High-frequency damping of collective excitations in fermion systems. I. Plasmon damping and frequency-dependent local-field factor in a two-dimensional electron gas

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Starting from a general expression of Glick and Long [Phys. Rev. B 4, 3455 (1971)] for the imaginary part of the frequency- and wave-number-dependent dielectric function $\epsilon(k,\omega)$ of an electron gas in a two-particle–hole pair excitation approximation, an asymptotic expression for $\epsilon_2(k,\omega)$ is derived, which is valid for *any wave number k* provided ω is much higher than a certain characteristic k-dependent frequency. The formula for $\epsilon_2(k,\omega)$ is cast in a form suitable for application to any arbitrary potential $v(k)$ in three-, two-, and one-dimensional space. In the case of $D=3$ and the Coulomb potential $v(k)$ our result is the same as that of Glick and Long, although its region of validity is now larger. As a specific example, the damping of plasmons in a 2D electron gas has been calculated. Also, an interpolation formula for the complex local-field factor $G(k,\omega)$ in a 2D electron gas has been derived. The latter immediately leads to an expression for the ω-dependent exchange-correlation potential—a result analogous to the one derived by Gross and Kohn in the 3D case. The asymptotic expression for $\epsilon_2(k,\omega)$ is used in paper II to calculate the damping of zero sound in normal liquid ^{3}He.

I. INTRODUCTION

Elementary excitations in Fermi liquids, like the plasmons in an electron gas or zero-sound phonons in liquid ^{3}He, may be found theoretically as complex poles of the density-density response function $\chi(k,\omega)$, or, alternatively, as peaks of the dynamic structure factor $S(k,\omega)$ viewed as a function of ω. These functions are written in terms of the proper polarizability $\Pi(k,\omega)$ and the interaction potential between particles $v(k)$ (see, e.g., Ref. 1):

$$\chi(k,\omega)=\Pi(k,\omega)/[1-v(k)\Pi(k,\omega)]=\Pi(k,\omega)/\epsilon(k,\omega)\ , \tag{1.1}$$

$$S(k,\omega)=-\frac{1}{\pi n}\mathrm{Im}\chi(k,\omega)=-\frac{1}{\pi n v(k)}\mathrm{Im}\left[\frac{1}{\epsilon(k,\omega)}\right]. \tag{1.2}$$

Here n is the particle number density. The complex dielectric function $\epsilon(k,\omega)$ has been introduced in (1.1), whose real and imaginary parts will be denoted as ϵ_1 and ϵ_2, respectively. In the random-phase approximation (RPA), $\Pi(k,\omega)$ is approximated by the 0-order diagram (Lindhard function) and the collective excitations are undamped. $S(k,\omega)$ has a delta-function-like peak. This is so because ϵ_2, in this approximation, is exactly zero for frequencies ω higher than $\omega_{\mathrm{SPE}}(k)$, the single-pair excitation (SPE) edge, given by

$$\omega_{\mathrm{SPE}}(k)=kv_F\left[1+\frac{k}{2k_F}\right]=\frac{\hbar}{2m}(k^2+2kk_F)\ , \tag{1.3}$$

where k_F is the Fermi wave number [e.g., $k_F=(3\pi^2 n)^{1/3}$ in a three-dimensional system]. This property is also shared by all the first-order diagrams,[2] and by such class of higher-order diagrams in which all the intermediate states consist of exactly one-particle–hole pair at any given moment of time [examples are given in Figs. 1(a)–1(d)]. Finite width of the elementary excitations arises due to diagrams which contain multiple-particle–hole excitations among their intermediate states, e.g., Figs. 1(e)–1(g). Glick and Long[3] (hereafter referred

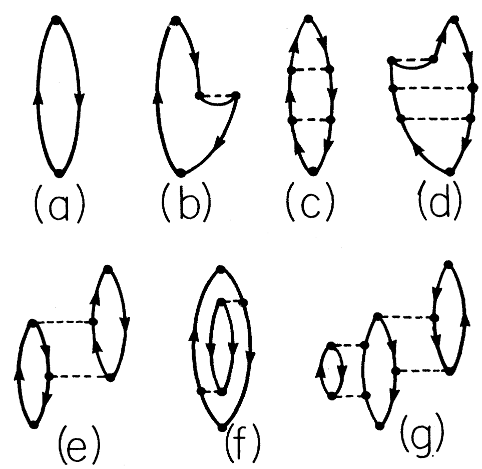

FIG. 1. Examples of diagrams for proper polarizability: (a) one pair, zeroth order; (b) one pair, first order; (c) one pair, second order; (d) one pair, third order; (e) two pair, second order; (f) two pair, second order; (g) three pair, fourth order.

to as GL) investigated in a systematic way the lowest terms of the perturbation theory which can give rise to damping, i.e., the second-order diagrams with two-particle–hole pair excitations (references to earlier works dealing with these problems may be found in GL). They were able to obtain in a closed form an asymptotic (valid at high frequencies, i.e., $\hbar\omega \gg E_F$) expression for $\epsilon_2(k,\omega)$ under the additional assumption that *k is small compared to k_F*. Their considerations were restricted to a *three-dimensional* electron gas and, therefore, to the *Coulomb interaction* among the fermions.

In the present paper we derive an asymptotic expression for $\epsilon_2(k,\omega)$, analogous to that derived by Glick and Long, but which is free from all the above-mentioned restrictions. This expression is cast in a form suitable to be used, for future applications, for any general form of the potential $v(k)$ in either three-, two-, or one-dimensional space and for arbitrary k. In the case of $D=3$ and $v(k)=4\pi e^2/k^2$ our result is the same as that of Glick and Long, although their restriction $k \ll k_F$ on its validity is removed now. As a specific case, we have used our expression to calculate the damping of plasmons in a two-dimensional (2D) electron gas and the damping of zero sound in liquid ^{3}He. Both these problems are of great physical interest. The latter forms the subject matter of part II (see the following paper).

We have also derived an expression for the frequency-dependent exchange-correlation potential $f_{xc}(k,\omega)$ in the 2D case–a result analogous to the one derived by Gross and Kohn[4] in the 3D case.

II. EXPRESSION FOR $\epsilon_2(k,\omega)$

In the Appendix we have derived the following asymptotic expression for the imaginary part of the dielectric function corresponding to ten second-order diagrams with two-pair excitations:

$$\epsilon_2(k,\omega) = -v(k)\,\mathrm{Im}\Pi_{2p}(k,\omega) , \tag{2.1a}$$

$$\mathrm{Im}\Pi_{2p}(k,\omega) = -m^3\hbar^{-6}\pi^{1-3D}2^{-1-3D}\Omega_{SD}^3 D^{-2}k_F^{2D} \times k^4 Q^{D-10} P_D(Q)[1+O(\eta)] , \tag{2.1b}$$

where

$$Q = \left[\frac{m\omega}{\hbar}\right]^{1/2} , \tag{2.1c}$$

and

$$P_D(Q) = v_{sc}^2(Q)\{1+2[A(Q)-2]\langle\mu^2\rangle_D + [A(Q)-2]^2\langle\mu^4\rangle_D\} . \tag{2.1d}$$

The subscript $2p$ on Π stands for two-pair excitations. In (2.1), D stands for the dimensionality of the space, Ω_{SD} is the full solid angle in D-dimension, $\langle\mu^2\rangle_D$ and $\langle\mu^4\rangle_D$ are given in Table I, and $A(Q)$ is defined by

$$A(Q) = \frac{Q}{v_{sc}(Q)}\frac{dv_{sc}(Q)}{dQ} . \tag{2.2}$$

$v_{sc}(Q)$ is the screened potential. It is the interaction occurring within the Feynman diagrams for proper polarizability, and can be approximated, as discussed by GL, by statically screened potential. If one is interested strictly in the second-order diagrams, v_{sc} should be the bare potential.

TABLE I. Angular characteristics in D dimensional space.

D	Ω_{SD}	$\langle\mu^2\rangle_D$	$\langle\mu^4\rangle_D$
1	2	1	1
2	2π	$\frac{1}{2}$	$\frac{3}{8}$
3	4π	$\frac{1}{3}$	$\frac{1}{5}$

As we have seen in the Appendix, Eq. (2.1) is valid for any k with an accuracy of order η, given by

$$\eta = \left[\frac{\omega_{\mathrm{asym}}(k)}{\omega}\right]^{1/2} , \tag{2.3a}$$

where

$$\omega_{\mathrm{asym}}(k) = \frac{\hbar}{2m}\left[k^2 + \frac{2D}{D+2}k_F^2\right] . \tag{2.3b}$$

We have seen in the Appendix that the asymptotic formula (2.1b) is accurate within 1% for $\eta \le 0.7$ in the case of 2D electron gas with screened or bare Coulomb interaction. In the Appendix, there is also a second restriction given defining the frequency range of validity of the formula (2.1b), namely,

$$\omega \gg \omega_{\mathrm{SPE}}(k) , \tag{2.3c}$$

which is connected with the nonanalyticity of $\epsilon_2(k,\omega)$ at $\omega=\omega_{\mathrm{SPE}}(k)$. Figure 2 permits one to gain some feeling of

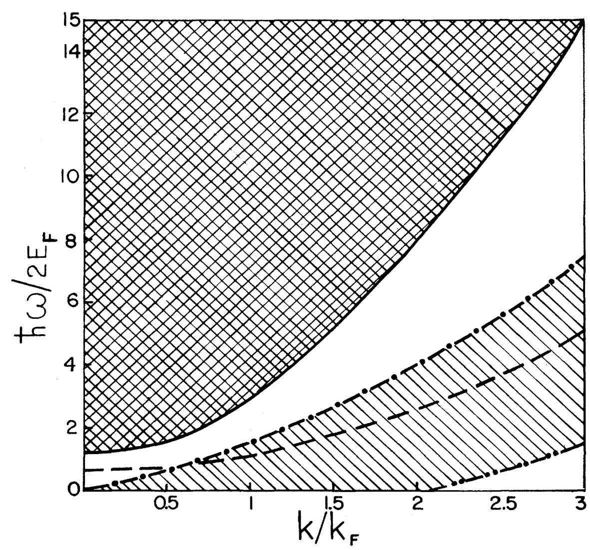

FIG. 2. Characteristics frequencies and frequency ranges for $D=3$. Dashed line: $\omega_{\mathrm{asym}}(k)$, Eq (2.3b); dashed-dotted line: $\omega_{\mathrm{SPE}}(k)$, Eq. (1.3); dashed-double-dotted line: $\omega_{\mathrm{SPE}}(k)$, Eq. (1.3) with + sign replaced by − sign, pertaining to the lower edge of single-particle excitations; cross-hatched area: region of validity of the asymptotic formula (2.1); hatched area: single-pair excitation continuum.

the restrictions mentioned previously. The region of validity of the formula (2.1), shown as the cross-hatched area, was chosen (according to the assumed 1% accuracy in the Coulomb case) to be

$$\omega \geq 2\max[\omega_{\text{asym}}(k), \omega_{\text{SPE}}(k)] .$$

A. $\epsilon_2(k,\omega)$ for the Coulomb potential in $D=3$ case

Here,

$$v(k)=\frac{4\pi e^2}{k^2}=\frac{4\pi\hbar^2}{ma_B}\frac{1}{k^2} . \tag{2.4}$$

Within the Thomas-Fermi (TF) approximation,

$$v_{\text{sc}}(k)=\frac{v(k)}{\epsilon_{\text{TF}}(k)}=\frac{v(k)}{1+(k/k_{\text{TF}})^2}=\frac{4\pi\hbar^2}{ma_B}\frac{1}{k^2+k_{\text{TF}}^2} , \tag{2.5}$$

where

$$k_{\text{TF}}^2=\frac{6\pi ne^2}{E_F} . \tag{2.6}$$

Using the above expressions, Eq. (2.2) and Table I in Eq. (2.1), it is easy to show that

$$\epsilon_2(k,\omega)=\frac{4}{9\pi^2}\frac{k_F^6}{a_B^3}\frac{k^2}{(m\omega/h)^{11/2}} \times F^2(\omega)[\tfrac{7}{15}+\tfrac{4}{15}F(\omega)+\tfrac{4}{5}F^2(\omega)] , \tag{2.7}$$

where

$$F(\omega)=\frac{m\omega/\hbar}{m\omega/\hbar+k_{\text{TF}}^2} . \tag{2.8}$$

Equation (2.7) is exactly the same as Eq. (19) of GL. In the limit that $k_{\text{TF}}^2 \to 0$, i.e., no screening, Eq. (2.7) reduces to

$$\epsilon_2(k,\omega)=\frac{92}{135\pi^2}\frac{1}{(k_Fa_B)^3}\frac{(k/k_F)^2}{(\hbar\omega/2E_F)^{11/2}} . \tag{2.9}$$

B. $\epsilon_2(k,\omega)$ for the Coulomb potential in $D=2$ case

Here

$$v(k)=\frac{2\pi e^2}{k}=\frac{2\pi\hbar^2}{ma_B}\frac{1}{k} , \tag{2.10}$$

$$v_{\text{sc}}(k)=\frac{v(k)}{\epsilon_{\text{TF}}(k)}=\frac{2\pi\hbar^2}{ma_B}\frac{1}{k+k_{\text{TF}}} , \tag{2.11}$$

where

$$k_{\text{TF}}=\frac{2\pi ne^2}{E_F} . \tag{2.12}$$

Using the above expressions and proceeding as in the $D=3$ case we have

$$\epsilon_2(k,\omega)=\frac{\pi}{8}\left[\frac{r_s}{\sqrt{2}}\right]^3\frac{(k/k_F)^3}{(\hbar\omega/2E_F)^5} \times f^2(\omega)[\tfrac{1}{2}+\tfrac{1}{2}f(\omega)+\tfrac{3}{8}f^2(\omega)] , \tag{2.13}$$

where

$$f(\omega)=\frac{(\hbar\omega/2E_F)^{1/2}}{(\hbar\omega/2E_F)^{1/2}+\sqrt{2}r_s} , \tag{2.14}$$

and $k_Fa_B=\sqrt{2}/r_s$. In the limit $r_s\to 0$ (i.e., neglecting screening) or, alternatively, in the limit $\omega\to\infty$

$$\epsilon_2(k,\omega)=\frac{11\pi}{64}\frac{1}{(k_Fa_B)^3}\frac{(k/k_F)^3}{(\hbar\omega/2E_F)^5} . \tag{2.15}$$

Comparing (2.9) and (2.15), we see that in the $D=3$ case the wave number and frequency dependence of ϵ_2 is, respectively, k^2 and $\omega^{-11/2}$, whereas the corresponding dependence of ϵ_2 in the $D=2$ case is k^3 and $\omega^{-10/2}$.

III. LOCAL-FIELD CORRECTION $G(k,\omega)$ FOR AN ELECTRON GAS

The function G is defined by

$$G(k,\omega)=1/Q^0(k,\omega)-1/Q(k,\omega) , \tag{3.1}$$

where

$$Q^0(k,\omega)=-v(k)\Pi^0(k,\omega) , \tag{3.2}$$

Π^0 is the Lindhard function, and

$$Q(k,\omega)=-v(k)\Pi(k,\omega)=\epsilon(k,\omega)-1 . \tag{3.3}$$

Then,

$$\text{Im}G=\text{Im}\epsilon/|Q|^2-\text{Im}Q^0/|Q^0|^2 . \tag{3.4}$$

For

$$\omega>\omega_{\text{SPE}}(k)=\frac{\hbar^2}{2m}(k^2+2kk_F),\quad \text{Im}Q^0=0 ,$$

so that,

$$\text{Im}G(k,\omega)=\text{Im}\epsilon(k,\omega)/|Q(k,\omega)|^2 . \tag{3.5}$$

In the case of large ω, we know that

$$Q(k,\omega)=-\frac{\omega_p^2(k)}{\omega^2}\left[1+O\left[\frac{1}{\omega^2}\right]\right] , \tag{3.6}$$

and that this leading contribution comes from Q^0 only because Q^0 above fulfills the first-moment sum rule. Hence,

$$\text{Im}G(k,\omega)=\text{Im}\epsilon(k,\omega)\left[\frac{\omega}{\omega_p(k)}\right]^4\left[1+O\left[\frac{1}{\omega^2}\right]\right] . \tag{3.7}$$

Up to this point the above considerations are valid for any dimensionality D. In the specific case of $D=2$ (see, e.g., Ref. 5)

$$\omega_p^2(k)=\frac{2\pi ne^2k}{m}=\frac{r_s}{\sqrt{2}}\left[\frac{k}{k_F}\right]\left[\frac{2E_F}{\hbar}\right]^2 . \tag{3.8}$$

Using (2.15) and (3.8) in (3.7) we have

$$\mathrm{Im}G(k,\omega)=\frac{11\pi}{64}\left[\frac{r_s}{\sqrt{2}}\right]^1\left[\frac{k}{k_F}\right]^1\left[\frac{\hbar\omega}{2E_F}\right]^{-1} \quad (3.9)$$

($D=2$ case).

The corresponding result[4,6] in the $D=3$ case is

$$\mathrm{Im}G(k,\omega)=\frac{23}{60}\left[\frac{4}{9\pi}\right]^{1/3} r_s\left[\frac{k}{k_F}\right]^2\left[\frac{\hbar\omega}{2E_F}\right]^{-3/2} \quad (3.10)$$

($D=3$ case). Notice that in both cases, the dependence of ImG on r_s is the same since one is considering second-order diagrams. Dependence of ImG on k is different in the two cases for two reasons: $\omega_p^2 \propto k^0$ in the $D=3$ case, while $\omega_P^2 \propto k^1$ in the $D=2$ case, and Im$\epsilon \propto k^2$ in $D=3$, while Im$\epsilon \propto k^3$ in $D=2$. Dependence on ω is dictated by that of ω^4Imϵ.

The function $f_{xc}^h(k,\omega)$, introduced by Gross and Kohn,[4] and which is related to the exchange-correlation potential, is given by

$$f_{\mathrm{xc}}^h(k,\omega)=-v(k)G(k,\omega) \; . \quad (3.11)$$

From (3.9) and (3.11) we have

$$\mathrm{Im}f_{\mathrm{xc}}^h(k,\omega)=-\frac{11\pi^2}{32}\frac{\hbar^3}{\mathrm{m}^2a_B^2}\omega^{-1} \; , \quad (3.12)$$

for large ω. This function, as in the $D=3$ case, does not depend on k and on r_s.

IV. MODEL OF $G(k,\omega)$ IN $D=2$ CASE

Following Gross and Kohn[4] we propose a simple form for Im$G(k,\omega)$, interpolating between its small-ω behavior, Im$G \propto \omega^{+1}$, and its large-ω behavior Im$G \propto \omega^{-1}$, Eq. (3.9). We write

$$\mathrm{Im}G(k,\omega)=\frac{\omega_1(k)\omega}{\omega_2^2(k)+\omega^2} \; , \quad (4.1)$$

where $\omega_1(k)$ and $\omega_2(k)$ are two functions of k to be determined. The full $G(k,\omega)$, which is analytic in the upper half of the complex ω plane can be written as

$$\begin{aligned} G(k,\omega)&=G(k,\infty)+\frac{i\omega_1(k)}{\omega+i\omega_2(k)} \\ &=G(k,\infty)+\frac{\omega_1(k)\omega_2(k)}{\omega^2+\omega_2^2(k)}+i\frac{\omega\omega_1(k)}{\omega^2+\omega_2^2(k)} \; . \end{aligned} \quad (4.2)$$

We must choose $\omega_2(k)>0$.

From (4.2) we see that the static local field is given by

$$G(k,0)=G(k,\infty)+\frac{\omega_1(k)}{\omega_2(k)} \; , \quad (4.3)$$

while

$$\lim_{\omega\to\infty}[\omega\,\mathrm{Im}G(k,\omega)]=\omega_1(k) \; . \quad (4.4)$$

Comparing (4.4) with (3.9) we find

$$\omega_1(k)=\frac{11\pi}{64}\frac{r_s}{\sqrt{2}}\frac{k}{k_F}\frac{2E_F}{\hbar} \; . \quad (4.5)$$

From (4.3) we have

$$\omega_2(k)=\frac{\omega_1(k)}{G(k,0)-G(k,\infty)} \; . \quad (4.6)$$

The value of $G(k,\infty)$ can be determined from the third-moment sum rule.[5] Expanding

$$-v(k)\chi(k,\omega)=\frac{Q^0(k,\omega)}{1+[1-G(k,\omega)]Q^0(k,\omega)} \quad (4.7)$$

in powers of ω^{-1}, we see that $G(k,\infty)$ will enter the coefficient of ω^{-4}, which, when compared to the third moment $M_3(k)$ gives

$$G(k,\infty)=I_d(k)-\frac{3}{4}\frac{\sqrt{2}}{r_s}\frac{k}{k_F}\delta_{\mathrm{kin}} \; , \quad (4.8)$$

where

$$\delta_{\mathrm{kin}}=(\langle E_{\mathrm{kin}}\rangle-\langle E_{\mathrm{kin}}\rangle_0)/\langle E_{\mathrm{kin}}\rangle_0 \; , \quad (4.9)$$

and $I_d(k)$ is defined in Ref. 5, Eq. (5.16), as an integral with Hartree-Fock structure factor $S_{\mathrm{HF}}(q)$ of the integrand to be replaced by exact $S(q)$. $\langle E_{\mathrm{kin}}\rangle$ is the exact kinetic energy of the interacting electron system and $\langle E_{\mathrm{kin}}\rangle_0$ that of the noninteracting system. Expression (4.8) is the same as Eq. (4.13) of Iwamoto[7] if we identify $G(k,\infty)$ with $G_3(k)$. Now $\langle E_{\mathrm{kin}}\rangle$ can be expressed in terms of ϵ_c—the correlation energy per particle, and is given by (see e.g., Ref. 7)

$$\langle E_{\mathrm{kin}}\rangle=\langle E_{\mathrm{kin}}\rangle_0-\frac{d}{dr_s}[r_s\epsilon_c(r_s)] \; . \quad (4.10)$$

Thus, $G(k,\infty)$ is known for arbitrary k provided we know $S(q)$ and $\epsilon_c(r_s)$. The latter quantity has been calculated by Ceperley,[8] for some values of r_s and an interpolation formula may be obtained. Unfortunately, Ceperley has not given the values of $S(k)$. It is, however, possible to calculate it in the STLS approximation.

In the limit of small k, $I_d(k)$ has a very simple form and is given by Eqs. (4.14) and (4.17) of Ref. 7. Therefore, combining this result with (4.10) we have

$$\lim_{k\to 0}\left[G(k,\infty)\frac{k_F}{k}\right]=\frac{5}{6\pi}+\frac{7}{8\sqrt{2}}r_s\epsilon_c+\frac{19}{16\sqrt{2}}r_s^2\frac{d\epsilon_c}{dr_s} \; , \quad (4.11)$$

ϵ_c in rydbergs.

From Ref. 7, Eq. (3.6c), we have

$$\lim_{k\to\infty}\left[G(k,0)\frac{k_F}{k}\right]=\frac{1}{\pi}+\frac{1}{8\sqrt{2}}r_s^2\frac{d\epsilon_c}{dr_s}-\frac{1}{8\sqrt{2}}r_s^3\frac{d^2\epsilon_c}{dr_s^2} \; . \quad (4.12)$$

We thus note that the model $G(k,\omega)$, as given by (4.2), in the limit $k\to 0$ can be calculated if one knows the correlation energy as a function of r_s. From (3.11) and the foregoing set of equations, we can calculate $f_{\mathrm{xc}}^h(k,\omega)$ in the limit $k\to 0$, a knowledge of which may prove useful in certain density-functional calculations.

V. PLASMON DAMPING

As mentioned in the Introduction, plasmon excitation may be investigated theoretically as the peak of the loss function $S(k,\omega)$, Eq. (1.2). In the case of a narrow peak (which is true for 2D plasmons, as will be shown), the peak position, i.e., plasmon frequency $\omega_{\rm pl}(k)$ is obtained as a solution of the equation involving the real part of ϵ:

$$\epsilon_1[k,\omega_{\rm pl}(k)]=1+{\rm Re}Q[k,\omega_{\rm pl}(k)]=0\ . \tag{5.1}$$

In RPA, where the role of terms higher than Q^0 is neglected, Eq. (5.1) becomes

$$1+{\rm Re}Q^0[k,\omega_{\rm pl}(k)]=0\ . \tag{5.2}$$

This equation may be solved analytically,[9]

$$\omega_{\rm pl}(k)=\omega_p(k)(1+X)\left[\frac{1}{1+\frac{1}{2}X}+\frac{1}{2}k^2X\right]^{1/2}, \tag{5.3}$$

where, according to (3.8)

$$\omega_p(k)=\left[\frac{r_s}{\sqrt{2}}k\right]^{1/2}, \tag{5.4}$$

and

$$X=X(k)=k/(\sqrt{2}r_s)\ . \tag{5.5}$$

We are using in this and the next section k_F as a unit for the wave vector and $2E_F/\hbar$ as a unit of frequency.

Now the width of the plasmon peak is determined from the loss function, written for frequencies in the vicinity of $\omega_{\rm pl}(k)$ as

$$\begin{aligned}{\rm Im}\frac{-1}{\epsilon(k,\omega)}&={\rm Im}\frac{-1}{\epsilon_1'[k,\omega_{\rm pl}(k)][\omega-\omega_{\rm pl}(k)]+i\epsilon_2[k,\omega_{\rm pl}(k)]}\\&=\frac{\epsilon_2[k,\omega_{\rm pl}(k)]}{\epsilon_1'[k,\omega_{\rm pl}(k)]}\frac{1}{[\omega-\omega_{\rm pl}(k)]^2+(\Gamma(k)/2)^2},\end{aligned} \tag{5.6}$$

where we have defined

$$\frac{\Gamma(k)}{2}=\frac{\epsilon_2[k,\omega_{\rm pl}(k)]}{\epsilon_1'[k,\omega_{\rm pl}(k)]}, \tag{5.7}$$

and

$$\epsilon_1'(k,\omega)=\frac{\partial}{\partial\omega}\epsilon_1(k,\omega)\ . \tag{5.8}$$

So the relative full width at half maximum is

$$\frac{\Gamma(k)}{\omega_{\rm pl}(k)}=\left.\frac{2\epsilon_2(k,\omega)}{\omega\epsilon_1'(k,\omega)}\right|_{\omega=\omega_{\rm pl}(k)}. \tag{5.9}$$

Using the RPA result of Ref. 9, we have

$$\epsilon_1'(k,\omega)=\frac{1}{2kX}\left[\frac{\frac{2\omega}{k^2}-1}{(P-\omega)^{1/2}}-\frac{\frac{2\omega}{k^2}+1}{(P+\omega)^{1/2}}\right], \tag{5.10}$$

where X was defined in (5.5) and

$$P=P(k,\omega)=\left[\frac{\omega}{k}\right]^2+\left[\frac{k}{2}\right]^2-1\ . \tag{5.11}$$

We mention that in the limit of small k Eqs. (5.3) and (5.10) reduce to

$$\omega_{\rm pl}(k)=\omega_p(k)[1+O(k)]\ , \tag{5.12}$$

$$\epsilon_1'[k,\omega_{\rm pl}(k)]=\frac{2}{\omega_p(k)}[1+O(k)]\ , \tag{5.13}$$

leading to a particularly simple form of the relative width (5.9)

$$\frac{\Gamma(k)}{\omega_{\rm pl}(k)}=\epsilon_2[k,\omega_p(k)][1+O(k)]\ . \tag{5.14}$$

In connection with the construction of the model $G(k,\omega)$, it may be convenient to have the width expression in terms of the local-field function. From (3.5) we have

$$\epsilon_2(k,\omega)=|Q(k,\omega)|^2{\rm Im}G(k,\omega)\ . \tag{5.15}$$

But at the plasmon frequency ${\rm Re}Q$ is -1 according to Eq. (5.1), while ${\rm Im}Q$ is very small as compared to 1 (narrow plasmon peak), so

$$\epsilon_2[k,\omega_{\rm pl}(k)]={\rm Im}G[k,\omega_{\rm pl}(k)][1+O({\rm Im}G)^2]\ , \tag{5.16}$$

and this result allows to calculate $\Gamma(k)$ in terms of ${\rm Im}G$ using (5.9).

In particular, we can find the behavior of the relative width in the region $k\to 0$. From (5.14), (5.16), and (4.1) we have

$$\frac{\Gamma(k)}{\omega_{\rm pl}(k)}=\frac{\omega_1(k)\omega_p(k)}{\omega_2^2(k)+\omega_p^2(k)}\ . \tag{5.17}$$

The numerator is $\propto k^1k^{1/2}=k^{3/2}$, see (4.5) and (5.4), while the denominator is the sum of a term $\propto k^0$ [ω_2^2, see (4.6), (4.11), and (4.12)] and a term $\omega_p^2\propto k^1$. So, finally, the relative width is $\propto k^{3/2}$ for small k, while the proportionality coefficient is given in terms of the correlation energy and its derivatives, as it follows from (4.5), (4.6), (4.11), (4.12), and (5.4). We must stress that the $k\to 0$ limit of the width cannot be found from (5.14) directly using the asymptotic expression (2.13) or (2.15). This is so because the plasmon frequency $\omega_p\propto k^{1/2}$ goes below $\omega_{\rm asym}(k)$, Eq. (2.3b),

$$\omega_{\rm asym}(k)=\tfrac{1}{2}(1+k^2)\ , \tag{5.18}$$

therefore η, Eq. (2.3a), grows to infinity as $\eta\propto k^{-1/2}$, which makes the expansion (2.1) meaningless.

VI. NUMERICAL ESTIMATE OF THE RELATIVE WIDTH OF PLASMON IN $D=2$

We have calculated plasmons for various densities and wave vectors, using the RPA formula (5.3)–(5.5) for peak position $\omega_{\rm pl}(k)$ and (5.10) and (5.11) for ϵ_1, while asymptotic formula (2.13) with (2.14) for ϵ_2, and (5.9) for the rel-

TABLE II. Plasmon frequency $\omega_{pl}(k)$ and its relative width $\Gamma(k)/\omega_{pl}(k)$ at various densities for a two-dimensional electron gas.

r_s	$\dfrac{k_c}{k_F}$	$\dfrac{k}{k_c}$	$\dfrac{\hbar\omega_{pl}(k)}{2E_F}$	$\dfrac{\Gamma(k)}{\omega_{pl}(k)}$ (%)	$\left[\dfrac{\omega_{asym}(k)}{\omega_{pl}(k)}\right]^{1/2}$	$\dfrac{\omega_{SPE}(k)}{\omega_{pl}(k)}$
4.08	2.2	0.1	0.82	0.27	0.80	0.30
		0.3	1.51	0.52	0.69	0.58
		0.5	2.15	0.45	0.72	0.79
2.81	1.8	0.1	0.62	0.30	0.91	0.32
		0.3	1.15	0.54	0.75	0.60
		0.5	1.63	0.47	0.74	0.80
1.77	1.4	0.1	0.44	0.33	1.08	0.34
		0.3	0.82	0.55	0.85	0.62
		0.5	1.17	0.46	0.80	0.81
0.97	1.0	0.3	0.53	0.52	1.02	0.65
		0.5	0.75	0.41	0.91	0.83

ative width $\Gamma(k)/\omega_{pl}(k)$. Some results are given in Table II. The cutoff vector, at which the plasmon dispersion curve touches the edge of single-pair excitation region [see (1.3)]

$$\omega_{SPE}(k)=\tfrac{1}{2}(k^2+2k) , \tag{6.1}$$

is denoted by k_c. It is related to the density parameter r_s via equation:[9]

$$r_s=\frac{k_c^2}{2\sqrt{2}}\left[1+\left[1+\frac{2}{k_c}\right]^{1/2}\right] . \tag{6.2}$$

As it is discussed in the Appendix, since the frequency $\omega_{SPE}(k)$ is also a point of nonanalyticity of $\epsilon_2(k,\omega)$, the use of the asymptotic expression of $\epsilon_2(k,\omega)$ requires the ratio ω_{SPE}/ω to be small. We see from the last column in Table II that this quantity grows to 1 with k tending to k_c, making our scheme inapplicable in this range of k. On the other hand, because [see (2.3a) and (5.18)] the quantity

$$\eta=[\omega_{asym}(k)/\omega_{pl}(k)]^{1/2} , \tag{6.3}$$

as given in the sixth column, increases sharply for k tending to 0, we cannot calculate the width for the small k region. The allowed k region narrows with decreasing r_s, making it impossible to have a meaningful estimate for $r_s \lesssim 0.9$. It is clear from Table II that in an electron gas in two dimensions the plasmon width arising from electron correlations is no more than 0.5%.

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APPENDIX: ASYMPTOTIC HIGH-FREQUENCY FORM FOR TWO-PAIR CONTRIBUTION TO $\epsilon_2(k,\omega)$

The purpose of this appendix is to derive in a rigorous way the asymptotic form for $\epsilon_2(k,\omega)$, corresponding to ten second-order diagrams with two-pair excitations, considered by GL. Our expression should be valid for arbitrary potential, be applicable to fermion systems in one-, two-, or three-dimensional space, and to serve for arbitrary k. Its range of validity for frequency ω is limited by the following k dependent inequalities: $\omega>\omega_{SPE}(k)$ [see (1.3)] and $\omega>\omega_{asym}(k)$ [see (A32)].

The full integral form of ϵ_2 is[3]

$$\epsilon_2(k,\omega)=C_D(k)\int d^Dq_1d^Dq_2d^Dq_3\Theta(k_F^2-q_1^2)\Theta(k_F^2-q_2^2)\Theta(q_3^2-k_F^2)\Theta(q_4^2-k_F^2)\delta\left[\frac{q_3^2+q_4^2-q_1^2-q_2^2}{2}-\frac{m\omega}{\hbar}\right]$$
$$\times\sigma(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{k})[\sigma(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{k})-\tfrac{1}{2}\sigma(\mathbf{q}_2,\mathbf{q}_1,\mathbf{q}_3,\mathbf{k})] , \tag{A1}$$

where

$$\mathbf{q}_4=\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_3 , \tag{A2}$$

$$C_D(k)=v(k)m^3\hbar^{-6}(2\pi)^{1-3D} , \tag{A3}$$

$$\sigma(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{k})=B(\mathbf{q}_3-\mathbf{q}_1,\mathbf{k},\mathbf{q}_3-\mathbf{q}_2-\mathbf{k},\mathbf{q}_3-\mathbf{q}_2)-B(\mathbf{q}_3-\mathbf{q}_1-\mathbf{k},\mathbf{k},\mathbf{q}_3-\mathbf{q}_2-\mathbf{k},\mathbf{q}_3-\mathbf{q}_2) , \tag{A4}$$

$$B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=v_{sc}(|\mathbf{p}_1|)(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{p}_1\cdot\mathbf{p}_3)^{-1}(\mathbf{p}_1\cdot\mathbf{p}_4)^{-1} , \tag{A5}$$

$$v_{sc}(p)=v(p)/\epsilon(p,0) , \tag{A6}$$

and $\Theta(x)$ is a unit step function. Equations (A1)–(A6) represent Eq. (16) of Gl, adapted by us to D dimensional space. The static screening of the interaction in (A6) involves, of course, selected higher-order diagrams, so, if exactly second-order diagrams are of interest, $\epsilon(p,0)$ in (A6) should be replaced by 1.

The first step in the evaluation of ϵ_2, Eq. (A1), consists in integrating over $q_3=|\mathbf{q}_3|$. It is convenient to introduce the notation

$$Q^2=m\omega/\hbar \ , \tag{A7}$$

$$\mathbf{K}=\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2 \ , \tag{A8}$$

$$\nu=\mathbf{q}_3\cdot\mathbf{K}/(q_3K) \ , \tag{A9}$$

in terms of which the delta function of (A1) may be written as

$$\delta\left[\frac{q_3^2+(\mathbf{K}-\mathbf{q}_3)^2-q_1^2-q_2^2}{2}-Q^2\right]=\delta\left[\left[q_3-\frac{K\nu}{2}\right]^2-Q^2(1+\eta_{12})\right]=\delta\left[q_3-\frac{K\nu}{2}-Q(1+\eta_{12})^{1/2}\right]/[2Q(1+\eta_{12})^{1/2}] \ , \tag{A10}$$

where

$$\eta_{12}=[q_1^2+q_2^2-K^2(1-\tfrac{1}{2}\nu^2)]/(2Q^2) \ , \tag{A11}$$

and we have assumed that

$$\eta_{12}>-1 \ . \tag{A12}$$

The integral in D dimensional space is represented as

$$\int d^Dq_i=\int_0^\infty dq_i q_i^{D-1}\int d^{D-1}\Omega_{\mathrm{SD}}(\mathbf{n}_i) \ , \tag{A13}$$

where $\mathbf{n}_i=\mathbf{q}_i/q_i$ and Ω_{SD} is the full solid angle in this space (see Table I). After integration over q_3, using (A10) and (A13), (A1) reduces to

$$\begin{aligned}\epsilon_2(k,\omega)=C_D(k)&\int d^Dq_1\Theta(k_F^2-q_1^2)\int d^Dq_2\Theta(k_F^2-q_2^2)\\&\times\int d^{D-1}\Omega_{\mathrm{SD}}(\mathbf{n}_3)q_3^{D-1}[2Q(1+\eta_{12})^{1/2}]^{-1}\\&\times\sigma(\cdots)[\sigma(\cdots)-\tfrac{1}{2}\sigma(\cdots)] \ ,\end{aligned} \tag{A14}$$

where

$$\mathbf{q}_3=q_3\mathbf{n}_3,\quad q_3=Q\left[(1+\eta_{12})^{1/2}+\frac{\nu K}{2Q}\right] \ , \tag{A15}$$

and it was assumed that Q is large enough to satisfy

$$q_3^2\geq k_F^2 \quad\text{and}\quad q_4^2=(q_3^2-2\nu Kq_3+K^2)\geq k_F^2 \ . \tag{A16}$$

Since $|\mathbf{q}_1|$ and $|\mathbf{q}_2|$ do not exceed k_F [due to theta functions in (A14)], the condition sufficient for inequalities (A12) and (A16) to be satisfied is such that Q is much larger than k and simultaneously much larger than k_F. Let us assume that this is fulfilled, leaving details to be worked out later. In this case such quantities as k/Q, $|\mathbf{q}_1|/Q$, $|\mathbf{q}_2|/Q$, K/Q, $|\eta_{12}|$, and $|q_3-Q|/q$ are very small for all allowed values of integration variables in (A14). Therefore we can expand the integrand in power series with respect to these small parameters.

The structure of (A4) suggests the following expansion:

$$B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)-B(\mathbf{p}_1-\mathbf{k},\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)=\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}_1}B(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4)+O\left[\left[\frac{k}{p_1}\right]^2\right] \ , \tag{A17}$$

where, according to (A5),

$$\frac{1}{B}\frac{\partial B}{\partial\mathbf{p}_1}=\frac{1}{v_{\mathrm{sc}}(p_1)}\frac{dv_{\mathrm{sc}}(p_1)}{dp_1}\frac{\mathbf{p}_1}{p_1}+\frac{\mathbf{p}_2}{\mathbf{p}_1\cdot\mathbf{p}_2}-\frac{\mathbf{p}_3}{\mathbf{p}_1\cdot\mathbf{p}_3}-\frac{\mathbf{p}_4}{\mathbf{p}_1\cdot\mathbf{p}_4} \ . \tag{A18}$$

Note that the error in (A17) is of the order of $(k/Q)^2$. It is not a small-k expansion. Finally, denoting $\mathbf{Q}=Q\mathbf{n}_3$, we have

$$\sigma(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{k})=\sigma(0,0,\mathbf{Q},\mathbf{k})[1+O(\eta)]=k^2Q^{-4}v_{\mathrm{sc}}(Q)\{1+[A(Q)-2]\mu^2\}[1+O(\eta)] \ , \tag{A19}$$

where

$$\mu=\mathbf{n}_3\cdot\mathbf{k}/k \ , \tag{A20}$$

$$A(Q)=\frac{Q}{v_{\mathrm{sc}}(Q)}\frac{dv_{\mathrm{sc}}(Q)}{dQ} \ . \tag{A21}$$

A set of small corrections, denoted by $O(\eta)$, contains the following terms,

$$O(\eta)=O\left[\frac{k}{Q},\frac{|\mathbf{Q}-\mathbf{q}_3-\mathbf{q}_1|}{Q},\frac{|\mathbf{Q}-\mathbf{q}_3-\mathbf{q}_1-\mathbf{k}|}{Q}\right], \tag{A22}$$

and also similar terms, with $\mathbf{q}_1$ replaced by $\mathbf{q}_2$. Power-series expansion of the remaining factors of the integrand in (A14) is straightforward. So we arrive at

$$\epsilon_2(k,\omega)=C_D(k)\int d^Dq_1\Theta(k_F^2-q_1^2)\int d^Dq_2\Theta(k_F^2-q_2^2)\int d^{D-1}\Omega_{\rm SD}(\mathbf{n}_3)Q^{D-1}(2Q)^{-1}2^{-1} \times(k^2Q^{-4}v_{\rm sc}(Q)\{1+[A(Q)-2]\mu^2\})^2[1+O(\eta)], \tag{A23}$$

where the correction term $O(\eta)$, besides the already listed terms (A22), includes also

$$O(\eta)=O\left[\eta_{12},\frac{\nu K}{2Q}\right]. \tag{A24}$$

Now the leading term of the integrand in (A23) [i.e., the full expression except the $O(\eta)$ term] does not depend on $\mathbf{q}_1$ and $\mathbf{q}_2$, so corresponding integrations are straightforwardly carried out [see (A13)] as

$$\int d^Dq_i\Theta(k_F^2-q_i^2)=\Omega_{\rm SD}D^{-1}k_F^D. \tag{A25}$$

Integration over the solid angle connected with the direction $\mathbf{n}_3$ leads to averaging of μ^2 and μ^4. Therefore, the final expression is

$$\epsilon_2(k,\omega)=v(k)k^4m^3\hbar^{-6}\pi^{1-3D}2^{-1-3D}\Omega_{\rm SD}^3D^{-2}k_F^{2D}Q^{D-10}P_D(Q)[1+O(\eta)], \tag{A26}$$

where

$$Q=\left[\frac{m\omega}{\hbar}\right]^{1/2}, \tag{A27}$$

$$P_D(Q)=v_{\rm sc}^2(Q)\Omega_{\rm SD}^{-1}\int d^{D-1}\Omega_{\rm SD}(\mathbf{n}_3)\{1+[A(Q)-2]\mu^2(\mathbf{n}_3)\}^2 =v_{\rm sc}^2(Q)\{1+2[A(Q)-2]\langle\mu^2\rangle_D+[A(Q)-2]^2\langle\mu^4\rangle_D\} \tag{A28}$$

with $A(Q)$ defined by (A21) and averages $\langle\mu^2\rangle_D$ and $\langle\mu^4\rangle_D$ given in Table I. Note that the first form of (A28) assures that

$$P_D(Q)>0. \tag{A29}$$

The error estimate $O(\eta)$ in (A26) contains a list of the same terms as in (A22) and (A24), but now those quantities are averaged out during integrations over d^Dq_1, d^Dq_2, and $d^{D-1}\Omega_{\rm SD}$. For example, the average $\mathbf{K}^2$ is

$$\langle\langle\mathbf{K}^2\rangle\rangle=\langle\langle(\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2)^2\rangle\rangle=k^2+2\frac{D}{D+2}k_F^2. \tag{A30}$$

We find that a general estimate

$$\eta=\frac{(\frac{1}{2}\langle\langle\mathbf{K}^2\rangle\rangle)^{1/2}}{Q}=\left[\frac{\omega_{\rm asym}(k)}{\omega}\right]^{1/2}, \tag{A31}$$

where

$$\omega_{\rm asym}(k)=\frac{\hbar}{2m}\left[k^2+\frac{2D}{D+2}k_F^2\right] \tag{A32}$$

covers the order of magnitude of all above-mentioned small quantities.

In order to have some feeling of what error may correspond to given η, we have analyzed the results shown by GL, which were obtained for three different potentials (bare Coulomb and screened Coulomb with two values of the screening parameter). We see that for $\eta\le 0.7$ the error of the asymptotic formula (as compared to the "exact" results of Monte Carlo integration) is less than the accuracy of their plot, about 1%. At $\eta=0.94$ the error is 30%, 60%, and 90% (for three potentials). At $\eta=1.1$ it is already in the range from 90% to 200%.

We also notice from GL Figs. 3 and 4 that $\epsilon_2(k,\omega)$ rapidly grows when ω approaches kv_F. In order to explain this fact let us investigate the question whether the integrand of $\epsilon_2(k,\omega)$, in (A1), at given k, may become singular for some ω. In (A4), (A5) in the denominators occur products of the following vectors:

$$\mathbf{d}_1=\mathbf{q}_3-\mathbf{q}_1,\quad \mathbf{d}_2=\mathbf{q}_3-\mathbf{q}_1-\mathbf{k}, \quad \mathbf{d}_3=\mathbf{q}_3-\mathbf{q}_2,\quad \mathbf{d}_4=\mathbf{q}_3-\mathbf{q}_2-\mathbf{k}. \tag{A33}$$

Now $|\mathbf{d}_1|$ can become zero ($\mathbf{q}_3=\mathbf{q}_1$) for the frequency ω determined by the delta function occurring in (A1):

$$\omega=\frac{\hbar}{2m}[q_3^2+(\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_3)^2-q_1^2-q_2^2] =\frac{\hbar}{2m}[\mathbf{q}_1^2+(\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2-\mathbf{q}_1)^2-q_1^2-q_2^2] =\frac{\hbar}{2m}[(\mathbf{k}+\mathbf{q}_2)^2-q_2^2]. \tag{A34}$$

Using the triangle inequality

$$|\mathbf{k}+\mathbf{q}| \le |\mathbf{k}|+|\mathbf{q}| \ , \tag{A35}$$

the above expression may be bounded by

$$\omega \le \frac{\hbar}{2m}[(|\mathbf{k}|+|\mathbf{q}_2|)^2-q_2^2]=\frac{\hbar}{2m}(k^2+2kq_2) \ . \tag{A36}$$

The theta function in (A1) gives the restriction $q_2 \le k_F$. So finally [compare with (1.3)]

$$\omega \le \frac{\hbar}{2m}(k^2+2kk_F)=\omega_{\mathrm{SPE}}(k) \ . \tag{A37}$$

Similar considerations apply when $|\mathbf{d}_i|=0$, $i=2,3,4$, and in each case we get the same restriction (A37). Thus the integrand has qualitatively different properties for frequencies below and above $\omega_{\mathrm{SPE}}(k)$, the latter is therefore the point of nonanalyticity of $\epsilon_2(k,\omega)$. Obviously, the asymptotic formula (A26), derived above, may be valid for frequencies being "far" from the point of nonanalyticity, i.e., for

$$\omega \gg \omega_{\mathrm{SPE}}(k) \ . \tag{A38}$$

Coming back to the explanation for the strange behavior of $\epsilon_2(k,\omega)$ at ω close to $k\,v_F$, we note that at $k=0.1\ k_F$ (used in GL calculations) $\omega_{\mathrm{SPE}}(k)\approx kv_F$ [see (1.3)], so the observed fact must be connected with nonanalyticity of $\epsilon_2(k,\omega)$ occurring there.

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