Drift and diffusion in a one-dimensional disordered system

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The low-frequency drift and diffusion coefficients in a one-dimensional diffusion model are determined. Of particular interest is the case of a constant bias and random hopping elements with a distribution which diverges for small values of the hopping elements. With use of the replica method this model is shown to be equivalent to a particle in a uniform magnetic field proportional to the bias. This model exhibits scaling behavior at low frequencies with crossover scaling behavior for small and large bias.

I. INTRODUCTION

The problem of drift and diffusion of a particle on a random, one-dimensional random lattice has been widely studied recently.[1–10] Various models have been proposed and in many of them the hopping rates are biased in one direction and the particle drifts in this direction with diffusion superimposed on this motion. Depending on the model the drift and diffusion may be normal or anomalous. These models may be of interest in neural networks, microcircuits in electric fields, diffusion in alloys, etc.

The motion of the particle is described by an equation of the form

$$\frac{\partial p_r}{\partial t}=W'_{r-1,r}p_{r-1}+W'_{r+1,r}p_{r+1}-(W'_{r,r-1}+W'_{r,r+1})p_r \ , \tag{1.1}$$

where $W'_{r-1,r}$ is the hopping rate from site $r-1$ to site r and p_r is the probability of finding the particle at site r. In the problem of diffusion with bias $W'_{r-1,r}\neq W'_{r,r-1}$ and in this paper we will study the case in which

$$\begin{aligned} W'_{r-1,r}&=W_{r-1,r}-\delta \ , \\ W'_{r,r-1}&=W_{r,r-1}+\delta \ , \end{aligned} \tag{1.2}$$

where δ is a constant bias and $W_{r-1,r}=W_{r,r-1}$ is a random variable chosen from the same distribution for each bond. We consider two types of distributions, $\rho(W)$, which have been widely studied[11] in the case $\delta=0$: (a) $\rho(W)$ is such that the inverse moments of W are finite, e.g.,

$$W_a^{-1}=\int_0^{\infty}dW\,\rho(W)W^{-1}<\infty \ , \tag{1.3}$$

and, case (b),

$$\rho(W)=\begin{cases}(1-\alpha)W^{-\alpha} \ \ (\text{with } 0<\alpha<1), \ \ 0<W<1 & (1.4a)\\ 0, \ \ 1<W. & (1.4b)\end{cases}$$

In case (a) [Eq. (1.3)], a number of methods can be applied to calculate the average drift and diffusion constant which are normal, i.e., at low frequencies the average drift and diffusion constants are frequency independent. Aslangul, Pottier, and Saint-James[12] have studied two models, not identical to (1.2), which fall into this category. This paper is devoted mainly to case (b) [Eq. (1.4)] and we extend the method of Stephen and Kariotis[13] (SK) who considered the same problem with $\delta=0$. Use of the replica method enables us to map this problem onto that of a particle in magnetic field proportional to the bias δ. We are then able to obtain results for the drift velocity and diffusion constant at low frequencies for small and large δ.

II. GENERATING FUNCTION

The Laplace transform of (1.1) with the initial condition $p_r(0)=\delta_{r,0}$ is

$$\omega P_r-W'_{r-1,r}P_{r-1}-W'_{r+1,r}P_{r+1}+(W'_{r,r-1}+W'_{r,r+1})P_r=\delta_{r,0}+U_r \ , \tag{2.1}$$

where

$$P_r(\omega)=\int_0^{\infty}e^{-\omega t}p_r(t)dt$$

and we have included an external potential U_r for convenience. The solution of (2.1) can be written in terms of the Green's functions G and up to terms linear in U, $P_r=P_r^{(0)}+P_r^{(1)}$ with

$$\begin{aligned} P_r^{(0)}&=G_{r0}(\omega) \ , \\ P_r^{(1)}&=\sum_r G_{0r}(\omega)U_r \ . \end{aligned} \tag{2.2}$$

These Green's functions are easily found in the ordered system in which $W_{r-1,r}=W$ for all r and for later reference we note the results

$$G_{00}^{(0)}(\omega)\frac{1}{(\omega^2+4W\omega+4\delta^2)^{1/2}} , \quad (2.3)$$

$$G_{(k,\omega)}^{(0)}=\frac{1}{\omega-2i\delta\sin k+2W(1-\cos k)} , \quad (2.4)$$

where the superscript zero refers to the ordered case and $G^{(0)}(k,\omega)$ is the Fourier transform of $G_{0r}^{(0)}(\omega)$. Thus δ determines the drift and W the diffusion of the particle. In case (a) to a first approximation in the inverse moments of W the results (2.3) and (2.4) are still valid with W replaced by W_a.

For the disordered system we expect $\langle G_{00}(\omega)\rangle$ and $\langle G(k,\omega)\rangle$ to show scaling behavior in the low-frequency limit:

$$\langle G_{00}(\omega)\rangle=\frac{1}{\omega^{2\beta}}f_0(\delta/\omega^{\phi}) , \quad (2.5)$$

$$\langle G(k,\omega)\rangle=\frac{1}{\omega}f(k\xi,\delta/\omega^{\phi}) , \quad (2.6)$$

where the angular brackets indicate an average over the disorder. From (2.3) and (2.4) in case (a) $\beta=\frac{1}{4}$ and the crossover exponent $\phi=\frac{1}{2}$. The correlation length $\xi=\omega^{2\beta-1}\xi_0(\delta/\omega^{\phi})$. In case (b) the exponent $\beta=1/[2(2-\alpha)]$, the same as in SK, and we will show below that $\phi=2\beta$.

To identify the drift velocity v and diffusion constant D we can expand (2.6) in powers of $k\xi$ and write the expansion in the form

$$\langle G(k,\omega)\rangle=\frac{1}{\omega}\left[1+\frac{2ik}{\omega}v-\frac{k^2}{\omega}D-\frac{4k^2}{\omega^2}v^2+\cdots\right] . \quad (2.7)$$

We now introduce a partition function

$$Z=\left\langle\int(dV)e^{H(V)}\right\rangle , \quad (2.8)$$

where the $\mathbf{V}$ are n-component complex variables, integration is over the real and imaginary parts of $\mathbf{V}$ between $-\infty$ and $+\infty$, and

$$H(V)=i\sum_{r,s}\mathbf{V}_r^*\cdot G^{-1}{}_{rs}\mathbf{V}_s-i\sum_r(\mathbf{V}_r\cdot\mathbf{U}_r+\mathbf{V}_r^*\cdot\mathbf{U}_r^*) \quad (2.9)$$

and

$$G^{-1}{}_{rs}=(\omega+W'_{r,r-1}+W'_{r,r+1})\delta_{rs} -\delta_{s,r-1}W'_{r-1,r}-\delta_{s,r+1}W'_{r+1,r} . \quad (2.10)$$

It is convenient to arrange that G^{-1} is Hermitian and to achieve this we replace δ by $i\delta'$. In the final results we will analytically continue δ' to $e^{-i\pi/2}\delta$ with no apparent difficulties. To determine the average Green's functions we introduce

$$s(p)=\left\langle\int(dV)e^{-i\mathbf{p}\cdot\mathbf{V}_0-i\mathbf{p}^*\cdot\mathbf{V}_0^*}e^{H(V)}\right\rangle , \quad (2.11)$$

where $\mathbf{p}$ is an n-component vector and it is understood that the $n=0$ limit is to be taken. Omitting terms in U^* which are not required

$$s(p)=\left\langle\exp\left[-i\mathbf{p}^*\cdot\mathbf{p}G_{00}-i\sum_r\mathbf{U}_r\cdot G_{r0}\mathbf{p}^*\right]\right\rangle . \quad (2.12)$$

The average Green's functions are then given by

$$\langle G_{00}(\omega)\rangle=\left[i\frac{\partial^2}{\partial p_\alpha\,\partial p_\alpha^*}s_0(p)\right]_{p=0} , \quad (2.13)$$

$$\langle G(k,\omega)\rangle=\frac{i}{U}\left[\frac{\partial}{\partial p_\alpha^*}\delta s(p)\right]_{p=0} , \quad (2.14)$$

where $s_0(p)$ is the part of s independent of U, δs is linear in U, and we have chosen $U_r=Ue^{-ikr}$. It is convenient to introduce the Fourier transform of $s(p)$

$$s(V)=\int d\mathbf{p}\,e^{i(\mathbf{p}\cdot\mathbf{V}+\mathbf{p}^*\cdot\mathbf{V}^*)}s(p) . \quad (2.15)$$

In terms of this function

$$\langle G_{00}(\omega)\rangle=-i\int d\mathbf{V}\,|V_\alpha|^2 s_0(V) , \quad (2.16)$$

$$\langle G(k,\omega)\rangle=\frac{1}{U}\int d\mathbf{V}\,V_\alpha^*\delta s(\mathbf{V}) \quad (2.17)$$

in the limit $n=0$.

III. INTEGRAL EQUATION

We focus attention on site 0 and in generating function (2.11) we represent everything to the right and left of site 0 by Q_r' and Q_L', respectively. Then

$$s_0(V)=e^{i\omega|\mathbf{V}|^2}Q_R'Q_L' . \quad (3.1)$$

From translational invariance the Q' satisfy the integral equation

$$Q_{R,L}'(V)=\int d\mathbf{V}'\,K_{R,L}'(\mathbf{V},\mathbf{V}')Q_{R,L}(V') , \quad (3.2)$$

where

$$K_{R,L}'(\mathbf{V},\mathbf{V}')=\langle e^{iW|\mathbf{V}-\mathbf{V}'|^2}\rangle e^{\pm\delta'(\mathbf{V}^*-\mathbf{V}'^*)\cdot(\mathbf{V}+\mathbf{V}')+i\omega|\mathbf{V}'|^2} . \quad (3.3)$$

It is convenient to put

$$Q_{R,L}'(V)=e^{(i\omega\pm\delta')|\mathbf{V}|^2}Q_{R,L}(V) \quad (3.4)$$

so that Q satisfies

$$e^{-i\omega|\mathbf{V}|^2}Q_{R,L}(V)=\int d\mathbf{V}'\,K_{R,L}(\mathbf{V},\mathbf{V}')Q_{R,L}(V) , \quad (3.5)$$

where

$$K_{R,L}(\mathbf{V},\mathbf{V}')=\langle e^{iW|\mathbf{V}-\mathbf{V}'|^2}\rangle e^{\pm\delta'(\mathbf{V}\cdot\mathbf{V}'-\mathbf{V}\cdot\mathbf{V}'^*)} . \quad (3.6)$$

We can reduce (3.5) to a differential equation by noting that apart from a multiplicative factor (which is unity when $n=0$) K is the Green's function, replicated n times, of a particle of unit mass in a magnetic field in two dimensions:[14]

$$D(\mathbf{r}',\mathbf{r},\beta)=\sum_\alpha \varphi_\alpha^*(\mathbf{r}')e^{-\beta H}\varphi_\alpha(\mathbf{r})$$
$$=\frac{\omega_c}{4\pi\sinh(\beta\omega_c/2)}\exp\left[\frac{i\omega_c}{2}(xy'-yx')-\frac{\omega_c}{4}\coth\left[\frac{\beta\omega_c}{2}\right](\mathbf{r}'-\mathbf{r})^2\right], \tag{3.7}$$

where

$$H=\frac{1}{2}\left[\mathbf{p}-\frac{e}{c}\mathbf{A}\right]^2, \quad \mathbf{A}=\frac{B}{2}(y,-x),$$

and φ_α are the eigenstates of H. Focusing attention on K_R, we identify the cyclotron frequency $\omega_c=4\delta'$ and the inverse temperature

$$\beta=\frac{1}{4\delta'}\ln\left[\frac{W+i\delta'}{W-i\delta'}\right] \tag{3.8}$$

and we can write the Hamiltonian in the form

$$H=-\frac{1}{2}\left[\frac{\partial}{\partial \mathbf{V}_x}-2i\delta'\mathbf{V}_y\right]^2-\frac{1}{2}\left[\frac{\partial}{\partial \mathbf{V}_y}+2i\delta'\mathbf{V}_x\right]^2. \tag{3.9}$$

Then $K(V,V')=D(\mathbf{V}',\mathbf{V},\beta)$ and the integral equation (3.5) can be written

$$\left\langle\left[\frac{W-i\delta'}{W+i\delta'}\right]^{H/4\delta'}\right\rangle Q=e^{-i\omega|V|^2}Q. \tag{3.10}$$

We omit the subscripts R and L because Q is the spherically symmetric solution of (3.10) and the sign of the field δ' is unimportant.

We now focus attention on case (b) in which case this equation simplifies considerably in the low-frequency scaling region. We introduce scaled variables $W=\omega^{2\beta}u$, $\mathbf{V}=\omega^{-\beta}(\mathbf{x}+i\mathbf{y})$, and $\Delta=\delta'/\omega^{2\beta}$ and after subtracting Q from both sides of (3.10) and retaining the leading terms in ω, as in SK, we find (with $\beta=1/[2(2-\alpha)]$)

$$F(H'/4\Delta)Q=-i\Delta^{\alpha-1}(\mathbf{x}^2+\mathbf{y}^2)Q, \tag{3.11}$$

where

$$F(x)=(1-\alpha)\int_0^\infty du\frac{1}{u^\alpha}\left[\left[\frac{u-i}{u+i}\right]^x-1\right] \tag{3.12}$$

and

$$H'(\Delta)=-\frac{1}{2}\left[\frac{\partial}{\partial\mathbf{x}}-2i\Delta\mathbf{y}\right]^2-\frac{1}{2}\left[\frac{\partial}{\partial\mathbf{y}}+2i\Delta\mathbf{x}\right]^2. \tag{3.13}$$

Thus the crossover exponent of (2.5) $\phi=2\beta$. Equation (3.11) cannot be solved in closed form but we can obtain results for large and small values of the scaled field Δ by expanding in the eigenstates of H'. Q is a spherically symmetrical function of $R^2=\mathbf{x}^2+\mathbf{y}^2$ in which case

$$H'=-\frac{1}{2}\frac{\partial^2}{\partial R^2}+\frac{1}{2R}\frac{\partial}{\partial R}+2\Delta^2R^2 \tag{3.14}$$

in the $n=0$ limit. It is easily shown that its normalized eigenfunctions and eigenvalues are

$$\psi_m=e^{-z/2}zL_{m-1}^{(1)}(z)/\sqrt{m}, \quad E_m=4\Delta m, \tag{3.15}$$

where $m\geq 1$ is an integer, $\psi_0=e^{-z/2}$, $L_m^{(1)}$ is a Laguerre function, and $z=2\Delta R^2$. We can now expand

$$Q=\sum_{m=0}^\infty a_m\psi_m \tag{3.16}$$

with $a_0=1$ to satisfy the condition $Q(0)=1$. When (3.16) is substituted in (3.1) and (2.17)

$$\langle G_{00}\rangle=\frac{1}{2\delta}\sum_{m=0}^\infty(\sqrt{m}\,a_m-\sqrt{m+1}a_{m+1})^2. \tag{3.17}$$

We adopt the convention here and below that $\lim_{m\to 0}a_m\sqrt{m}=-a_0=-1$. The equations for the coefficients a_m are obtained by substituting (3.16) in (3.10) and using the recursion relations of the Laguerre functions. This gives

$$F(m)a_m=-\tfrac{1}{2}i\Delta^{\alpha-2}[-\sqrt{m(m-1)}a_{m-1}+2ma_m-\sqrt{m(m+1)}a_{m+1}] \tag{3.18}$$

for $m\geq 1$.

(a) $\Delta\gg 1$. In this case we only need the first few coefficients and from (3.18) with $a_0=1$ we find

$$a_1=-i/2\Delta^{2-\alpha}F(1), \quad F(1)=\frac{2\pi(1-\alpha)}{\sin(\pi\alpha)}e^{-i\pi(1+\alpha)/2}. \tag{3.19}$$

When these results are substituted in (3.17) we find

$$\langle G_{00}\rangle=\frac{1}{2\delta}\left[1-\frac{\omega\sin(\pi\alpha)}{2\pi(1-\alpha)\delta^{2-\alpha}}+\cdots\right], \tag{3.20}$$

where we have replaced δ' by $e^{i\pi/2}\delta$. This is consistent with scaling form (2.5). We can also identify the diffusion constant as $D=[\sin(\pi\alpha)/\pi(1-\alpha)]\delta^\alpha$ from the second term of (3.20).

(b) $\Delta<1$. In this case we require a_m for large m and (3.18) can be replaced by a differential equation. For large m (see Appendix)

$$F(m)=-\Gamma(\alpha)(2me^{i\pi/2})^{1-\alpha} \tag{3.21}$$

and (3.18) becomes

$$m^2a_m''+ma_m'-\tfrac{1}{4}a_m=\Gamma(\alpha)(2m\Delta e^{i\pi/2})^{2-\alpha}. \tag{3.22}$$

The substitution $m=(At)^{4\beta}/4\Delta e^{i\pi/2}$ with $A^2=(2-\alpha)^2/2^\alpha\Gamma(\alpha)$ reduces this to Bessel's equation and the solution satisfying the boundary condition $\lim_{m\to 0}\sqrt{m}\,a_m=-1$ and vanishing at ∞ is

$$a_m = -\frac{4(\Delta e^{i\pi/2})^{1/2}}{\Gamma(2\beta)(2A)^{2\beta}} K_{2\beta}(t) , \tag{3.23}$$

where K is a Bessel function. Equation (3.17) can now be evaluated by replacing the sum by an integral with the result

$$\langle G_{00} \rangle = \frac{\pi C_\beta^{(0)}}{\sin(2\pi\beta)\omega^{2\beta}} , \tag{3.24}$$

where $C_\beta^{(0)}$ is given in SK. This result agrees exactly with SK. We have not carried it to higher order in δ because it is easier to determine the drift velocity and diffusion constant for weak fields using the results of the next section.

IV. DRIFT AND DIFFUSION

In this section we discuss the Green's function (2.14) and determine the drift and diffusion constants using (2.7). When we include a field $U_r = Ue^{-ikr}$ the part of the generating function (2.12) linear in U can be written

$$\delta s(V) = e^{i\omega(\mathbf{V})^2}(\delta Q_L' Q_R' + Q_L' \delta Q_R' - i\mathbf{U}\cdot\mathbf{V} Q_L' Q_R') , \tag{4.1}$$

where $\delta Q_{R,L}'$ are the changes in $Q_{R,L}'$ due to the field. We again put

$$\delta Q_{R,L}'(V) = e^{(i\omega \pm \delta')|\mathbf{V}|^2} \delta Q_{R,L}(V) \tag{4.2}$$

so that δQ satisfies

$$e^{\pm ik} \delta Q_{R,L}(V) = \int K_{R,L}(\mathbf{V},\mathbf{V}') \times [\delta Q_{R,L}(\mathbf{V}') - i\mathbf{V}'\cdot\mathbf{U} Q_{R,L}(V')] , \tag{4.3}$$

where K is given by (3.6). We now adopt the convention in the following equations that the upper and lower signs refer to δQ_R and δQ_L, respectively. Following the same methods as in Sec. III, in case (b) in the scaling region (4.3) can be replaced by

$$\Delta^{1-\alpha} F(H''/4\Delta)\delta Q = [-iR^2 + \omega^{2\beta-1}(e^{\pm ik}-1)]\delta Q + i\omega^{\beta-1}\mathbf{U}\cdot(\mathbf{x}+i\mathbf{y})Q \tag{4.4}$$

when $H'' = H'(\pm\Delta)$ where $H'(\Delta)$ is given by (3.13). The solution of (4.4) for $k=0$, $\delta Q^{(0)}$ say, is given by

$$\delta Q^{(0)} = -\tfrac{1}{2}\omega^{\beta-1}\mathbf{U}\cdot\left[\left[\frac{\partial}{\partial \mathbf{x}} \pm 2i\Delta \mathbf{y}\right] + i\left[\frac{\partial}{\partial \mathbf{y}} \mp 2i\Delta \mathbf{x}\right]\right] Q . \tag{4.5}$$

This is easily verified by applying the operator on the right-hand side of (4.5) to (3.11) and noting that this operator commutes with H''. It is thus convenient to put

$$\delta Q = \delta Q^{(0)} + M' \tag{4.6}$$

and after using the equation satisfied by $\delta Q^{(0)}$ we obtain

$$[\Delta^{1-\alpha}F(H''/4\Delta) + iR^2]M' = \omega^{2\beta-1}(e^{\pm ik}-1) \times (\delta Q^{(0)} + M') . \tag{4.7}$$

From (4.5) M' must be of the form

$$M' = -2\Delta\omega^{\beta-1}\mathbf{U}\cdot(\mathbf{x}+i\mathbf{y})M , \tag{4.8}$$

where M is a function of R^2 only. In the $n=0$ limit H'' acting on a function of this symmetry replaces (3.14) by

$$H'' = -\frac{1}{2}\frac{\partial^2}{\partial R^2} - \frac{1}{2R}\frac{\partial}{\partial R} + 2\Delta^2 R^2 \pm 2\Delta . \tag{4.9}$$

This Hamiltonian has normalized eigenfunctions and eigenvalues

$$\phi_m = e^{-z/2} L_m^{(0)}(z); \quad E_m = 4\Delta(m + \tfrac{1}{2} \pm \tfrac{1}{2}) \tag{4.10}$$

with $m \geq 0$. We now expand M in terms of this complete set of eigenfunctions

$$M = \sum_{m=0}^{\infty} b_m \phi_m . \tag{4.11}$$

After expressing $\delta Q^{(0)}$ given by (4.5) and (3.16) in terms of these eigenfunctions, we find the equations for the coefficients b_m

$$F(m + \tfrac{1}{2} \pm \tfrac{1}{2})b_m + \frac{i}{2}\Delta^{\alpha-2}[-mb_{m-1} + (2m+1)b_m - (m+1)b_{m+1}] = \Delta^{\alpha-1}\omega^{2\beta-1}(e^{\pm ik}-1)[a_{m+1/2\pm 1/2}(m + \tfrac{1}{2} \pm \tfrac{1}{2})^{1/2} + b_m] , \tag{4.12}$$

where again we use the convention that $\lim_{m\to 0}\sqrt{m}\,a_m = -a_0 = -1$.

The Green's function (2.14) can be expressed in terms of the coefficient b_m:

$$\langle G(k,\omega) = \frac{1}{\omega}\left[1 - \sum_{m=0}^{\infty}(b_m^{(L)} + b_m^{(R)}) \times (\sqrt{m+1}\,a_{m+1} - \sqrt{m}\,a_m)\right] . \tag{4.13}$$

Again Eq. (4.12) can be solved in the limits of large and small fields.

(a) $\Delta > 1$. In this case we only need to consider the coefficients $b_0^{(L)}$ and $b_l^{(L)}$ and after expansion in powers of k we find

$$\langle G(k,\omega) \rangle = \frac{1}{\omega}(1 - A_k + A_k^2 - a_1 A_k^2 + \cdots) , \tag{4.14}$$

where $A_k = 2\Delta\omega^{2\beta-1}k$. Comparison with (2.7) gives the drift velocity and diffusion constant for $\Delta > 1$,

$$v=\delta, \quad D=\frac{\sin(\pi\alpha)}{\pi(1-\alpha)}\delta^{\alpha} . \tag{4.15}$$

(b) $\Delta<1$. In this case we require b_m for large m and we approximate (4.8) by a differential equation as in (3.22). This equation can be solved as a power series in k following the methods of SK and we find the drift velocity

$$v=\frac{2^{4-4\beta}\beta}{\Gamma^2(2\beta)}I\delta , \tag{4.16}$$

where

$$I=-\int_0^{\infty}dt\int_0^{\infty}dt'\,(tt')^{2\beta}K_{2\beta-1}(t)K_{2\beta-1}(t')\mathcal{G}(t,t') \tag{4.17}$$

and $\mathcal{G}$ is the Green's function introduced by SK. The drift is linear in δ. The diffusion constant in this limit has been calculated in SK and is proportional to $\omega^{4\beta-1}$. Again these results are consistent with the scaling form (2.6).

V. DISCUSSION

The low-frequency behavior of a one-dimensional diffusion model with constant bias and random hopping elements has been determined. The replica method shows that this problem is equivalent to that of a particle in a magnetic field proportional to the bias and allows us to readily calculate the behavior for large and small fields. The scaled field is $\Delta=\delta/\omega^{2\beta}$ so that the results at large Δ give the behavior at low frequencies or at long times. At low frequencies we have confirmed the scaling behavior of this model and determined the crossover scaling exponent. For both large and small values of $\delta/\omega^{2\beta}$ the drift is proportional to δ and the diffusion constant varies as δ^{α} and $\omega^{4\beta-1}$, respectively.

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APPENDIX

With the substitution $x=(u-i)/(u+i)$ in (3.12) we find

$$F(m)=-C_{\alpha}\int_{-1}^{1}dx\frac{1-x^m}{(1-x)^{2-\alpha}(1+x)^{\alpha}} , \tag{A1}$$

where $C_{\alpha}=2(1-\alpha)e^{i\pi(1-\alpha)/2}$. We break the integral up into two parts:

$$F(m)=-C_{\alpha}\left[\int_0^1 dx\frac{1-x^m}{(1-x)^{2-\alpha}(1+x)^{\alpha}} + \int_0^1\frac{dx\,[1-(-x)^m]}{(1+x)^{2-\alpha}(1-x)^{\alpha}}\right] . \tag{A2}$$

The second integral is bounded as $m\to\infty$ ($\alpha<1$) and we omit it. In the first integral for large m the main contribution comes from $x\sim 1$ so we set $1-x=y/m$ and

$$F(m)\simeq -C_{\alpha}2^{-\alpha}m^{-\alpha}\int_0^m dy\frac{1-(1-y/m)^m}{y^{2-\alpha}(1-y/2m)^{\alpha}} . \tag{A3}$$

We may now take the limit $m\to\infty$ of the integral and obtain

$$F(m)\simeq -C_{\alpha}2^{-\alpha}m^{1-\alpha}\int_0^{\infty}dy\,y^{\alpha-2}(1-e^{-y}) . \tag{A4}$$

This is easily evaluated to give (3.21). This limiting procedure can be justified by dividing the interval of integration in (A3) into two parts:

$$\int_0^m dy=\int_0^{m^{\beta}}dy+\int_{m^{\beta}}^m dy , \tag{A5}$$

where $0<\beta<\frac{1}{2}$. Then

$$\lim_{m\to\infty}\int_0^{m^{\beta}}dy\frac{1-(1-y/m)^m}{y^{2-\alpha}(1-y/2m)^{\alpha}}=\int_0^{\infty}dy\,y^{\alpha-2}(1-e^{-y}) . \tag{A6}$$

In this interval $y/m\ll 1$ for large m so that the integrand converges to the right-hand side of (A6). The contribution of the second interval in (A5) is

$$\int_{m^{\beta}}^m dy\frac{1-(1-y/m)^m}{y^{2-\alpha}(1-y/2m)^{\alpha}}<2^{\alpha}\int_{m^{\beta}}^m dy\frac{1}{y^{2-\alpha}} =O(1/m^{(1-\alpha)\beta}) \tag{A7}$$

and vanishes as $m\to\infty$.

[1] For recent reviews see J. W. Haus and K. W. Kehr, Phys. Rep. **150**, 263 (1987) and S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987); and references therein.
[2] M. J. Stephen, J. Phys. C **14**, L1077 (1981).
[3] B. Derrida and Y. Pomeau, Phys. Rev. Lett. **48**, 627 (1982).
[4] B. Derrida and R. Orbach, Phys. Rev. B **27**, 4694 (1983).
[5] B. Derrida, J. Stat. Phys. **31**, 433 (1983).
[6] R. Biller, Z. Phys. B **55**, 7 (1984).
[7] J. Bernasconi and W. R. Schneider, J. Phys. A **15**, L729 (1983).
[8] J. Bernasconi and W. R. Schneider, Helv. Phys. Acta **58**, 597 (1985).
[9] T. Schneider, M. P. Soerensen, A. Politi, and M. Zannetti, Phys. Rev. Lett. **56**, 2341 (1986).
[10] J. P. Bouchaud, A. Comtet, A. Georges, and P. LeDoussal, Europhys. Lett. **3**, 653 (1987).
[11] For a review see S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981).
[12] C. Aslangul, N. Pottier, and D. Saint-James, J. Phys. (Paris) (to be published).
[13] M. J. Stephen and R. Kariotis, Phys. Rev. B **26**, 1917 (1982).
[14] E. H. Sondheimer and A. H. Wilson, Proc. R. Soc. London, Ser. A **210**, 173 (1951).