\frac{1}{8\pi^2}\int d^3x\ \mathrm{Tr}[\tfrac{1}{2}\epsilon^{\mu\nu\rho}A^\mu F^{\nu\rho}-\tfrac{1}{3}\epsilon^{\mu\nu\rho}A^\mu A^\nu A^\rho]\ . \tag{24}$$

$S_0[A]$ is the part of the action containing a higher order of derivatives. The $\pm$ signs in front of $W[A]$ correspond to two degenerate ground states, respectively. The coefficient 2 comes from the fermion doubling and from our parity-breaking regularization scheme. Should we use the regularization scheme (22), the Chern-Simons term $W[A]$ from two fermions would have opposite signs and they would have canceled each other. Whether or not the θ term is zero (in our case $\theta=2\pi$) implies very different physical properties. $W[A]$ is the mass of the gauge field which is not gauge invariant under a large gauge transformation with winding number n, but the effective action changes by $2\pi n$. It is well known that S_{eff} is periodic in $\theta\to\theta+2\pi n$, but the actual value of the θ is determined dynamically. In the present case θ is fixed to be 2π because of the two fermions. For $\theta=0$, the gauge field is massless and the excitations of A field will interact with each other via the long-range force. For $\theta\neq 0$, the gauge field is massive and the long-range force between quasiparticles is pure gauge force. In the next section, we shall see that $\theta=2\pi$ causes the spinons in the RVB obeying $\frac{1}{2}$ statistics.

IV. $\frac{1}{2}$ STATISTICS OF THE SPINONS

Under a topologically nontrivial gauge transformation

$$A_\mu \to A_\mu^g=g^{-1}(A_\mu-\partial_\mu)g\ , \tag{25}$$

the variation of $W[A]$ is given by[18]

$$\begin{aligned}\omega(g)&=W[-g^{-1}\partial g]\\ &=\frac{1}{24\pi^2}\int d^3x\ \epsilon^{\mu\nu\rho}\mathrm{Tr}[g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\rho g]\ ,\end{aligned} \tag{26}$$

which is just the winding number of the transformation g. Wu *et al*,[19] have shown that this variation is identical to the Hopf invariant in 2+1 dimensions

$$H=\frac{1}{8\pi^2}\int d^3x\ \epsilon^{\mu\nu\rho}\tilde{A}_\mu\tilde{F}_{\nu\rho}\ , \tag{27}$$

where $\tilde{A}_\mu$ is a $U(1)$ gauge field and

$$\tilde{F}_{\nu\rho}=\partial_\nu\tilde{A}_\rho-\partial_\rho\tilde{A}_\nu\ .$$

This identification is made possible by introducing the SU(2) representation of the gauge transformation g_n:

$$g=\begin{bmatrix}z_1 & z_2\\ -z_2^* & z_1^*\end{bmatrix}, \tag{28}$$

with z_1,z_2 satisfying

$$|z_1|^2+|z_2|^2=1\ ,$$

and the gauge potential $\tilde{A}_\mu$ can be written as

$$\tilde{A}_\mu=i(z^\dagger\partial_\mu z),\quad z=(z_1,z_2)\ . \tag{29}$$

We recogonize that this representation is the same as that of the CP^1 form representation for the nonlinear σ model with $\mathbf{n}=z^\dagger\sigma z$.

We now create spinons in the Gutzwiller-projected Slater determinant. The spinons can be obtained by first creating a hole in the Slater determinant and then making a Gutzwiller projection. The spinons are nothing but the ordinary projected holes. When the mass of the gauge field is large enough, we can regard the spinons as pointlike particles $j_0(x)\sim\delta(x-x_0)$. They interact with the gauge field via the standard coupling $j_\mu\cdot A_\mu$. In one spinon sector, the effective action is taken to be

$$\mathcal{L}=j_{\mu}\cdot\tilde{A}_{\mu}+2\pi H=j_{\mu}\cdot\tilde{A}_{\mu}+\frac{1}{4\pi}\epsilon^{\mu\nu\rho}\tilde{A}_{\mu}\tilde{F}_{\nu\rho}\ . \quad (30)$$

The first term describes the coupling between spinon current and the gauge field. The equation of motion for $\tilde{A}_{\mu}$ gives

$$j_{\mu}=\frac{\mu}{2}\epsilon^{\mu\nu\rho}\tilde{F}_{\nu\rho}\ , \quad (31)$$

where $\mu=1/\pi$. The zeroth component of this equation relates the "charge" of the spinon to the flux

$$j_0=\mu\tilde{F}_{12}\ . \quad (32)$$

Thus we see

$$e^*=\int d^2\mathbf{x}\, j_0=\mu\int ds\, \tilde{F}_{12}\ . \quad (33)$$

The conclusion is that the "charged" particle is also a flux tube carrying $\tilde{A}$ flux, of magnitude

$$\Phi=\frac{e^*}{\mu}=\pi\ . \quad (34)$$

Here we have set the factitious charge e^* to unity ($e^*=1$). The field strength $\tilde{F}$ vanishes at large distance ($\gg\mu^{-1}$) from the flux tube. The remarkable result of Eq. (34) tells us that the spinon also carries the flux of the gauge field, i.e., a spinon is a charged (spin) particle attached to a flux line. Thus the spinons are also vortices of the gauge field.

Suppose the bare charge of the spinon is q_0. Because of the screening effect of the gauge field in the presence of the Chern-Simons term, the effective charge of the spinon will be reduced to $q=q_0/2$. This subtle point was clarified in Ref. 20. Charged particle ($q=q_0/2$) moving around the flux tube with flux $\Phi=\pi$ acquires Aharonov-Bohm phase. We therefore conclude that the spinons obey θ statistics with θ given by

$$\theta=q^2/2\mu=\pi/2\ , \quad (35)$$

namely, when interchanging two spinons at large distance, the wave function is accompanied by a phase $e^{i\pi/2}$. Thus the spinons are $\frac{1}{2}$ fermions. We emphasize that $\theta=\pi/2$ results from the fact that we have two species of fermion in (23) and that we have used the parity-breaking regularization scheme that is consistent with the ground-state degeneracy. Should one of the fermions be absent, the spinon would be a fermion.

It is worth noting that one can also arrive at the same conclusion from a U(1) gauge theory. In the U(1) gauge theory, we would have four fermions, two from the reduction of the Brillouin zone and two from the spins. But the coefficient of the Chern-Simons term for the U(1) field is $\frac{1}{2}$ of that for the SU(2) field.[12] Thus the final result is identical. The relation between SU(2) and U(1) theories was clarified in Ref. 19.

V. CONCLUSIONS AND GENERALIZATION

We have so far discussed only the flux phase with $\frac{1}{2}$-flux quantum per square. We want to stress again that the term "flux state" really means the Gutzwiller-projected Hofstadter-Slater determinant,

$$\Psi(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=P_G\mathrm{Det}\{\psi(\mathbf{k}_\alpha,\mathbf{r}_j)\}\ , \quad (36)$$

where the entries of the determinant are the Hofstadter states corresponding to $\frac{1}{2}$-flux quantum per square ($p/q=\frac{1}{2}$). This is obviously a spin-singlet RVB wave function. We have shown that this state is at least twofold degenerate when the time-reversal symmetry is broken.

Although it is first obtained by a mean-field theory for the nearest neighbor Heisenberg model (NNHM), we do not expect this state be the ground state for NNHM. The purpose of this work is to study the properties of the wave function (36) in the continuum limit, the statistics of the quasiparticle excitations in particular. The important feature of this wave function is that before the Gutzwiller projection, the free Hofstadter state possesses two low-energy fermion excitations, which is described by relativistic quantum field theory. The Gutzwiller projection in this work is implemented by coupling the fermions to an SU(2) gauge field. The equivalence of the Gutzwiller projection and the SU(2) gauge symmetry was explicitly demonstrated in Ref. 8. Therefore, the low-energy physics is described by a relativistic quantum field theory involving fermions coupled to a gauge field. This enables us to utilize some of the well-known field theory results, especially the result on anomaly.

Although the anomaly in the modern gauge field theory is still poorly understood, it is commonly believed that the anomaly has its origin in spontaneous parity breaking. The ground state is at least twofold degenerate in the presence of the anomaly. One conclusion drawn from this work based on the analog with the continuum theory is that the spinons in the flux state (3) obey $\frac{1}{2}$ statistics when parity symmetry is broken and the ground state is then at least twofold degenerate.

Our approach can be generalized to other flux states with $\phi/\phi_0=p/q$. It was proved by Hofstadter[11] using numerical calculation, and recently by Wen and Zee[21] using topological argument that for even q, the corresponding Hofstadter model possesses q gapless Fermi points, which correspond to q families of Dirac fermions in the continuum limit. In this case, one can prove that the single-particle states are q-fold degenerate and that the q degenerate ground states are related by the magnetic translations

$$\Psi,\hat{T}(\mathbf{a}_2)\Psi,\ldots,\hat{T}^{q-1}(\mathbf{a}_2)\Psi\ , \quad (37)$$

one can immediately conclude that the quasiparticles obey θ statistics with θ given by

$$\theta=\pi/q\ . \quad (38)$$

Thus, for $q=4$, we would have $\frac{1}{4}$ fermion.

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