Energy spectrum of a layered system in a strong magnetic field

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We investigate the excitation spectrum of two- and three-layer electron systems in a strong perpendicular magnetic field with $\nu=\frac{1}{2}$ and $\frac{1}{3}$, respectively, in each layer. For layer separation $z=0$ the dispersion relations $\omega(k)$ vanish as k^2 for $k\rightarrow 0$, as one expects for Goldstone modes. For $z>0$, $\omega(k)$ behaves as an acoustic mode, vanishing linearly for small k. For large values of k one finds that the dispersion relations have the form $\Delta(z)-e^2/\kappa k l_0^2$, where l_0 is the magnetic length and κ the dielectric constant of the medium. At kl_0 of order unity, the dispersion relations develop a dip as z is increased. These become soft modes at certain critical values of z, indicating that the system undergoes a phase transition as the layer spacing is increased.

I. INTRODUCTION

In recent months, there has been some interest in the behavior of layered two-dimensional electron systems in the presence of a strong perpendicular magnetic field B. In particular, a numerical calculation by Chakraborty and Pietiläinen[1] strongly suggests that a two-layer system in which each layer has filling factor $\nu=\frac{1}{2}$ [where $\nu=2\pi\rho l_0^2$, ρ is the electron density in each layer, and $l_0=(\hbar c/eB)^{1/2}$ is the magnetic length] will exhibit the fractional quantized Hall effect (FQHE). In their calculation, Chakraborty and Pietiläinen diagonalize within the lowest Landau level the Hamiltonian for two layers of electrons, separated by a distance $z=2.0l_0$. They find that the excitation spectrum has a form very similar to that associated with the FQHE in single-layer systems;[2] in particular, there is a nonvanishing gap between the ground state and all the excited states of the system.

The purpose of this paper is to study in the strong-magnetic-field limit the excitation spectrum of two- and three-layer systems with filling factor $\nu=\frac{1}{2}$ and $\frac{1}{3}$, respectively, for layer separations $z\sim l_0$. We will consider an idealized model in which electrons in each layer are δ-function localized in the $\hat{\mathbf{z}}$ direction, move freely in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane, and interact only with one another and planes of uniform neutralizing background charge. Tunneling between layers is not allowed.

Throughout this paper we will work within the lowest Landau level, and assume all the electrons to be in the lowest spin state, which is appropriate in the strong-field limit. For the remainder of this work we ignore the spin variables of the electrons. We choose the Landau gauge $\mathbf{A}=(0,Bx,0)$, so that the single-electron wave functions take the form

$$\phi_{n,k}(x,y;\sigma)=\left[\frac{1}{\pi l_0^2 L^2}\right]^{1/4}\times\exp[iky-(x+kl_0^2)/2l_0^2]\chi_n(\sigma)\ ,$$

where L is the linear sample dimension, and by imposing periodic boundary conditions in the $\hat{\mathbf{y}}$ direction, k becomes a discrete variable with $L/2\pi l_0^2$ possible values. The variable σ is discrete and labels the layer in which the electron resides; for a system of m layers, σ has the values $1,2,\ldots,m$. The function $\chi_n(\sigma)=1$ if $\sigma=n$, 0 if $\sigma\neq n$.

To find the excitation spectrum of this system, we will calculate "valley" density response functions (defined in Sec. II) in a self-consistent approximation discussed by Kallin and Halperin.[3] The poles of the response functions correspond to the excitation energies of the system. The accuracy of our results is difficult to estimate, because in our situation there is no small expansion parameter (in contrast to the work of Ref. 3.) Nevertheless, our results for $z=0$ in the two-layer system agree with past calculations,[4] and the dispersion curves $\omega(k)$ have sensible features for $kl_0\ll 1$ and $kl_0\gg 1$.

We first discuss our results for the two-layer system. When $z=0$ the ground state may be written in the form[5]

$$\psi=\prod_k (ua_k^\dagger+vb_k^\dagger)|0\rangle\ , \tag{1}$$

where $a_k^\dagger$ ($b_k^\dagger$) creates an electron in layer 1 (2) with wave function $\phi_{1\,(2),k}$, and $|u|^2=|v|^2=\frac{1}{2}$ guarantees that each layer is half-filled. This wave function may be written in the following suggestive form:

$$\psi=\prod_k (u+vb_k^\dagger d_{-k}^\dagger)\Psi_0\ , \tag{2}$$

where $\Psi_0\equiv\prod_k a_k^\dagger|0\rangle$, and $d_{-k}^\dagger\equiv a_k$. The operators $b_k^\dagger d_{-k}^\dagger$ create electron-hole pairs, and the wave function ψ may then be interpreted as a Bose condensate of excitons in their lowest-energy state.[4] The energy of the ground state divided by the number of electron-hole pairs turns out to be exactly the binding energy of a single exciton. This is easily understood: the spatial wave functions created by the operators $b_k^\dagger$ and $d_{-k}^\dagger$ are identical, so that the excitons associated with $b_k^\dagger d_{-k}^\dagger$ have a vanishing charge density at every point in space. There is then no interaction between different excitons, and it follows that all the energy of this state is associated with the binding energy of the individual electron-hole pairs.[4]

A few words about why Eqs. (1) and (2) represent good ground-state wave functions for the $z=0$ case are in order. As pointed out by Yoshioka *et al.*,[1] one can think of this system as a single layer of spin-$\frac{1}{2}$ electrons, where the Zeeman energy is taken to vanish; the layer index in this case plays the role of spin. If we set $u=v=1/\sqrt{2}$, each factor of $(1/\sqrt{2})(a_k^\dagger+b_k^\dagger)$ in Eq. (1) creates an electron in the kth Landau orbital with "spin" wave function $(1/\sqrt{2})(|\uparrow\rangle+|\downarrow\rangle)$. It is then easy to see that Eq. (1) factorizes into a product of wave functions, one involving only the spatial coordinates and the other only the "spin" coordinates. The spatial part of the wave function has the form of a filled Landau level, so that the wave function vanishes linearly whenever two electron coordinates are brought together. From this property, it is easy to show that the spatial part of the wave function is the same (up to a gauge transformation) as that of the wave function $\psi_{1,1,1}$ investigated by Yoshioka *et al.*[1] Using a form of $\psi_{1,1,1}$ appropriate for electrons on a sphere, these authors calculated the overlap of this wave function with that obtained by numerical diagonalization of the Hamiltonian (projected onto the lowest Landau level) for finite-size systems. For system sizes up to 10 electrons, they found that the overlap of $\psi_{1,1,1}$ with the exact wave function was extremely close to unity.

The spatial part of the wave function in Eq. (1) is thus very close to that of the ground-state wave function for $z=0$. The "spin" portion of this wave function, on the other hand, is not correct; in fact, it has the unfortunate property that it is not an eigenfunction of the number operators for "spin"-up or "spin"-down electrons. Stated differently, Eq. (1) does not have the number of electrons in each layer as good quantum numbers. However, because for $z=0$ the interactions are independent of the layers in which the electrons reside—i.e., they are "spin" independent—the energy of this state will be determined only by the spatial part of the wave function. Thus, we expect Eq. (1) to give a very good estimate for the ground-state energy.

The lowest-lying excitations of this system are higher-energy excitons. Because excitons have no net charge, these excitations have a well-defined momentum $\mathbf{K}$, in spite of the presence of a magnetic field.[3] To understand the nature of these excitations, it is instructive to write down the wave function for a single exciton:

$$\psi_{\mathbf{K}}=\left[\frac{1}{N}\right]^{1/2}\sum_q e^{-iK_x q l_0^2} b^\dagger_{K_y/2+q} d^\dagger_{K_y/2-q}\Psi_0 .$$

Physically, this wave function may be pictured as a bound electron-hole pair, separated by a vector[3,4] $\langle\mathbf{r}\rangle=l_0^2\hat{\mathbf{z}}\times\mathbf{K}$. From this result, one expects the energy of the exciton to behave as $\Delta-e^2/\kappa K l_0^2$ in the large-K limit.

To generate an actual excited state of our system, one needs to eliminate a $\mathbf{K}=\mathbf{0}$ exciton from the ground state and replace it with a higher $\mathbf{K}$ exciton. Because of the neutral nature of the $\mathbf{K}=\mathbf{0}$ excitons, the dispersion relation of these excitations is given to a good approximation by[4]

$$\omega(k)=E_{\rm ex}(k)-E_{\rm ex}(0) ,$$

where $E_{\rm ex}(k)$ is the binding energy of a single exciton with momentum k. This result was derived by the authors of Ref. 4 using a random-phase-approximation (RPA) technique; we obtain precisely the same result with our self-consistent approximation. The dispersion relation may be written explicitly as

$$\omega(k)=\frac{e^2}{\kappa l_0}\left[\frac{\pi}{2}\right]^{1/2}[1-e^{-k^2l_0^2/4}I_0(k^2l_0^2/4)] , \qquad (3)$$

where I_0 is a modified Bessel function. Equation (3) is plotted in Fig. 1. For $kl_0\ll 1$, $\omega(k)\sim k^2$, which is expected for a system of two identical layers in which the electron interactions within the layers are the same as those between layers.[5] For $kl_0\gg 1$, we find $\omega(k)\cong(e^2/\kappa l_0)[(\pi/2)^{1/2}-1/kl_0]$, which is precisely the form one expects for large-k excitons.

In Fig. 2 we plot the dispersion relations for the two-layer system with $z>0$. For large kl_0 we find $\omega(k)\cong\Delta(z)-e^2/\kappa k l_0^2$, where $\Delta(z)$ is a decreasing function of z that will be specified in the Appendix. For small k the dispersion curve rises linearly, with a slope that is proportional to z for $z/l_0\ll 1$.

This acoustic behavior of the dispersion relation may be understood in the following way. For small separations, we expect the wave function in Eq. (2) to be a good variational estimate for the ground state. However, because the attractive interaction between electrons and holes $[(e^2/\kappa)(r^2+z^2)^{-1/2}]$ is weaker than the electron-electron interaction within a plane $(e^2/\kappa r)$, the condensed excitons now have a weak repulsion. In analogy with the dispersion relation of a standard Bose gas with weak repulsive interactions, one expects the dispersion relation to rise linearly at small k. Alternatively, the linear behavior may be understood by mapping the problem onto an equivalent spin system; this is discussed in Sec. III and analyzed in detail in Ref. 5.

In the intermediate region $(kl_0\sim 1)$, the dispersion curve develops a dip for separations $z\sim l_0$, which becomes a soft mode $[\omega^2(k)<0]$ for $z>1.21l_0$. Within our

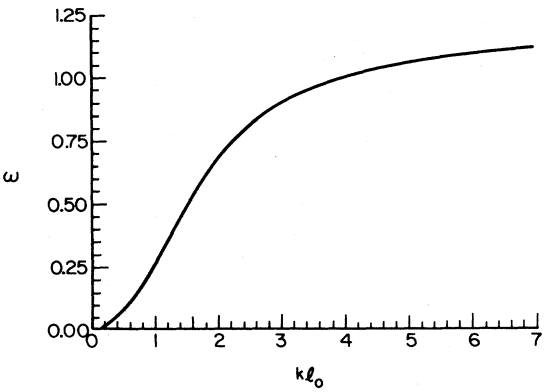

FIG. 1. Dispersion relation for a two-layer system with $z=0$. ω is in units of $e^2/\kappa l_0$.

approximations, this indicates that the system undergoes a phase transition at this separation. It is possible that for $z > 1.21 l_0$ the new phase will exhibit the gap observed in the numerical work of Chakraborty and Pietiläinen;[1] however, since we cannot calculate the dispersion curve for larger values of z with this formalism, it is not obvious that the new state will have an energy gap. Nevertheless, we emphasize that since that acoustic behavior of the dispersion at small values of z seems sensible, it is clear that there must be a phase transition between the "Bose-condensed exciton" ground state and one that is separated by a gap from all the excited states.

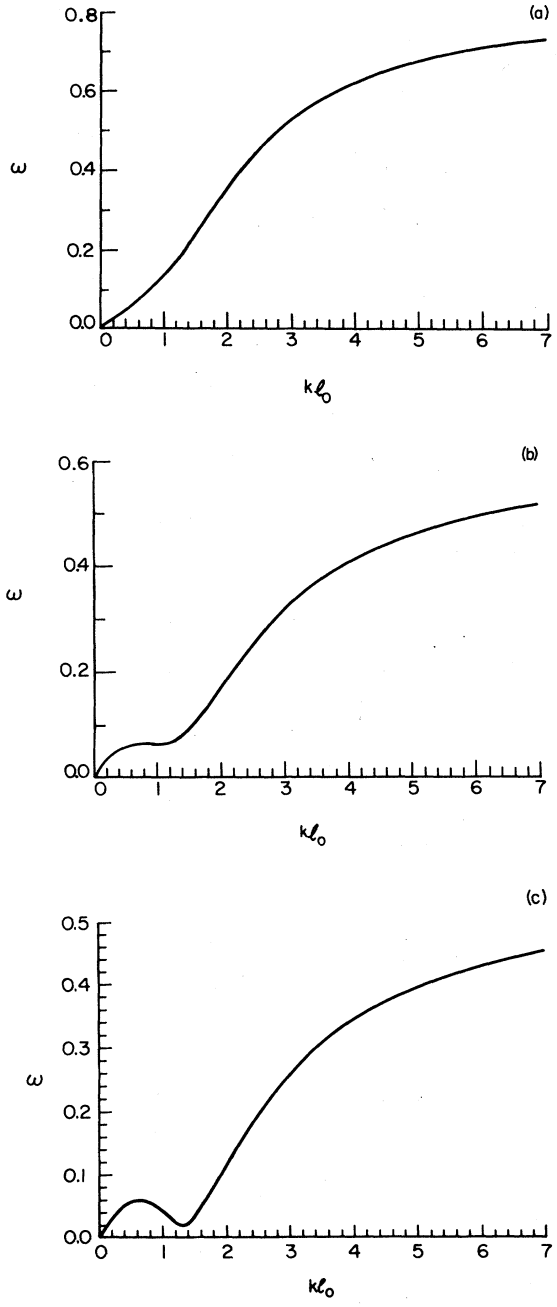

FIG. 2. Dispersion relations for a two-layer system with (a) $z=0.5l_0$, (b) $z=1.0l_0$, and (c) $z=1.2l_0$. ω is in units of $e^2/\kappa l_0$.

Our results for the three-layer system are qualitatively the same as those of the two-layer system, and are described in detail in Sec. III. We study the specific situation in which the middle layer is equidistant from the two outer layers. One finds two excitation modes within the lowest Landau level, which we denote as $\omega_1(k)$ and $\omega_2(k)$. For both modes, $\omega_i(k)\sim k^2$ for $z=0$, and $\omega_i(k)\sim k$ for $z>0$ when $k\to 0$. The large-k dependence of $\omega_i(k)$ is given by $\Delta(z)-e^2/\kappa k l_0^2$, as expected for excitonic modes. Finally, for $z=0.92l_0$ and $1.51l_0$, ω_1 and ω_2, respectively, develop soft modes near $kl_0=1.2$, indicating that the system undergoes a phase transition as the layer spacing increases.

This article is organized as follows. In Sec. II we describe the approximations used to find the two- and three-layer dispersion curves. In Sec. III, we present our results, and conclude with a summary in Sec. IV. The Appendix outlines some of the details of the calculation for the two-layer system.

II. FORMALISM

The Hamiltonian for a system of N layers of electrons in a strong perpendicular magnetic field may be written as

$$H_{\rm int}=\sum_{\sigma_1,\sigma_2}\sum_{p_1,p_2}\sum_{s\neq 0}F_{\sigma_1\sigma_2}(s,p_1-p_2)\times a^\dagger_{\sigma_1 p_1-s}a^\dagger_{\sigma_2 p_2}a_{\sigma_2 p_2-s}a_{\sigma_1 p_1}\ , \tag{4}$$

where σ_1,σ_2 label the layers, and $a^\dagger_{\sigma p}$ creates an electron with wave function $\phi_{\sigma p}$. As is appropriate in the strong-magnetic-field limit, we ignore coupling to higher Landau levels. The $s=0$ terms are excluded from the sum because of the interaction with the uniform neutralizing background charge. The interaction matrix elements $F_{\sigma_1\sigma_2}(s,p_1-p_2)$ are given by[4]

$$F_{\sigma_1\sigma_2}(q_x,p_1-p_2)=\frac{1}{2L^2}\sum_{q_y}e^{-q^2l_0^2/2}\tilde v_{\sigma_1\sigma_2}(q)\times e^{iq_y(p_1-p_2)l_0^2}\ , \tag{5}$$

where $\tilde v_{\sigma_1\sigma_2}(q)=(2\pi e^2/q)e^{-z_{\sigma_1\sigma_2}q}$ is the two-dimensional Fourier transform of the Coulomb interaction between electrons in layer σ_1 and layer σ_2, and $z_{\sigma_1\sigma_2}$ is the distance between these layers. For electrons in the same layer, $z_{\sigma_1\sigma_2}=0$.

We will first describe the calculation for the two-layer system. It is convenient to define $F_1\equiv F_{11}=F_{22}$, $F_2\equiv F_{12}=F_{21}$, and the operators $\alpha_p^\dagger=(1/\sqrt{2})(a_{1p}^\dagger+a_{2p}^\dagger)$, $\beta_p^\dagger=(1/\sqrt{2})(a_{1p}^\dagger-a_{2p}^\dagger)$. We add a small symmetry-breaking term to the Hamiltonian so that $H=H_{\rm int}+H_{\rm SB}$, where

$$H_{\rm SB}=h\sum_k(\beta_k^\dagger\beta_k-\alpha_k^\dagger\alpha_k)\ . \tag{6}$$

In the absence of interactions ($H_{\rm int}=0$), the ground state is just $\psi=\prod_k\alpha_k^\dagger|0\rangle$. The parameter h will be set to zero at the end of the calculation. To find the dispersion rela-

tion of this system, we use the formalism developed by Kallin and Halperin.[3] The calculation closely follows their work because one can map our problem directly onto that associated with a single-layer system with spin $\frac{1}{2}$; in this case, $\alpha_k^\dagger$ creates spin-down electrons, $\beta_k^\dagger$ creates spin-up electrons, and the Zeeman splitting is given by $2h$. The only complication that arises in this mapping is that the interaction is spin dependent. We emphasize that this "spin" picture is unrelated to the real spins of the electrons; these are taken to be polarized by the magnetic field. Closely related mappings between valley and spin degrees of freedom have been used to investigate the properties of both two-[5,6] and three-dimensional[7] multivalley systems in strong magnetic fields. To proceed, we define the "valley-density" operator[7]

$$\rho(\mathbf{q},t)=\sum_{s_1,s_2}\sum_k e^{-q^2l_0^2/4+iq_xq_yl_0^2/2+ikq_xl_0^2} \times c_{s_1k}^\dagger(t)(2\mathbf{S})_{s_1s_2}c_{s_2,k+q_y}(t)\ , \tag{7}$$

where $c_{s_1k}(t)=e^{iHt}c_{s_1k}(0)e^{-iHt}$ are Heisenberg operators, $c_{s_1k}(0)=\alpha_k$ if $s_1=1$, β_k if $s_1=2$, and the components of $\mathbf{S}$ are the usual Pauli spin matrices:

$$S_x=\frac{1}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix},$$

$$S_y=\frac{1}{2}\begin{bmatrix}0&-i\\i&0\end{bmatrix},$$

$$S_z=\frac{1}{2}\begin{bmatrix}1&0\\0&-1\end{bmatrix}.$$

Defining valley raising and lowering operators

$$\rho^\pm(\mathbf{q},t)=\frac{1}{\sqrt{2}}[\rho_x(\mathbf{q},t)\pm i\rho_y(\mathbf{q},t)]\ , \tag{8}$$

we can study the valley-density reponse functions

$$\chi_\pm(\mathbf{q},\omega)=-i\int_0^\infty dt\, e^{i\omega t}\langle[\rho^\mp(\mathbf{q},t),\rho^\pm(-\mathbf{q},0)]\rangle\ . \tag{9}$$

The poles of this function represent neutral excitations of the system. In the valley picture they may be thought of as valley-density waves, and in the spin picture one would interpret them as spin-density waves.

We note that one can also study within this formalism the valley-density response function

$$\chi_z(\mathbf{q},\omega)=-i\int_0^\infty dt\, e^{i\omega t}\langle[\rho_z(\mathbf{q},t),\rho_z(-\mathbf{q},0)]\rangle$$

or the full density response function

$$\chi_{\rho_F}(\mathbf{q},\omega)=-i\int_0^\infty dt\, e^{i\omega t}\langle[\rho_F(\mathbf{q},t),\rho_F(-\mathbf{q},0)]\rangle\ ,$$

where

$$\rho_F(\mathbf{q},t)=\sum_k e^{-q^2l_0^2/4+iq_xq_yl_0^2/2+iq_xkl_0^2} \times e^{iHt}(\alpha_k^\dagger\alpha_{k+q_y}+\beta_k^\dagger\beta_{k+q_y})e^{-iHt}\ .$$

However, these functions have no poles when we consider excitations within the lowest Landau level. The excitations associated with these functions are higher in energy than the one we will calculate by an amount of order $\hbar\omega_c$, where $\omega_c=eB/mc$ is the cyclotron frequency.

The diagrammatic expansion by which one calculates $\chi_\pm(\mathbf{q},w)$ is described in detail in Ref. 3. One writes the response function in terms of a vertex function $\Gamma_{\mu\nu}^\pm(q_1,q_2;k\omega)$ via the relation

$$\chi_\pm(k,\omega)=\sum_{\mu,\nu}\sum_{q_1,q_2}\langle q_1\mu|e^{i\mathbf{k}\cdot\mathbf{r}}2S^\pm|q_2\nu\rangle \times\int\frac{d\omega_1}{2\pi}G_\mu(\omega+\omega_1)G_\nu(\omega_1) \times\Gamma_{\mu\nu}^\pm(q_1,q_2;k\omega)\ , \tag{10}$$

where $S_\pm=(1/\sqrt{2})(S_x\pm iS_y)$, $|q_2\nu\rangle=\alpha_{q_2}^\dagger|0\rangle$ if $\nu=1$, $\beta_{q_2}^\dagger|0\rangle$ if $\nu=2$, and the Green's functions are given by

$$G_\mu(\omega)=\frac{1}{\omega-\Sigma_\mu-(-1)^\mu h+i\eta_\mu}\ .$$

The quantities Σ_μ are self-energies that are determined self-consistently, $\eta_\mu=0^-$ for $\mu=1$ and $\eta_\mu=0^+$ for $\mu=2$. The diagrammatic representations of the self-energy and the vertex function are presented in Fig. 3(a). We note that this expansion does not include bubble diagrams of the form shown in Fig. 3(b); in the work of Kallin and Halperin,[3] this approximation was justified because each particle-hole bubble has an energy denominator $\hbar\omega_c$ associated with it, and thus makes a negligible contribution in the strong-field limit. In our situation there is no such gap, so we are on less sure footing in using this approximation. A fully-self-consistent calculation of the vertex

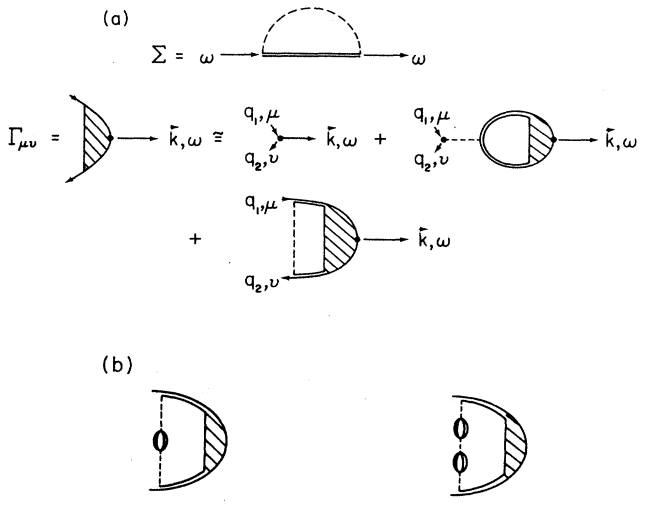

FIG. 3. (a) Approximation scheme used to calculate the self-energy and the vertex function. (b) Some diagrams not included in the approximation.

function including particle-hole bubbles is quite difficult, and most probably would give qualitatively similar results. We thus stay within the approximations defined by Fig. 3(a).

To evaluate the reponse functions, we write the interaction Hamiltonian in the form

$$H_{\rm int}=\tfrac{1}{2}\sum_{\substack{s_1,s_2\\ s_3,s_4}}\sum_{\substack{p_1,p_2\\ p_3,p_4}} V_{s_1s_2s_3s_4}(p_1p_2p_3p_4) \times c^{\dagger}_{s_1p_1}c^{\dagger}_{s_2p_2}c_{s_3p_3}c_{s_4p_4} , \tag{11}$$

where

$$V_{s_1s_2s_3s_4}(p_1p_2p_3p_4)=V'_{s_1s_2s_3s_4}(p_2-p_3,p_4-p_2)\times\delta_{p_1+p_2,p_3+p_4} ,$$

and

$$V'_{s_1s_2s_3s_4}(p_2-p_3,p_4-p_2)=F_1(p_2-p_3,p_4-p_2)+F_2(p_2-p_3,p_4-p_2) \tag{11a}$$

if $s_2=s_3$ and $s_1=s_4$,

$$V'_{s_1s_2s_3s_4}(p_2-p_3,p_4-p_2)=F_1(p_2-p_3,p_4-p_2)-F_2(p_2-p_3,p_4-p_2) \tag{11b}$$

if $s_1=s_2\neq s_3=s_4$ or $s_1=s_3\neq s_2=s_4$, and $V'_{s_1s_2s_3s_4}$ vanishes for all other combinations of $s_1,s_2,\ldots,s_4$.

In terms of these coefficients, the vertex function satisfies the equation[3]

$$\tilde{\Gamma}^{\pm}_{\mu\nu}(\mathbf{k},\omega)=[M^{\pm}_{\mu\nu}(\mathbf{k},\omega)]^{*}+i\sum_{\lambda,\epsilon}[\tilde{V}^{(1)}_{\mu\epsilon\nu\lambda}(\mathbf{k})-\tilde{V}^{(2)}_{\mu\epsilon\lambda\nu}(\mathbf{k})]\times D_{\lambda\epsilon}(\omega)\tilde{\Gamma}^{\pm}_{\lambda\epsilon}(\mathbf{k},\omega) , \tag{12}$$

where

$$\tilde{\Gamma}^{\pm}_{\mu\nu}(\mathbf{k},\omega)\delta_{p,k_x}\delta_{k_y,-\Delta q}=\sum_{q} e^{ipql_0^2}\Gamma^{\pm}_{\mu\nu}\left[q+\frac{\Delta q}{2},q-\frac{\Delta q}{2};\mathbf{k}\omega\right] ,$$

$$D_{\lambda\epsilon}(\omega)=\int\frac{d\omega_1}{2\pi}G_{\lambda}(\omega+\omega_1)G_{\epsilon}(\omega_1) ,$$

$$M^{\pm}_{\mu\nu}(\mathbf{k})\delta_{p,k_x}\delta_{k_y,-\Delta q}=\sum_{\substack{p_1,p_2\\ q}} e^{ipql_0^2}\delta_{p_1+p_2,2q}\delta_{p_1-p_2,\Delta q}\langle p_1\mu|e^{i\mathbf{k}\cdot\mathbf{r}}2S^{\pm}|p_2\nu\rangle ,$$

and

$$\tilde{V}^{(1)}_{\mu\epsilon\nu\lambda}(\mathbf{k})=\sum_{p} e^{ipk_xl_0^2}V'_{\mu\epsilon\nu\lambda}(p,k_y) , \tag{13a}$$

$$\tilde{V}^{(2)}_{\mu\epsilon\lambda\nu}(\mathbf{k})=\sum_{p} e^{ipk_yl_0^2}V'_{\mu\epsilon\lambda\nu}(k_x,p) . \tag{13b}$$

The self-energies Σ_{μ} needed to evaluate $D_{\lambda\epsilon}(\omega)$ are given by

$$\Sigma_{\mu}=-\sum_{p_1,p_2} V_{\mu1\mu1}(p,p_1,p_2,p_1) . \tag{14}$$

Finally, the response functions $\chi^{\pm}(\mathbf{k},\omega)$ may be written in the form

$$\chi^{\pm}(\mathbf{k},\omega)=\sum_{\mu,\nu}M_{\mu\nu}(\mathbf{k})D_{\mu\nu}(\omega)\tilde{\Gamma}^{\pm}_{\mu\nu}(k,\omega) ,$$

so that the poles of the response functions coincide with those of the vertex functions. Our results for the two-layer problem are discussed in Sec. III.

For the three-layer problem, the formalism outlined above is essentially the same. We define the operators $\boldsymbol{\alpha}^{\dagger}_k=(\alpha^{\dagger}_k,\beta^{\dagger}_k,\gamma^{\dagger}_k)$ by the transformation $\boldsymbol{\alpha}^{\dagger}_k=\underline{t}\,\mathbf{a}^{\dagger}_k$, where the components of $\mathbf{a}^{\dagger}_k=(a^{\dagger}_{1k},a^{\dagger}_{2k},a^{\dagger}_{3k})$ create an electron in layers 1, 2, or 3. The transformation matrix $\underline{t}$ is chosen to be

$$\underline{t}=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\\ 1 & e^{2\pi i/3} & e^{-2\pi i/3}\\ e^{-2\pi i/3} & e^{2\pi i/3} & 1\end{bmatrix} , \tag{15}$$

and we take for the symmetry-breaking Hamiltonian

$$H_{\rm SB}=h\sum_{k}(\gamma^{\dagger}_k\gamma_k-\alpha^{\dagger}_k\alpha_k) .$$

In the absence of interactions, the ground state is uniquely given by $\psi=\prod_k\alpha^{\dagger}_k|0\rangle$. We again look at the valley-density response functions $\chi^{\pm}(\mathbf{k},\omega)$, which are determined by Eqs. (7)–(9) with one modification: the Pauli spin matrices in Eq. (7) must be replaced by their three-dimensional versions,

$$S_x=\frac{1}{2}\begin{bmatrix}0&1&0\\1&0&1\\0&1&0\end{bmatrix} ,$$

$$S_y=\frac{1}{2}\begin{bmatrix}0&-i&0\\i&0&-i\\0&i&0\end{bmatrix} ,$$

$$S_z=\frac{1}{2}\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix} .$$

The vertex function for this case may be calculated using Eqs. (11) and (12)–(14); one only need remember that the indices now run from 1 to 3 instead of from 1 to 2. The coefficients $V_{s_1s_2s_3s_4}(p_1p_2p_3p_4)$ in Eq. (11) are not specified by (11a) and (11b) in this case. They may be calculated by substituting $\mathbf{a}^{\dagger}_k=\underline{t}^{\dagger}\boldsymbol{\alpha}^{\dagger}_k$ into Eq. (4), and reading off the resulting coefficients; this is tedious but manageable because not all of the coefficients are needed to find the poles of the vertex function. The relevant matrix elements associated with these coefficients will be presented in Sec. III.

III. RESULTS

A. Two-layer system

For the two-layer case, the dispersion relation for separation z is

$$\omega(k)=\{[\Delta(z)-2\tilde{F}_2^{(1)}(k)] \times[\Delta(z)-2\tilde{F}_1^{(1)}(k)+2\tilde{F}_1^{(2)}(k)-2\tilde{F}_2^{(2)}(k)]\}^{1/2} . \tag{16}$$

The definitions of $\Delta(z)$ and $\tilde{F}_j^{(i)}(k)$ as well as some details of the calculation of Eq. (16) are presented in the Appendix. We plot $\omega(k)$ for several values of z in Figs. 1 and 2. For small values of k we find that $\omega(k)\cong Akl_0+C(kl_0)^2$, with

$$A=\left\{A_0\left\{\Delta(z)+\frac{e^2}{\kappa l_0}\left[\frac{z}{l_0}-\left[\frac{\pi}{2}\right]^{1/2}\right]\right\}\right\}^{1/2}$$

and

$$A_0=\frac{e^2l_0^2}{4\kappa}\int_0^\infty q^2e^{-q^2l_0^2-zq}dq .$$

For $z/l_0\ll 1$ the coefficient A is, to lowest order in z,

$$A\simeq\frac{e^2}{\kappa l_0}\frac{\sqrt{\pi}}{4}\left[\frac{z}{l_0}\right] .$$

For $z>0$ the dispersion relation vanishes linearly with kl_0 for $kl_0\ll 1$, and the slope of $\omega(k)$ varies linearly with z for small separations. This behavior is similar to that of the acoustic-plasmon mode of a two-layer system in the absence of a magnetic field;[8] in the latter case, however, the acoustic mode is damped by the particle-hole excitations if z is too small, and does not persist down to zero separation. In our situation, the electron-hole excitations are replaced by excitations to higher Landau levels. The coupling of these excitations to the linear mode vanishes in the strong-field limit, so that the linear behavior of $\omega(k)$ remains for all $z>0$.

For $z=0$ the coefficient of the linear term vanishes, so that $\omega(k)\sim k^2$. The general behavior of $\omega(k)$ at small k has been discussed previously,[5,6] and may be understood in the following way. For $z=0$ the Hamiltonian $H_{\rm int}$ is invariant under transformations of the form $\boldsymbol{\alpha}_k\rightarrow\underline{U}\boldsymbol{\alpha}_k'$, where U is a 2×2 unitary matrix with unit determinant [i.e., a representation of SU(2)], and $\boldsymbol{\alpha}_k=(\alpha_k,\beta_k)$. The specific choice of ground state $\prod_k\alpha_k^\dagger|0\rangle$ is a broken-symmetry state, and one expects a Goldstone mode, i.e., a collective excitation with dispersion relation $\omega(k)\sim k^2$ for small k.

For $z>0$ there is a symmetry-breaking term added to the interaction Hamiltonian. If z is small one may think of this term as a small perturbation. It is then possible to map the problem onto an equivalent spin system, for which there are known results that show that the dispersion relation rises linearly at small wave vectors.[5] Alternatively, one may recall the Bose-condenstate interpretation[4] of the ground-state wave function for $z=0$ discussed in the Introduction. We then think of the perturbation as a weak repulsion between the condensed excitons,[4] so that one expects $\omega(k)$ to have a linear behavior for small k.

For large values of k, one finds $F_i^{(1)}(k)\sim(e^2/2\kappa)(1/kl_0^2)$, $F_i^{(2)}(k)\sim e^{-k^2l_0^2/2}$, so that

$$\omega(k)\sim\Delta(z)-\frac{e^2}{\kappa kl_0^2} .$$

As discussed in the Introduction, this is exactly the form one expects for magnetic excitons: the quantity $\Delta(z)$ represents the binding energy of an individual electron-hole pair in the ground state, and the term $-e^2/\kappa kl_0^2$ comes from the Coulomb attraction between these particles when they are separated by a distance kl_0^2.

At intermediate values of k ($kl_0\sim 1$), the dispersion relation develops a dip when $z>1.0$. For values of $z>z_c\equiv 1.21l_0$ this dip becomes a soft mode ($\omega^2<0$) near $kl_0\sim 1.2$. We interpret this as indicating that the system undergoes a phase transition when z is increased past z_c. Based on the numerical work of Chakraborty and Pietiläinen,[1] it is tempting to interpret this as a transition to a state that has an energy gap; however, we caution the reader that, in the absence of more information about the state into which the system develops for $z>z_c$, we cannot be sure that such an interpretation would be correct. The results reported in Ref. 1 are for $z=2.0l_0$, in a system of eight electrons. If we assume that the energy gap exhibited there persists in the limit of infinite particle number, then it is clear that a phase transition must take place between $z=0$ and $z=2.0l_0$, since the acoustic behavior of the "valley" mode at small separations seems qualitatively correct. However, one cannot immediately rule out, based on our study, the possibility of intermediate phases in the interval $1.21l_0<z<2.0l_0$ (e.g., charge- or valley-density waves).

Finally, we note that the dispersion relation can be written explicitly for $z=0$ as

$$\omega(k)=\frac{e^2}{\kappa l_0}\left[\frac{\pi}{2}\right]^{1/2}[1-e^{-k^2l_0^2/4}I_0(k^2l_0^2/4)] , \tag{17}$$

which is in precise agreement with the results of Ref. 4.

B. Three layers

For the three-layer case, Eq. (12) reduces to a 4×4 matrix equation involving $\Gamma_{12}^{\pm},\Gamma_{21}^{\pm},\Gamma_{13}^{\pm},\Gamma_{31}^{\pm}$. A full solution of these equations is difficult to achieve analytically for general layer spacings. Fortunately, for the case of equal spacing between adjacent layers, ($z_{12}=z_{23}=z$, $z_{13}=2z$), they can be separated into two independent 2×2 matrix equations. In the representation of the interaction Hamiltonian shown in Eq. (4), one writes for this situation

$$\begin{aligned}
&F_{\sigma_1\sigma_1}(s,p_1-p_2)\equiv F_1(s,p_1-p_2) ,\\
&F_{12}(s,p_1-p_2)=F_{21}(s,p_1-p_2)\\
&\qquad=F_{23}(s,p_1-p_2)\\
&\qquad=F_{32}(s,p_1-p_2)\\
&\qquad\equiv F_2(s,p_1-p_2) ,\\
&F_{13}(s,p_1-p_2)=F_{31}(s,p_1-p_2)\equiv F_3(s,p_1-p_2) ,
\end{aligned} \tag{18}$$

where the functions $F_{\sigma_1\sigma_2}(s, p_1-p_2)$ are defined in Eq. (5). Following the prescription outlined in Sec. II, the dispersion relations associated with this Hamiltonian take the form

$$\omega_i(k)=\{[\Delta\Sigma-U_i^a(k)]^2-[U_i^b(k)]^2\}^{1/2} ,$$

where $i=1,2$ for the two modes,

$$\Delta\Sigma=\tfrac{2}{3}[2\tilde{F}_2^{(1)}(0)+\tilde{F}_3^{(1)}(0)] ,$$

$$U_1^a=\tfrac{2}{3}[\tilde{F}_1^{(1)}(k)+\tfrac{2}{3}\tilde{F}_2^{(1)}(k)+\tfrac{4}{3}\tilde{F}_3^{(1)}(k) -\tilde{F}_1^{(2)}(k)+\tfrac{4}{3}\tilde{F}_2^{(2)}(k)-\tfrac{1}{3}\tilde{F}_3^{(2)}(k)] ,$$

$$U_1^b=\tfrac{2}{3}[\tilde{F}_1^{(1)}(k)-\tfrac{4}{3}\tilde{F}_2^{(1)}(k)+\tfrac{1}{3}\tilde{F}_3^{(1)}(k) -\tilde{F}_1^{(2)}(k)+\tfrac{4}{3}\tilde{F}_2^{(2)}(k)-\tfrac{1}{3}\tilde{F}_3^{(2)}(k)] ,$$

$$U_2^a=\tfrac{2}{3}[\tilde{F}_1^{(1)}(k)+2\tilde{F}_2^{(1)}(k)-\tilde{F}_1^{(2)}(k)+\tilde{F}_3^{(2)}(k)] ,$$

$$U_2^b=\tfrac{2}{3}[\tilde{F}_1^{(1)}(k)-\tilde{F}_3^{(1)}(k)-\tilde{F}_1^{(2)}(k)+\tilde{F}_3^{(2)}(k)] ,$$

and the quantities $\tilde{F}_j^{(i)}(k)$ for $j=1,2$ are defined in Eq. (A4). The matrix elements $\tilde{F}_j^{(i)}(k)$ for $j=3$ have the same form as for $j=2$, except one has to replace z with $2z$ in Eqs. (A4b) and (A4d).

The dispersion relations $\omega_1(k)$ and $\omega_2(k)$ are plotted in Figs. 4 and 5, respectively. We note that these excitations may be interpreted as valley waves, and that there must be two excitation modes within the lowest Landau level because there are two orthogonal valleys above the one occupied in the (noninteracting) ground state $\prod_k \alpha_k^\dagger|0\rangle$. The qualitative features of these curves are similar to those found in the preceding section. As in the two-layer case, $\omega_i(k)\propto k^2$ for $z=0$, and $\omega_i(k)\propto k$ for $z>0$ for small values of k. For large values of k one has $\omega_i(k)\sim\Delta(z)-e^2/\kappa k l_0^2$, where here $\Delta(z)\equiv\Delta\Sigma$. Finally, for $z/l_0\sim 1$ both modes have a dip in $\omega_i(k)$ near $kl_0\sim 1$; these become soft modes for $z>0.92l_0$ for $i=1$ and $z>1.51l_0$ for $i=2$. As in the two-layer case, this indicates that the system undergoes a phase transition. Such a transition is expected, because in the limit $z\to\infty$ the system is equivalent to three uncoupled layers with $\nu=\frac{1}{3}$. In that situation the system exhibits the FQHE, and has a gap over the entire energy spectrum, rather than the behavior $\omega(k)\to 0$, as $k\to 0$ typical of these systems with $z\ll l_0$. However, as in the two-layer case, one cannot rule out the possibility that intermediate phases exist between $z=0.92l_0$ and $z=\infty$ based on our calculation.

IV. CONCLUSIONS

We have studied the dispersion relation for two- and three-layer electron systems with $\nu=\frac{1}{2}$ and $\frac{1}{3}$, respective-

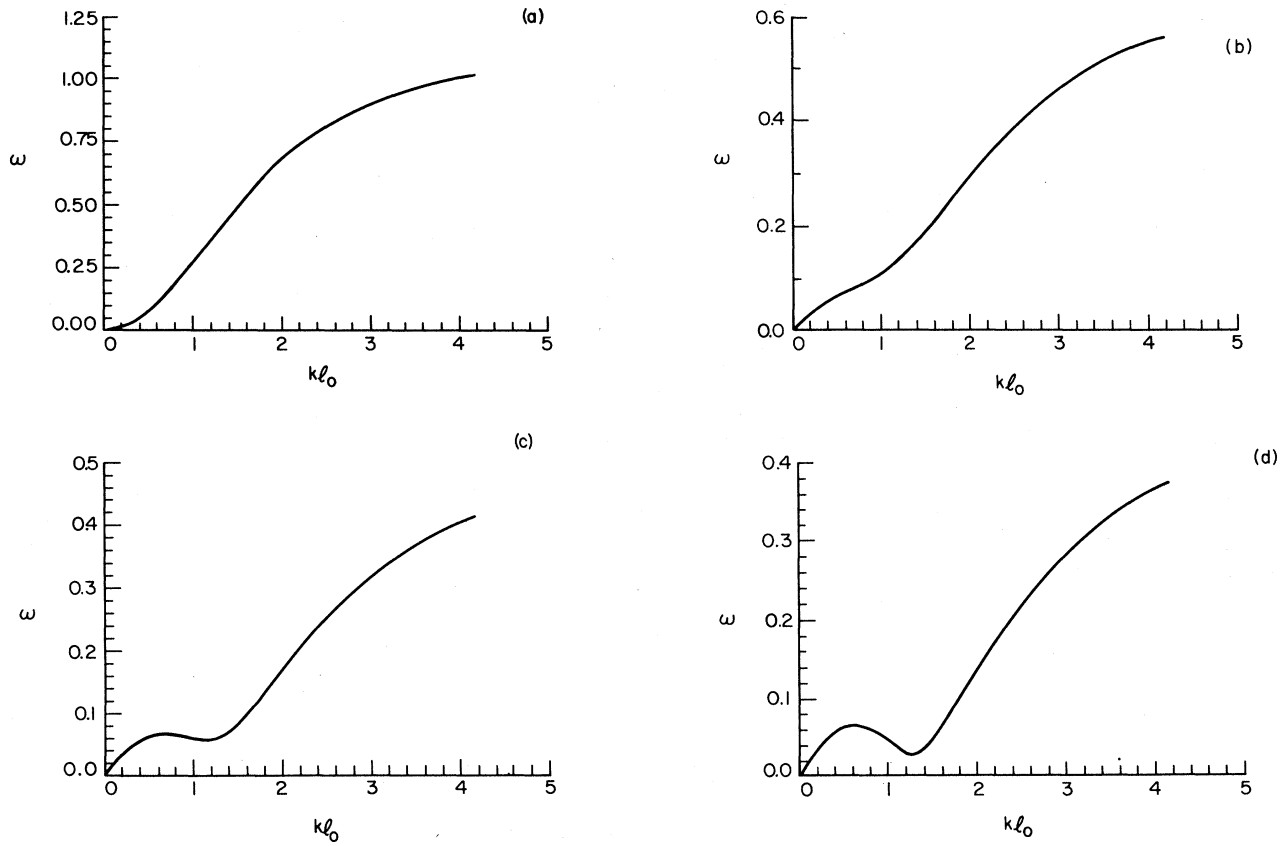

FIG. 4. Dispersion relation $\omega_1(k)$ for the three-layer system: (a) $z=0.0$, (b) $z=0.5l_0$, (c) $z=0.8l_0$, and (d) $z=0.9l_0$. ω_1 is in units of $e^2/\kappa l_0$.

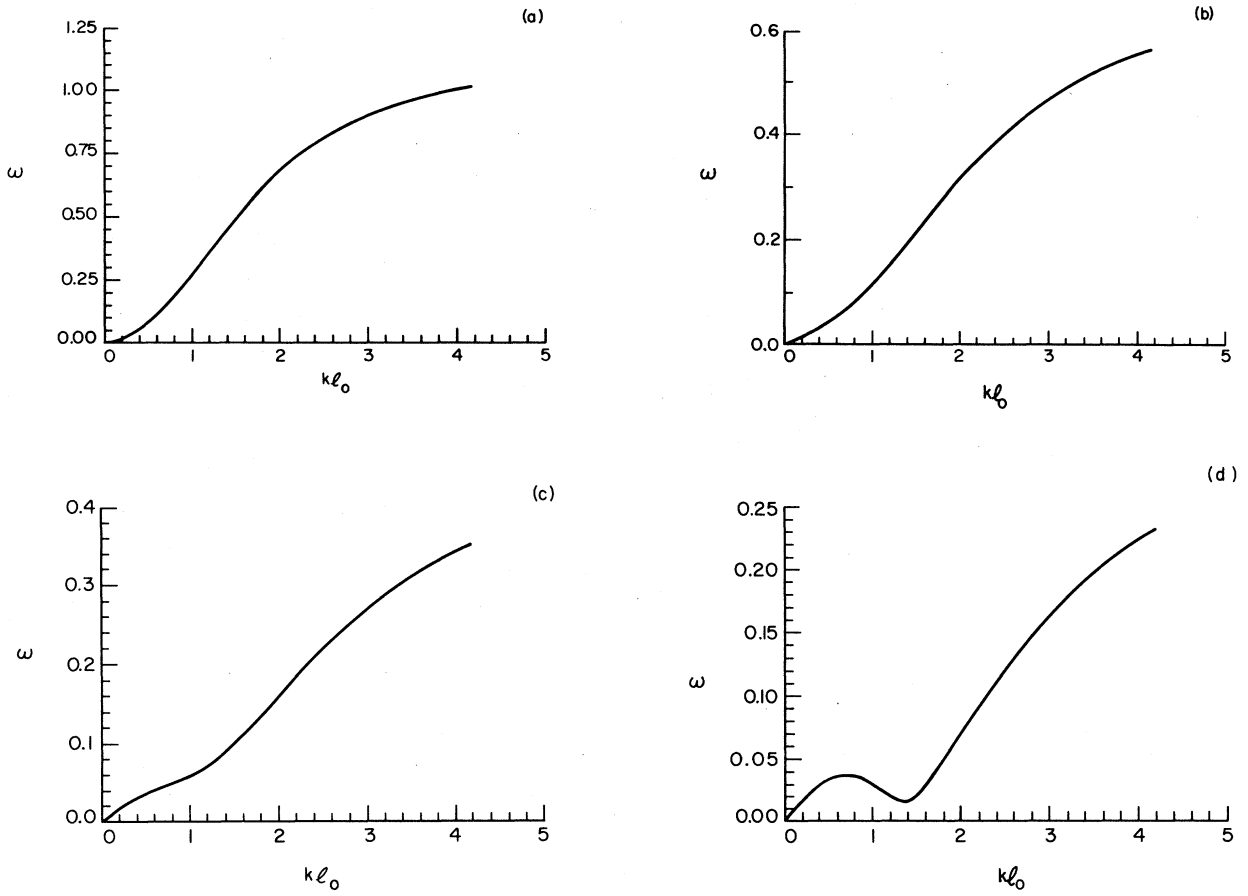

FIG. 5. Dispersion relation $\omega_2(k)$ for the three-layer system: (a) $z=0.0$, (b) $z=0.5l_0$, (c) $z=1.0l_0$, and (d) $z=1.5l_0$. ω_2 is in units of $e^2/\kappa l_0$.

ly, in the strong-magnetic-field limit. The qualitative features of the excitation spectra have the forms expected[5,6] for multivalley semiconductor systems for small kl_0. For large kl_0 the spectra have a form that clearly indicates the excitonic nature of the excited states. For $z\sim l_0$ we found that the dispersion relations develop a dip around $kl_0\sim 1$. These become soft modes when $z>1.21l_0$ for the two-layer system, and $z_1>0.92l_0$, $z_2>1.51l_0$ for the three-layer system. This indicates that the system undergoes a phase transition as the layer spacing is increased through these critical separations.

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APPENDIX: DISPERSION RELATION FOR TWO LAYERS

For the two-layer case, Eq. (12) reduces to a 2×2 matrix equation for $\tilde{\Gamma}^{\pm}_{12}$ and $\tilde{\Gamma}^{\pm}_{21}$, which turn out to be the only nonzero elements of $\tilde{\Gamma}^{\pm}_{\lambda\epsilon}$. Solving for these vertex functions, one finds that they have poles at frequencies

$$\begin{aligned}\omega(k)=\{&[\Delta\Sigma-\tilde{V}^{(1)}_{1221}(\mathbf{k})+\tilde{V}^{(2)}_{1212}(\mathbf{k})]\\&\times[\Delta\Sigma-\tilde{V}^{(1)}_{2112}(\mathbf{k})+\tilde{V}^{(2)}_{2121}(\mathbf{k})]\\&-[\tilde{V}^{(1)}_{1122}(\mathbf{k})-\tilde{V}^{(2)}_{1122}(\mathbf{k})]\\&\times[\tilde{V}^{(1)}_{2211}(\mathbf{k})-\tilde{V}^{(2)}_{2211}(\mathbf{k})]\}^{1/2},\end{aligned}\tag{A1}$$

where $\Delta\Sigma=\Sigma_2-\Sigma_1$, and we have set $h=0$. The matrix elements $\tilde{V}^{(i)}_{\mu\lambda\epsilon\nu}(\mathbf{k})$ are evaluated as follows. From Eqs. (5) and the discussion following it, we recall that the functions $F_i(q_y,p_1-p_2)$ are given by

$$F_i(q_y,p_1-p_2)=\frac{1}{2L}\int\frac{dq_x}{2\pi}e^{-q^2l_0^2/2}\tilde{v}_i(q)e^{iq_x(p_1-p_2)l_0^2},\tag{A2}$$

where $\tilde{v}_1(q)=2\pi e^2/q$, $\tilde{v}_2(q)=(2\pi e^2/q)e^{-zq}$, z is the distance between layers, and we have replaced the sum in (5) by an integral. Substituting Eq. (A2) into Eqs. (11a) and (11b), one can evaluate the matrix elements from the resulting coefficients using Eq. (13). The results are

$$\begin{aligned}\tilde{V}^{(i)}_{1221}(\mathbf{k})&=\tilde{V}^{(i)}_{2112}(\mathbf{k})=\tilde{F}^{(i)}_{1}(\mathbf{k})+\tilde{F}^{(i)}_{2}(\mathbf{k}) ,\\ \tilde{V}^{(i)}_{1212}(\mathbf{k})&=\tilde{v}^{(i)}_{2121}(\mathbf{k})=\tilde{V}^{(i)}_{1122}(\mathbf{k})\\ &=\tilde{V}^{(i)}_{2211}(\mathbf{k})\\ &=\tilde{F}^{(i)}_{1}(\mathbf{k})-\tilde{F}^{(i)}_{2}(\mathbf{k}) ,\end{aligned} \tag{A3}$$

where $i=1,2$, and

$$\tilde{F}^{(1)}_{1}(\mathbf{k})=\frac{e^2}{2\kappa l_0}\left[\frac{\pi}{2}\right]^{1/2} e^{-k^2l_0^2/4} I_0\left[\frac{k^2l_0^2}{4}\right] , \tag{A4a}$$

$$\tilde{F}^{(1)}_{2}(\mathbf{k})=\frac{e^2}{2\kappa}\int_0^\infty dq\, e^{-q^2l_0^2/2} e^{-zq} J_0(kql_0^2) , \tag{A4b}$$

$$\tilde{F}^{(2)}_{1}(\mathbf{k})=\frac{e^2}{2\kappa k l_0^2} e^{-k^2l_0^2/2} , \tag{A4c}$$

$$\tilde{F}^{(2)}_{2}(\mathbf{k})=\frac{e^2}{2\kappa k l_0^2} e^{-k^2l_0^2/2-kz} , \tag{A4d}$$

where J_0 and I_0 are Bessel functions. Substituting Eqs. (A3) and (A4) into Eq. (A1), one finds after some algebra

$$\begin{aligned}\omega(k)=\{&[\Delta\Sigma-2\tilde{F}^{(1)}_{2}(\mathbf{k})]\\ &\times[\Delta\Sigma-2\tilde{F}^{(1)}_{1}(\mathbf{k})+2\tilde{F}^{(2)}_{1}(\mathbf{k})-2\tilde{F}^{(2)}_{2}(\mathbf{k})]\}^{1/2} .\end{aligned} \tag{A5}$$

The self-energy difference $\Delta\Sigma=\Sigma_2-\Sigma_1$ is easily found by substituting Eqs. (A3) and (A4) into Eq. (14):

$$\begin{aligned}\Delta\Sigma\equiv\Delta(z)&\\ =2\tilde{F}^{(1)}_{2}(0)&\\ =\frac{e^2}{\kappa l_0}\left[\frac{\pi}{2}\right]^{1/2} e^{z^2/2l_0^2}&\mathrm{erfc}\left[\frac{z}{\sqrt{2}l_0}\right] ,\end{aligned} \tag{A6}$$

where erfc is the complementary error function. From its definition, one may show that $\Delta(z)$ is a decreasing function of z.

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