Theory of critical first sound near the λ transition of ^{4}He. II. Attenuation and dispersion for $T \geq T_\lambda$

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The critical dynamics of the superfluid transition are studied on the basis of a complete stochastic model that includes both the thermal diffusion and the first-sound mode. A two-loop calculation is carried out for the sound attenuation and dispersion above and at T_λ. Static and dynamic effects are properly separated. Renormalized field theory is used to describe the critical behavior. It is shown that the static and dynamic renormalization-group functions of the simpler model F are closely related to those of the present model. This provides the basis for novel quantitative tests of the dynamic renormalization-group theory as a function of frequency.

I. INTRODUCTION

In the preceding paper[1] (hereafter referred to as I) we have discussed a recently introduced model[2,3] for the coupled critical dynamics of the thermal-diffusion and first-sound modes near the superfluid transition in ^{4}He. In the present paper we shall use this model in order to develop a quantitative renormalization-group (RG) theory of first-sound propagation in homogeneous bulk ^{4}He for $T \geq T_\lambda$.

The need for such a theory is quite obvious. We shall not attempt here to give a detailed account for the merits and failures of previous theoretical work[4–13] on this subject (above T_λ) but rather refer to review articles.[14–19] Suffice it to say that, for various reasons, these theories cannot be considered as being fully developed and quantitatively reliable. One of the reasons is that the nonasymptotic properties of the critical behavior[18–20] are not (or not properly) contained in the earlier theories; the nonasymptotic aspects have turned out to be crucial in the description of sound propagation.[2,3] Another reason is that these earlier treatments were not based on the complete equations of motion including all relevant static and dynamic couplings. Both types of couplings are indispensable for a satisfactory theory; they are also necessary for a quantitative analysis of experimental results on critical first sound as will be shown in a subsequent part of this work.[21]

A notable exception among the earlier approaches is the phenomenological theory by Ferrell and Bhattacharjee[22–24] (FB) who correctly identified the leading physical mechanism of critical sound attenuation in terms of a frequency-dependent specific heat[25,26] and appropriately recognized the nonasymptotic nature of the critical frequency dependence, as confirmed by our RG theory.[2] Their approach, however, does not provide a statistical foundation for the concept of a frequency-dependent specific heat. Furthermore the underlying dynamics of ^{4}He have been characterized only in terms of one-loop expressions[22–24] (rather than in terms of a complete set of interaction vertices or corresponding equations of motion), and no general computational prescription for the sound attenuation and dispersion is available. In the present paper we shall go beyond the FB approach both conceptually and quantitatively by (i) treating the coupled sound and heat modes on the basis of a proper statistical-dynamical model, (ii) deriving two-loop results for the critical sound attenuation and dispersion by means of renormalized perturbation theory, and (iii) incorporating in a consistent fashion the detailed quantitative knowledge on critical statics[27–30] and on low-frequency dynamics.[18–20,31–36]

In Sec. II the basic thermodynamic quantities (specific heat, adiabatic and isothermal compressibilities) are identified in terms of correlation functions of our model. The statistical-dynamical definitions of the sound velocity and damping are given in Sec. III on the basis of the dynamic structure factor in the limit of small wave numbers. In the zero-frequency limit an exact separation between static and dynamic parts is implied by a dissipation-fluctuation theorem; the latter serves as an important guide for the appropriate representation of the perturbative two-loop results at finite frequencies presented in Sec. IV. This section also includes a discussion of the zero-frequency limit as well as a comparison with the one-loop result of FB. In Sec. V the field-theoretic renormalization of both static and dynamic quantities is performed. Thereby the connection with the RG functions of model F (Refs. 34 and 37) is established up to two-loop order. Section VI contains a brief summary. Details of the calculations are given in the appendixes.

II. THERMODYNAMICS AND STATIC CORRELATION FUNCTIONS

It is well known that thermodynamics plays a fundamental role in the theory of sound propagation.[25,38] Therefore, before developing our dynamic theory in Secs. III–V, it is necessary to establish the connection between thermodynamic quantities and the corresponding statistical quantities of our model. These are the static two-point correlation functions at wave number $k = 0$, e.g.,

$$\mathring{C}_{mm}=\int d^dx\langle[m_0(x)-\langle m_0\rangle][m_0(0)-\langle m_0\rangle]\rangle \ , \tag{2.1}$$

and the quantities $\mathring{C}_{pp}$, $\mathring{C}_{\rho\rho}$, $\mathring{C}_{mp}$, and $\mathring{C}_{pm}$ defined similarly. The results are summarized in (2.10)–(2.12) later. Here and in the following the brackets $\langle \cdots \rangle$ denote averages with the distribution $\sim\exp(-H)$, where H is given by I (2.79). (Equation numbers preceded by "I" are those of Ref. 1.)

All thermodynamic quantities can be derived from the thermodynamic potential[15,39,40]

$$\Omega(T,\mu)=\Omega^{(0)}(T,\mu)-\frac{k_BT}{V}\ln Z_1 \tag{2.2}$$

with

$$Z_1=\int D[m_0,p_0,\psi_0,j_0]\exp(-H) \ . \tag{2.3}$$

For simplicity we assume that $k_BT\ln Z_1$ depends on T and μ only via $r_0[T,\mu]$, i.e., we take the static couplings as constants. Then we obtain the entropy per unit volume

$$\langle s_0\rangle=-\left[\frac{\partial\Omega}{\partial T}\right]_\mu=-\Omega'^{(0)}-\tfrac{1}{2}k_BT\,r_0'\langle|\psi_0|^2\rangle \ , \tag{2.4}$$

and the mass density

$$\langle\rho_0\rangle=-\left[\frac{\partial\Omega}{\partial\mu}\right]_T=-\dot{\Omega}^{(0)}-\tfrac{1}{2}k_BT\,\dot{r}_0\langle|\psi_0|^2\rangle \ , \tag{2.5}$$

where the prime and the dot denote $\partial/\partial T$ at constant μ and $\partial/\partial\mu$ at constant T, respectively. Furthermore we assume that $r_0[T,\mu]$ depends only linearly on T [see I (2.30)]. Then the specific heat per unit volume at constant μ is given by

$$C_\mu=T\left[\frac{\partial\langle s_0\rangle}{\partial T}\right]_\mu=-T\Omega''^{(0)}+\tfrac{1}{4}k_BT^2r_0'^2\mathring{C}_\psi \tag{2.6}$$

with

$$\mathring{C}_\psi=\int d^dx\langle[|\psi_0(x)|^2-\langle|\psi_0|^2\rangle]\times[|\psi_0(0)|^2-\langle|\psi_0|^2\rangle]\rangle \ . \tag{2.7}$$

For the purpose of a comparison with experiments we are interested in the constant-pressure specific heat per unit volume C_p rather than in C_μ. A corresponding thermodynamic calculation yields approximately (see Appendix A)

$$C_p=k_B\mathring{\chi}_m-T\left[\frac{\partial\bar{\rho}_0}{\partial T}\right]_P(\langle\sigma_0\rangle-\bar{\sigma}_0)+k_B\mathring{\gamma}_m^2\mathring{\chi}_m^2\mathring{C}_\psi \ . \tag{2.8}$$

We wish to compare (2.8) with the two-point correlation function $\mathring{C}_{mm}$, which can be expressed in terms of $\mathring{C}_\psi$ as

$$\mathring{C}_{mm}=\mathring{\chi}_m+\mathring{\gamma}_m^2\mathring{\chi}_m^2\mathring{C}_\psi \ . \tag{2.9}$$

We see that the coefficient of the singular contribution $\sim\mathring{C}_\psi$ of C_p agrees with that of $k_B\mathring{C}_{mm}$. [This agreement is only approximate; see (A 18) in Appendix A.] The additional critical contribution $\sim(\langle\sigma_0\rangle-\bar{\sigma}_0)$ in (2.8), however, is not contained in $\mathring{C}_{mm}$. This term has not been noticed previously[17–20] where no attention has been paid to the difference between C_μ and C_p and where C_p has been identified with $k_B\mathring{C}_{mm}$. This can be justified approximately because of the smallness of the term $\sim(\langle\sigma_0\rangle-\bar{\sigma}_0)$ (see also the quantitative thermodynamic analysis of Ahlers[41]). In the application to bulk properties we shall ignore this term as well and thus maintain the previous identification

$$C_p\simeq k_B\mathring{C}_{mm} \ . \tag{2.10}$$

Nevertheless the more precise identification of $\mathring{C}_\psi$ and $\mathring{C}_{mm}$ in terms of C_μ according to (2.9) and (2.6) should be kept in mind for the application to finite-size calculations of the specific heat[42] where the nonequivalence of thermodynamic ensembles[43] may be of relevance.

In the same spirit the connection of other thermodynamic quantities with the remaining two-point correlation functions is made in Appendix A. Here we only present the results for the adiabatic and isothermal compressibilities and for the constant-volume specific heat, respectively,

$$\kappa_\sigma=\langle\rho_0\rangle^{-1}\left[\frac{\partial\langle\rho_0\rangle}{\partial P}\right]_{\langle\sigma_0\rangle}=(\mathring{C}_{pp}-\mathring{C}_{mp}^2\mathring{C}_{mm}^{-1})\frac{\bar{\rho}_0^2k_BT}{\langle\rho_0\rangle^2\mathring{\chi}_p^2} \ , \tag{2.11}$$

$$\kappa_T=\langle\rho_0\rangle^{-1}\left[\frac{\partial\langle\rho_0\rangle}{\partial P}\right]_T=\mathring{C}_{\rho\rho}(k_BT)^{-1}\langle\rho_0\rangle^{-2} \ , \tag{2.12}$$

$$C_v=C_p\frac{\kappa_\sigma}{\kappa_T} \ . \tag{2.13}$$

Equations (2.10)–(2.13) provide the connection between the static correlation functions of our model and the measurable thermodynamic quantities C_p, κ_σ, κ_T, C_v. These relations constitute the starting point for the quantitative identification of the nonuniversal static parameters.[21] Equation (2.11) is related to the thermodynamic expression for the velocity of first sound, see (4.39) later. The critical behavior of all quantities (2.10)–(2.13) is related to that of $\mathring{C}_\psi$ according to I (3.14)–(3.18).

III. DYNAMIC STRUCTURE FACTOR FOR $k\to 0$

A. General form

We start from the dynamic structure factor[44]

$$S(k,\omega)=\int d^dx\int dt\,e^{-i(kx-\omega t)}[\langle\rho_0(x,t)\rho_0(0,0)\rangle-\langle\rho_0\rangle^2]=\mathring{C}_{\rho\rho}(k,\omega) \ . \tag{3.1}$$

Since, by construction, $\rho_0(x,t)$ is a linear combination of our model variables $m_0(x,t)$ and $p_0(x,t)$ we obtain $\mathring{C}_{\rho\rho}$ from I (2.65) as

$$S(k,\omega)=b_\rho^{-2}[\mathring{C}_{pp}(k,\omega)-2b_m\mathring{C}_{pm}(k,\omega)+b_m\mathring{C}_{mm}(k,\omega)] , \quad (3.2)$$

where b_ρ and b_m are thermodynamic coefficients given by I (2.66) and I (2.67). One way of calculating $\mathring{C}_{\alpha\beta}(k,\omega)$ is via the response functions $\mathring{R}_{\alpha\beta}(k,\omega)$ according to I (4.20)–(4.24). This will be most efficient for the purpose of a calculation of the complete structure factor. As we are interested mainly in the damping and velocity of first sound we shall use here an alternative way by expressing the right-hand side of (3.2) in terms of two-point vertex functions according to I (4.17). This yields $S(k,\omega)$ in the form of the ratio

$$S(k,\omega)=-\frac{\mathring{N}_{\rho\rho}(k,\omega)}{|\mathring{\Delta}(k,\omega)|^2} \quad (3.3)$$

with the numerator

$$\mathring{N}_{\rho\rho}(k,\omega)=b_\rho^{-2}(\mathring{N}_{pp}-2b_m\mathring{N}_{pm}+b_m^2\mathring{N}_{mm}) . \quad (3.4)$$

The explicit expressions of $\mathring{N}_{\alpha\beta}(k,\omega)$ in terms of the various vertex functions $\mathring{\Gamma}_{\tilde{\alpha}\tilde{\beta}}$ and $\mathring{\Gamma}_{\alpha\tilde{\beta}}$ can be obtained from I (4.17). For our purpose it suffices to restrict the discussion to the denominator of (3.3). The quantity $\mathring{\Delta}(k,\omega)$ represents the determinant of the 3×3 matrix $[\mathring{\Gamma}(k,\omega)]$ of the vertex functions $\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)$,

$$\mathring{\Delta}(k,\omega)=\det[\mathring{\Gamma}(k,\omega)] , \quad (3.5)$$

$$[\mathring{\Gamma}(k,\omega)]=\begin{vmatrix} \mathring{\Gamma}_{m\tilde{m}} & \mathring{\Gamma}_{p\tilde{m}} & \mathring{\Gamma}_{j\tilde{m}} \\ \mathring{\Gamma}_{m\tilde{p}} & \mathring{\Gamma}_{p\tilde{p}} & \mathring{\Gamma}_{j\tilde{p}} \\ \mathring{\Gamma}_{m\tilde{j}} & \mathring{\Gamma}_{p\tilde{j}} & \mathring{\Gamma}_{j\tilde{j}} \end{vmatrix} . \quad (3.6)$$

Since $\mathbf{j}_0$ enters the equations of motion I (2.75)–(2.79) in a particularly simple way we have

$$\mathring{\Gamma}_{j\tilde{m}}(k,\omega)=0 , \quad (3.7)$$

$$\mathring{\Gamma}_{j\tilde{p}}(k,\omega)=ic_0\mathring{\chi}_j^{-1}k , \quad (3.8)$$

and

$$\mathring{\Gamma}_{j\tilde{j}}(k,\omega)=-i\omega+\mathring{\lambda}_j\mathring{\chi}_j^{-1}k^2 \quad (3.9)$$

as exact (trivial) results. Perturbative expressions of the remaining vertex functions $\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)$ are given in Appendix B up to two-loop order.

In order to define the damping coefficient D_1 and velocity c_1 of sound we first consider the hydrodynamic limit $k\to 0$ and $\omega\to 0$ well above T_λ. Then we know from ordinary hydrodynamics[45] that $S(k,\omega)$ has the form

$$S(k,\omega)=2k_BT\frac{\langle\rho_0\rangle k^2}{\omega}\mathrm{Im}\frac{-i\omega+D_T(C_p/C_v)k^2}{(-i\omega+D_Tk^2)(-\omega^2+c_1^2k^2-i\omega D_1k^2)} , \quad (3.10)$$

where D_T is the thermal diffusivity and C_p/C_v is the thermodynamic specific-heat ratio. It is straightforward to verify that in the absence of all nonlinear couplings $(\hat{u}_0,\mathring{\gamma}_m,\mathring{\gamma}_p,\mathring{g}_m,\mathring{g}_p)$ and for $(k,\omega)\to 0$ one indeed obtains the hydrodynamic form (3.10), after substituting the zeroth-order part of the vertex functions [see (B1) and I (4.6)] into (3.3). Then the various quantities in (3.10) are replaced by the zeroth-order expressions

$$\langle\rho_0\rangle^{(0)}=\bar{\rho}_0 , \quad (3.11)$$

$$C_p^{(0)}/C_v^{(0)}=(\partial\bar{\sigma}_0/\partial T)_{\bar{P}_0}(\partial\bar{\sigma}_0/\partial T)_{\bar{\rho}_0}^{-1} , \quad (3.12)$$

$$D_T^{(0)}=\mathring{\lambda}_m\mathring{\chi}_m^{-1}=\kappa_0/(\bar{\rho}_0C_p^{(0)}) , \quad (3.13)$$

$$D_1^{(0)}=\mathring{\lambda}_j\mathring{\chi}_j^{-1}+\mathring{\lambda}_p\mathring{\chi}_p^{-1} =\frac{1}{\bar{\rho}_0}\left[\zeta_0+\tfrac{4}{3}\eta_0+\kappa_0\left[\frac{1}{C_v^{(0)}}-\frac{1}{C_p^{(0)}}\right]\right] , \quad (3.14)$$

$$c_1^{(0)}=c_0(\mathring{\chi}_j\mathring{\chi}_p)^{-1/2}=(\partial\bar{P}_0/\partial\bar{\rho}_0)_{\bar{\sigma}_0}^{1/2} . \quad (3.15)$$

For the relation between our model parameters and the corresponding thermo-hydrodynamic quantities see I.

Near T_λ the nonlinear couplings lead to a much more complicated k and ω dependence of $S(k,\omega)$. In this paper we shall keep only the hydrodynamic k dependence as given in (3.10) but allow for arbitrary frequencies ω. This is a systematic approximation which is relevant for the application to first-sound experiments. Thus we shall focus upon the critical t and ω dependence of the coefficients $D_1(t,\omega)$ and $c_1(t,\omega)$ in (3.10), where t denotes the relative temperature

$$t=[T-T_\lambda(P)]/T_\lambda(P) . \quad (3.16)$$

The effect due to the finiteness of k in real sound propagation will be discussed elsewhere. Inspection of the general k dependence of all vertex functions indicates that for $k\to 0$, even in the presence of the nonlinear couplings, the expression (3.3) still exhibits the hydrodynamic k dependence given in (3.10), now with frequency-dependent coefficients. This provides the possibility of a unique definition of D_T, D_1, and c_1 at finite frequencies. In particular we have verified that for $k\to 0$ the denominator of (3.3) corresponds to the denominator of (3.10) (at finite ω and in the presence of nonlinear couplings). This implies that the frequency-dependent generalizations of the coefficients D_T, D_1, and c_1 can be expressed entirely in terms of the vertex functions appearing in (3.6), i.e., the vertex functions $\mathring{\Gamma}_{\tilde{\alpha}\tilde{\beta}}$ do not enter our calculation of D_1 and c_1. This is a nontrivial simplification which will not remain valid below T_λ. Then a combined calculation of both the numerator and the denominator of (3.3), as well as an appropriate rearrangement of perturbative contributions will be necessary in order to arrive at a consistent definition of D_1. This can be inferred from the corresponding analysis[46] in case of second-sound damping D_2.

Taking the $k \to 0$ limit and keeping only the leading (hydrodynamic) k dependence of the determinant (3.5) we obtain

$$\mathring{\Delta}(k,\omega)=[-i\omega+z_2(t,\omega)k^2][-\omega^2+y(t,\omega)k^2 -i\omega z_1(t,\omega)k^2] \; . \quad (3.17)$$

Here y and z_i are *complex* functions

$$y(t,\omega)=-\frac{\partial}{\partial k^2}(\mathring{\Gamma}_{j\tilde p}\mathring{\Gamma}_{p\tilde j}) \; , \quad (3.18)$$

$$z_1(t,\omega)=\frac{\partial}{\partial k^2}(\mathring{\Gamma}_{j\tilde j}+\mathring{\Gamma}_{p\tilde p}+\mathring{\Gamma}_{m\tilde j}\mathring{\Gamma}_{p\tilde m}\mathring{\Gamma}_{p\tilde j}^{-1}) \; , \quad (3.19)$$

$$z_2(t,\omega)=\frac{\partial}{\partial k^2}(\mathring{\Gamma}_{m\tilde m}-\mathring{\Gamma}_{m\tilde j}\mathring{\Gamma}_{p\tilde m}\mathring{\Gamma}_{p\tilde j}^{-1}) \; . \quad (3.20)$$

In (3.18)–(3.20), $\mathring{\Gamma}_{\alpha\tilde\beta}$ stands for $\mathring{\Gamma}_{\alpha\tilde\beta}(k,\omega)$, and all derivatives are taken at $k=0$. Obviously the second factor in (3.17) implies the dispersion relation

$$\omega^2/k^2=y(t,\omega)-i\omega z_1(t,\omega) \; . \quad (3.21)$$

The identification of the physical quantities $D_T(t,\omega)$, $D_1(t,\omega)$, and $c_1(t,\omega)$ follows from the requirement that $\mathring{\Delta}(k,\omega)$ can be represented as

$$\mathring{\Delta}(k,\omega)=[-i\omega+D_T(t,\omega)k^2][-\omega^2+c_1(t,\omega)^2k^2 -i\omega D_1(t,\omega)k^2] \quad (3.22)$$

with D_T, D_1, and c_1 being *real* functions of t and ω. This requirement determines c_1, D_1, and D_T uniquely as[2]

$$c_1(t,\omega)^2=\mathrm{Re}[y(t,\omega)-i\omega z_1(t,\omega)] \; , \quad (3.23)$$

$$D_1(t,\omega)=-\frac{1}{\omega}\mathrm{Im}[y(t,\omega)-i\omega z_1(t,\omega)] \; , \quad (3.24)$$

$$D_T(t,\omega)=\mathrm{Re}\, z_2(t,\omega) \; . \quad (3.25)$$

Equations (3.23)–(3.25) constitute the basic connection between our theory and the measurable dynamic quantities c_1, D_1, and D_T, in the small k limit. These relations should be valid for $T \ge T_\lambda$ in all orders of the nonlinear couplings and for arbitrary ω [up to a background frequency $\omega_B \sim O(10^{11}\ \mathrm{Hz})$].

B. Zero-frequency limit

In the zero-frequency limit, an exact information on the structure of c_1, D_1, and D_T can be inferred from the dissipation-fluctuation relation I (4.33). In this relation the response functions $\mathring{C}_{\alpha\tilde\beta}(k,\omega)$ appear which constitute a 3×3 matrix denoted by $[\mathring{\tilde C}(k,\omega)]$. The corresponding 3×3 matrix of vertex function $[\mathring{\Gamma}(k,\omega)]$, (3.6), is related to $[\mathring{\tilde C}(k,\omega)]$ according to

$$[\mathring{\Gamma}^0(k,\omega)]=[\mathring{\tilde C}(k,\omega)]^{-1} \; . \quad (3.26)$$

Thus, for $\omega \to 0$, I (4.33) and (3.26) yield the exact result

$$[\mathring{\Gamma}(k,0)]=[\mathring{\Gamma}(-k,0)]^* =\{[\mathring{L}(k)]+[\mathring{\phi}(k,0)]\}[\mathring{C}(k)]^{-1} \; , \quad (3.27)$$

where the matrix of transport coefficients $[\mathring{L}(k)]$ is given by I (4.6) and $[\mathring{C}(k)]$ is the 3×3 matrix of the static correlation functions

$$\mathring{C}_{\alpha\beta}(k)=\delta_{\alpha\beta}+\gamma_\alpha\gamma_\beta\mathring{C}_\psi(k) \quad (3.28)$$

with α and β representing one of the variables $m_0\mathring{\chi}_m^{-1/2}$, $p_0\mathring{\chi}_p^{-1/2}$, or $\mathrm{j}_0\mathring{\chi}_j^{-1/2}$ (we use the notation of I). The matrix elements

$$\mathring{\phi}_{\alpha\tilde\beta}(k,\omega)=\mathring{g}_\alpha\mathring{\Gamma}_{\tilde\beta\tilde\varphi}(k,\omega) \quad (3.29)$$

of the 3×3 matrix $[\mathring{\phi}(k,\omega)]$ are determined by the composite-field vertex functions $\mathring{\Gamma}_{\tilde\beta\tilde\varphi}$ which have been introduced in I (4.24) and are specified diagrammatically in Appendix B. Since $\mathring{g}_j \equiv 0$ [see I (A6)] we have

$$\mathring{\phi}_{j\tilde\beta}(k,\omega)=0 \; . \quad (3.30)$$

The significance of relation (3.27) is the separation of the purely static part $[\mathring{C}(k)]^{-1}$ from the genuine dynamic parts $[\mathring{L}(k)]$ and $[\mathring{\phi}(k,0)]$. For the corresponding separation in case of model F see (4.5) of Ref. 34. Since this property is an exact nonperturbative result, the structure of (3.27) serves as an important guide for an appropriate rearrangement of the expressions for $\mathring{\Gamma}_{\alpha\tilde\beta}(k,\omega)$ obtained within a strict perturbation expansion.

IV. SOUND ATTENUATION AND DISPERSION

Previously[2,3] we have presented the expressions for the sound attenuation and dispersion only up to one-loop order, with effective parameters taken from the two-loop model-F flow equations. While this description is adequate in order to explain the main features, it is nevertheless desirable to improve upon the one-loop approximation. Here we shall proceed to a two-loop calculation of the velocity and damping within our complete model equations I (2.75)–(2.79). We shall make use of the limit $c_0 \to \infty$ in the two-loop contributions which appears to be a reasonable approximation for the application to experimental data. As discussed in I, there exists a close connection between our model and model F in the limit $c_0 \to \infty$ which will greatly facilitate our calculation.

A. One-loop results

Ordinary diagrammatics with the dynamic functional I (A1)–(A8) yields the one-loop expressions for $\mathring{\Gamma}_{\alpha\tilde\beta}(k,\omega)$ given in (B1)–(B5) of Appendix B. According to (3.18)–(3.20) we need explicit results only up to $O(k^2)$. To this order we find

$$[\mathring{\Gamma}(k,\omega)] = -i\omega[1] + [\mathring{L}(k)] + k^2[\mathring{g},\mathring{g}]\mathring{P}(t,\omega) - [\mathring{L}(k)][\mathring{\gamma},\mathring{\gamma}]\mathring{F}_+(t,\omega) . \tag{4.1}$$

Here [1] denotes the 3×3 unit matrix and $[\mathring{x},\mathring{y}]$ represents a 3×3 matrix with elements $\mathring{x}_\alpha \mathring{y}_\beta$. For the components $\mathring{g}_\alpha$ and $\mathring{\gamma}_\alpha$ of $\mathring{g}$ and $\mathring{\gamma}$ see I (A6). The frequency-dependent parts of (4.1) are

$$\mathring{P}(t,\omega) = \frac{2}{\Gamma_0' d}\int_p \frac{p^2}{(p^2+r_0)^2(p^2+r_0-i\Omega_0)} \tag{4.2}$$

and

$$\mathring{F}_+(t,\omega) = 4\int_p \frac{1}{(p^2+r_0)(p^2+r_0-i\Omega_0)} \tag{4.3}$$

with

$$\Omega_0 = \frac{\omega}{2\Gamma_0'} \tag{4.4}$$

and

$$\Gamma_0' = \mathrm{Re}\Gamma_0, \quad \int_p \equiv (2\pi)^{-d} d^d p .$$

Now we invoke the exact relation (3.27) which dictates that for $\omega=0$ the purely static part in the last term of (4.1) *must appear in the denominator.* Therefore it is obvious that (4.1), although being a correct result of a strict perturbative expansion, should be rewritten in the partially "resummed" form

$$[\mathring{\Gamma}(k,\omega)] = -i\omega[1] + \{[\mathring{L}(k)] + k^2[\mathring{g},\mathring{g}]\mathring{P}(t,\omega)\} \times \{[1] + [\mathring{\gamma},\mathring{\gamma}]\mathring{F}_+(t,\omega)\}^{-1} . \tag{4.5}$$

For a corresponding rearrangement of perturbative results of $\mathring{\Gamma}_{m\tilde{m}}$ at $\omega=0$ up to two-loop order see Ref. 34. For $\omega=0$ the expression in the last curly brackets of (4.5) is indeed identical with the static correlation function (3.28) at $k=0$, in accord with the general structure (3.27). Comparison with the form I (3.2) yields

$$\mathring{C}_\psi(k=0) = \mathring{F}_+(t,0) = 4\int_p (p^2+r_0)^{-2} + O(u_0) , \tag{4.6}$$

in agreement with statics in one-loop order [see (A9) and (A10) of Ref. 29 for $n=2$]. We note that the parameter c_0 (which is proportional to the background sound velocity) appears only in $[\mathring{L}(k)]$ but not in the one-loop part of (4.5).

A more convenient representation of the inverse matrix in (4.5) is

$$\{[1] + [\mathring{\gamma},\mathring{\gamma}]\mathring{F}_+[t,\omega)\}^{-1} = [1] - [\mathring{\gamma},\mathring{\gamma}]\mathring{\Sigma}(t,\omega) , \tag{4.7}$$

where

$$\mathring{\Sigma}(t,\omega) = \frac{\mathring{F}_+(t,\omega)}{1+\mathring{\gamma}^2\mathring{F}_+(t,\omega)} \tag{4.8}$$

with

$$\mathring{\gamma}^2 = \mathring{\gamma}_m^2 \mathring{\chi}_m + \mathring{\gamma}_p^2 \mathring{\chi}_p . \tag{4.9}$$

In terms of $\mathring{\Sigma}(t,\omega)$ the relevant matrix elements of (4.5) read

$$\mathring{\Gamma}_{p\tilde{j}}(k,\omega) = ic_0 \mathring{\chi}_p^{-1} k[1 - \mathring{\gamma}_p^2 \mathring{\chi}_p \mathring{\Sigma}(t,\omega)] , \tag{4.10}$$

$$\mathring{\Gamma}_{m\tilde{j}}(k,\omega) = -ic_0 k \mathring{\gamma}_p \mathring{\gamma}_m \mathring{\Sigma}(t,\omega) , \tag{4.11}$$

$$\mathring{\Gamma}_{p\tilde{m}}(k,\omega) = k^2[L_0 \mathring{\chi}_p^{-1} - (L_0 \mathring{\gamma}_p^2 + \mathring{\lambda}_m \mathring{\gamma}_p \mathring{\gamma}_m)\mathring{\Sigma}(t,\omega][1 + \mathring{g}_m \mathring{g}_p L_0^{-1} \mathring{P}(t,\omega)] , \tag{4.12}$$

$$\mathring{\Gamma}_{p\tilde{p}}(k,\omega) = -i\omega + k^2[\mathring{\lambda}_p \mathring{\chi}_p^{-1} - (\mathring{\lambda}_p \mathring{\gamma}_p^2 + L_0 \mathring{\gamma}_p \mathring{\gamma}_m)\mathring{\Sigma}(t,\omega)][1 + \mathring{g}_p^2 \mathring{\lambda}_p^{-1} \mathring{P}(t,\omega)] . \tag{4.13}$$

We skip the analogous expressions for $\mathring{\Gamma}_{m\tilde{p}}$ and $\mathring{\Gamma}_{m\tilde{m}}$.

We call attention to the fact that the frequency dependence of the various vertex functions is entirely contained in the two functions $\mathring{P}(t,\omega)$ and $\mathring{\Sigma}(t,\omega)$. This simplifying feature is a consequence of the $k\to0$ limit whereas at finite k also mixed terms $\sim\mathring{\gamma}_\alpha \mathring{g}_\beta$ with a different frequency dependence would arise as seen from the one-loop result (B2)–(B5). For example there exists an $O(k^2)$ correction to the expression in the square brackets of (4.10) of the form

$$k^2 \frac{2\mathring{\gamma}_p \mathring{g}_p \Gamma_0''}{\Gamma_0'^2 d}\int_p \frac{p^2}{(p^2+r_0-i\Omega_0)^2(p^2+r_0)^2} , \tag{4.14}$$

which even at $\omega=0$ represents a genuine dynamic quantity, in contrast to the purely static quantities $\mathring{\gamma}_p^2 \mathring{\Sigma}(t,0)$ or $\mathring{\gamma}^2 \mathring{F}_+(t,0)$.

Substitution of (4.10)–(4.13) into (3.18) and (3.19) yields

$$y(t,\omega) = c_1^{(0)^2}[1 - \mathring{\gamma}_p^2 \mathring{\chi}_p \mathring{\Sigma}(t,\omega)] = c_1^{(0)^2}\left[1 - \frac{\mathring{\gamma}_p^2 \mathring{\chi}_p}{\mathring{\gamma}^2}\left[1 - \frac{1}{1+\mathring{\gamma}^2 \mathring{F}_+(t,\omega)}\right]\right] \tag{4.15}$$

and

$$z_1(t,\omega) = D_1^{(0)} + \Delta z_1(t,\omega) \tag{4.16}$$

with $D_1^{(0)}$ and $c_1^{(0)}$ given by (3.14) and (3.15), and

$$\Delta z_1(t,\omega)=[\mathring{\lambda}_p\mathring{\chi}_p^{-1}-(\mathring{\lambda}_p\mathring{\chi}_p^2+L_0\mathring{\gamma}_p\mathring{\gamma}_m)\mathring{\Sigma}(t,\omega)]\times[1+\mathring{g}_p^2\mathring{\lambda}_p^{-1}\mathring{P}(t,\omega)]-[L_0\mathring{\chi}_p^{-1}-(L_0\mathring{\gamma}_p^2+\mathring{\lambda}_m\mathring{\gamma}_p\mathring{\gamma}_m)\mathring{\Sigma}(t,\omega)]\times[1+\mathring{g}_p\mathring{g}_m L_0^{-1}\mathring{P}(t,\omega)] \,. \quad (4.17)$$

From (3.23) and (3.24) we obtain the result for the sound velocity and damping in one-loop order

$$c_1(t,\omega)^2=c_1^{(0)^2}\{1-\mathring{\gamma}_p^2\mathring{\chi}_p\mathrm{Re}[\mathring{\Sigma}(t,\omega)]\}+\omega\,\mathrm{Im}[\Delta z_1(t,\omega)] \quad (4.18)$$

and

$$D_1(t,\omega)=D_1^{(0)}+c_1^{(0)^2}\mathring{\gamma}_p^2\mathring{\chi}_p\mathrm{Im}[\mathring{\Sigma}(t,\omega)/\omega]+\mathrm{Re}[\Delta z_1(t,\omega)] \,. \quad (4.19)$$

An analogous expression for the thermal diffusivity D_T can be easily derived from (3.23) and (4.5).

The results (4.15)–(4.17) and (4.8) have been presented already in Ref. 2, where (the renormalized counterparts of) $\mathring{\Sigma}(t,\omega)$ and $\mathring{F}_+(t,\omega)$ have been denoted by $\Sigma(t,\omega)$ and $G_+(t,\omega)$, respectively. An equivalent formulation is given by the dispersion relation (3.21) which after substitution of (4.15) reads

$$\omega^2/k^2=c_1^{(0)^2}[1-\mathring{\gamma}_p^2\mathring{\chi}_p\mathring{\Sigma}(t,\omega)]-i\omega z_1(t,\omega) \,. \quad (4.20)$$

If we drop z_1 and use (4.8), this result is obviously identical in structure with (19a) and (20) of Ref. 47.

B. Two-loop result

The perturbation contributions to $\mathring{\Gamma}_{\alpha\beta}(k,\omega)$ start to depend explicitly on c_0 in two-loop order via internal propagators as discussed in Appendix B. We shall confine ourselves to the limit $c_0\to\infty$, which turns out to be well behaved, as expected from the general discussion in Sec. IV C of I. The corresponding integral expressions for $\mathring{\Gamma}_{\alpha\beta}(k,\omega)$ are given in (B12)–(B16). For the calculation of C_1, D_1, and D_T we need these results only to $O(k^2)$, which can be obtained by a straightforward (but lengthy) expansion around $k=0$.

In the following we focus upon c_1 and D_1. As noted in Ref. 2 and as will be shown in quantitative detail in the subsequent part of this work, the contribution of the function $y(t,\omega)$, (3.18), is by several orders of magnitude more important than that of $z_1(t,\omega)$, (3.19), in the t and ω range of interest. We shall therefore restrict the presentation of explicit two-loop results only to the vertex functions $\mathring{\Gamma}_{p\tilde{j}}(k,\omega)$ [for $\mathring{\Gamma}_{j\tilde{p}}(k,\omega)$ see the exact result (3.8)].

The perturbative expression of $\mathring{\Gamma}_{p\tilde{j}}(k,\omega)$ to $O(k)$ is given in (B21). Again we invoke the exact result (3.27) and argue along the lines of Sec. IV A earlier that a rearrangement also of the two-loop terms is necessary. Unlike the one-loop approximation, the two-loop approximation [with contributions up to $O(\mathring{\gamma}_p^4)$] provides the possibility of a nontrivial check for the consistency of this rearrangement *at finite* ω. Our results indeed verify this point explicitly for $\mathring{\Gamma}_{p\tilde{j}}(k,\omega)$ up to two-loop order. The important consequence is that the structure of $\mathring{\Gamma}_{p\tilde{j}}$, as given by (4.10) and (4.8), remains unaltered and that the two-loop contributions can be absorbed completely in the single function $\mathring{F}_+(t,\omega)$. The two-loop result for $\mathring{F}_+(t,\omega)$ reads

$$\mathring{F}_+(t,\omega)=4\int_{p_1}\frac{1}{\pi_1(\pi_1-i\Omega_0)}-64u_0\left[\int_{p_1}\frac{1}{\pi_1(\pi_1-i\Omega_0)}\right]^2-64u_0\int_{p_1}\int_{p_2}\frac{1}{\pi_1\pi_2(\pi_1-i\Omega_0)}\left[\frac{1}{\pi_1}+\frac{1}{\pi_1-i\Omega_0}\right]+8\int_{p_1}\int_{p_2}\frac{1}{(\pi_2-i\Omega_0)}[\mathring{Q}(\mathbf{p}_1,\mathbf{p}_2,i\Omega_0)+\text{c.c.}] \,, \quad (4.21)$$

where $\pi_1\equiv p_1^2+r_0$, $\pi_2\equiv p_2^2+r_0$. The function $\mathring{Q}$ is given by

$$\mathring{Q}(\mathbf{p}_1,\mathbf{p}_2,i\Omega_0)=\frac{v_0^*}{w_0'N(\mathbf{p}_1,\mathbf{p}_2,i\Omega_0)}\left[\frac{v_0^*}{\pi_1(\pi_2-i\Omega_0)}+\frac{v_0}{\pi_2(\pi_1-i\Omega_0)}-\frac{2\mathring{\gamma}_m w_0'}{\pi_1\pi_2}\right] \,, \quad (4.22)$$

where

$$N(\mathbf{p}_1,\mathbf{p}_2,i\Omega_0)=w_0^*\pi_1+w_0\pi_2+(\mathbf{p}_1+\mathbf{p}_2)^2-2i\Omega_0 w_0' \quad (4.23)$$

and

$$v_0=\mathring{\gamma}_m w_0-\frac{i}{2}\mathring{g}_m\mathring{\lambda}_m^{-1},\quad v_0^*=\mathring{\gamma}_m w_0^*+\frac{i}{2}\mathring{g}_m\mathring{\lambda}_m^{-1} \,, \quad (4.24)$$

$$w_0=\frac{\Gamma_0\mathring{\chi}_m}{\mathring{\lambda}_m}=w_0'+iw_0'',\quad w_0^*=w_0'-iw_0'' \,. \quad (4.25)$$

In the last term of (4.21), + c.c. means that the same function $\mathring{Q}$ must be added with v_0,v_0^*,w_0,w_0^*, replaced by v_0^*,v_0,w_0^*,w_0, respectively, but with the sign of $i\Omega_0$ left unaltered.

The first and the third term on the right-hand side of (4.21) can be combined as

$$4\int_{p_1}\frac{1}{\bar{\pi}_1(\bar{\pi}_1-i\Omega_0)}+O(u_0^2) \quad (4.26)$$

with

$$\tilde{\pi}_1 = p_1^2 + r_0 + 16u_0 \int_{p_2} \frac{1}{p_2^2 + r_0} . \tag{4.27}$$

Thus the third term in (4.21) can be considered as being a perturbation contribution to r_0 of purely static origin. In the renormalized version of the theory the last term in (4.27) corresponds to the $O(u)$ correction to ξ^{-2}, where ξ is the correlation length. An extension of this interpretation beyond two-loop order essentially corresponds to replacing the upper order-parameter propagator in diagram (1) in Fig. 1 of Ref. 34 by a "dressed" propagator, in analogy to Fig. 1 of Nelson.[48] This point will be taken up in the subsequent part of this work[21] and will lead to an improved treatment of the RG flow parameter $l(t,\omega)$ determined previously by Eq. (17) of Ref. 2 [where $r(l)$ should be replaced by ξ^{-2}].

Because of $v_0 + v_0^* = 2\mathring{\gamma}_m w_0'$ we have

$$\mathring{Q}(\mathbf{p}_1, \mathbf{p}_2, 0) = 0 , \tag{4.28}$$

therefore at $\omega=0$ (4.21) is reduced to the static specific-heat function

$$\mathring{F}_+(t,0) = 4J_2(r_0) - 64u_0[2J_1(r_0)J_3(r_0) + J_2(r_0)^2] \tag{4.29}$$

with

$$J_n(r_0) = \int \frac{1}{(p^2+r_0)^n} . \tag{4.30}$$

Equation (4.29) agrees with the two-loop expression for the specific-heat function $\mathring{C}_\psi(k=0)$ as given in (A10) of Ref. 29, again in accord with the exact relation (3.27).

In summary, in terms of $\mathring{F}_+(t,\omega)$ given by (4.21)–(4.25), we obtain the (square of the) sound velocity and the damping coefficient as

$$c_1(t,\omega)^2 = A_0^2 + B_0 \mathrm{Re} \frac{1}{1+\mathring{\gamma}^2 \mathring{F}_+(t,\omega)} + \omega \, \mathrm{Im}[\Delta z_1(t,\omega)] \tag{4.31}$$

and

$$D_1(t,\omega) = D_1^{(0)} - \frac{B_0}{\omega} \mathrm{Im} \frac{1}{1+\mathring{\gamma}^2 \mathring{F}_+(t,\omega)} + \mathrm{Re}[\Delta z_1(t,\omega)] , \tag{4.32}$$

where

$$A_0^2 = c_1^{(0)^2}(1 - \mathring{\gamma}_p^2 \mathring{\chi}_p / \mathring{\gamma}^2) \tag{4.33}$$

and

$$B_0 = c_1^{(0)^2} \mathring{\gamma}_p^2 \mathring{\chi}_p / \mathring{\gamma}^2 . \tag{4.34}$$

Instead of D_1 we shall also consider the attenuation coefficient

$$\alpha(t,\omega) = \frac{\omega^2}{2c_1(t,\omega)^3} D_1(t,\omega) \tag{4.35}$$

and its critical contribution

$$\alpha^c(t,\omega) = \frac{\omega^2}{2c_1(t,\omega)^3}[D_1(t,\omega) - D_1^{(0)}] . \tag{4.36}$$

We conjecture that, for $k \to 0$, the structure of the velocity dispersion and of the attenuation, (4.31) and (4.32), remains valid in all orders of perturbation theory, i.e., we expect that the higher-loop contributions to the frequency dependence can be absorbed completely in the two functions $\mathring{F}_+(t,\omega)$ and $\mathring{P}(t,\omega)$. Note, again, that a more complicated structure arises at finite k as seen already within the one-loop approximation, [see (4.14)].

C. Zero-frequency limit

It is worthwhile to summarize our results explicitly in the zero-frequency limit. This limit is of conceptual interest as one expects on general grounds[14] that thermodynamic-hydrodynamic relations should remain valid for $\omega \to 0$ even near criticality.[49] From (4.31), I (3.14), I (3.17), and (2.11) we obtain

$$c_1(t,0)^2 = A_0^2 + B_0[1 + \mathring{\gamma}^2 \mathring{C}_\psi]^{-1} \tag{4.37}$$

$$= \frac{c_0^2}{\mathring{\chi}_j} \frac{\mathring{C}_{mm}}{\mathring{C}_{pp}\mathring{C}_{mm} - \mathring{C}_{mp}^2} \tag{4.38}$$

$$= \left[\frac{\partial P}{\partial \langle \rho_0 \rangle} \right]_{\langle \sigma_0 \rangle} \equiv \langle \rho_0 \rangle^{-1} \kappa_\sigma^{-1} . \tag{4.39}$$

Thus we have established the usual relation[38] [compare (3.15)] between the sound velocity c_1 and the adiabatic compressibility κ_σ within our complete stochastic model—including the critical temperature dependence of these quantities (for $\omega \to 0$, $k \to 0$, $c_0 \to \infty$). Quantitative results for c_1 near the λ line are planned to be given in paper III of this work.

The zero-frequency expression for the sound damping D_1 is obtained from (4.32) as

$$D_1(t,0) = D_1^{(0)} + \Delta z_1(t,0) - \frac{c_1^{(0)^2} \mathring{\gamma}_p^2 \mathring{\chi}_p}{(1+\mathring{\gamma}^2 \mathring{C}_\psi)^2} \frac{\partial F_+(t,\omega)}{2\Gamma_0' \partial(i\Omega_0)} \tag{4.40}$$

with the derivative taken at $\omega=0$. A discussion of the critical temperature dependence of the last two terms in (4.40) will be deferred to a planned paper III of this work.

Here we only note that the last term $\sim \mathring{\gamma}_p^2/\Gamma_0'$, although small, does not vanish in the noncritical region well above T_λ where it can be absorbed in a redefinition of the background value ζ_0 of the bulk viscosity in Eq. (3.14). (There is no contribution to the shear viscosity within our model.) This term arises from the existence of a coupling $\mathring{\gamma}_p$ between the fluctuations of the order parameter and of the pressure. Within ordinary hydrodynamics above T_λ (where the order-parameter fluctuations are not included explicitly in the hydrodynamic equations) the last term in (4.40) can be interpreted as a relaxational contribution to the bulk viscosity of a fluid with internal degrees of freedom.[25,38] In the present case the order-parameter fluctuations represent such internal degrees of freedom.[26,50] Thus, in the zero-frequency limit, the last term in (4.40) is in accord with the general hydrodynamic relation [see Sec. 77 of Ref. 38; compare Eq. (3.14)] between D_1 and the viscosities—even close to T_λ, where this term develops a critical temperature dependence. For realistic finite frequencies the dominant critical contribution to D_1 emerges just from this term rather than from Δz_1 or from other critical contributions to the viscosities.

D. Comparison with Ferrell and Bhattacharjee

In the following we compare the form of (4.31) and (4.32) with the result of the phenomenological approach by Ferrell and Bhattacharjee (FB).[22–24] First we note that FB neglect the small contribution Δz_1 which is related to the thermal conductivity. According to Ref. 22 and to (2.1) of Ref. 24 FB describe the critical part (4.36) of the attenuation per wavelength, $\alpha^c c_1/\omega$, in terms of a frequency-dependent specific heat $C(t,\omega)_{\rm FB}$,

$$\alpha^c c_1/\omega = -c_T \,{\rm Im}\, C(t,\omega)_{\rm FB}^{-1} , \tag{4.41}$$

with c_T being a thermodynamic quantity. So far FB have specified $C(t,\omega)_{\rm FB}$ explicitly, in case of ^{4}He, only by the one-loop expression [see (2.8) of Ref. 24]

$$C(t,\omega)_{\rm FB} = \hat{c} \int_p \frac{1}{(p^2+\xi^{-2})(p^2+\xi^{-2}-i\omega/2B_\psi)} + B \tag{4.42}$$

supplemented by a background term B [see (12) of Ref. 22], where ξ is the correlation length and $\hat{c}$ some constant. Equations (4.41) and (4.42) are to be compared with (4.32) and (4.3). Using the obvious correspondence $r_0 \hat{=} \xi^{-2}$ and $\Gamma_0' \hat{=} B_\psi$ we see that our theory confirms explicitly the starting point of the FB approach in one-loop order for $k \to 0$. The correspondence between the two theories is

$$C(t,\omega)_{\rm FB} \hat{=} {\rm const} \times [1+\mathring{\gamma}^2 F_+(t,\omega)] . \tag{4.43}$$

We have shown by explicit computation in Sec. IV B earlier that the structure of the right-hand side (rhs) of (4.43) remains valid at least up to two-loop order. It is probably valid to all orders. Thus our theory provides a general statistical definition and computational prescription for $C(t,\omega)_{\rm FB}$ in terms of the vertex function $\mathring{\Gamma}_{p\tilde{j}}(t,\omega)$ in the framework of our complete stochastic model according to (4.43), (4.8), and (4.10), i.e.,

$$F_+(t,\omega) = \frac{\mathring{\Sigma}(t,\omega)}{1-\mathring{\gamma}^2 \mathring{\Sigma}(t,\omega)} \tag{4.44}$$

with

$$\mathring{\Sigma}(t,\omega) = \lim_{k\to 0} \lim_{c_0\to\infty} \left[\frac{1}{\mathring{\gamma}_p^2 \mathring{\chi}_p} \left[1 - \frac{\mathring{\Gamma}_{p\tilde{j}}(k,\omega)}{i c_0 \mathring{\chi}_p^{-1} k} \right] \right] . \tag{4.45}$$

The physical interpretation of $C(t,\omega)_{\rm FB}$ as a frequency-dependent specific heat is justified by the fact that for $\omega \to 0$ we have $\mathring{F}_+(t,0) = \mathring{C}_\psi$ as an exact relation [for $\mathring{C}_\psi$ see (2.7)]. We note, however, that the characterization of $C(t,\omega)_{\rm FB}$ as a frequency-dependent generalization of the *constant-pressure* specific heat C_p (Refs. 22 and 24) is inaccurate. As seen from (2.7)–(2.10) this interpretation would be correct only if $\mathring{\gamma}_0^2$ in (4.13) were replaced by $\mathring{\gamma}_m^2 \mathring{\chi}_m$. The difference $\sim \mathring{\gamma}_p^2 \mathring{\chi}_p \mathring{F}_+(t,\omega)$ yields a small but finite deviation from the FB interpretation.

An alternative representation of the rhs of (4.43) can be given in terms of the response function $\mathring{R}_\psi(k,\omega)$ defined in I (4.27). We start from the general form I (4.3) and I (4.7) for the matrix of two-point correlation functions. Inverting this relation and comparing with the expression I (4.16) for the two-point vertex functions we obtain the relation

$$\underline{\mathring{\Sigma}}(k,\omega) = -\underline{\mathring{M}}(k,\omega) \underline{\mathring{\Pi}}(k,\omega) \underline{\mathring{M}}(k,\omega) . \tag{4.46}$$

For the present purpose we are interested only in the matrix element $\mathring{\Sigma}_{p\tilde{j}}$ which constitutes the perturbation part of $\mathring{\Gamma}_{p\tilde{j}}$. From (4.10) we see that, for $k \to 0$, $\mathring{\Sigma}_{p\tilde{j}}$ does not have external vertices proportional to the dynamic couplings $\mathring{g}_\alpha$. For this reason and for simplicity we drop those dynamic couplings in $\underline{\mathring{\Pi}}^{-1}$ that enter through the last terms of the composite fields $\mathring{s}_\mu$ and $\tilde{\mathring{s}}_\mu$, I (4.9)–(4.12).[51] In the limit $c_0 \to \infty$ and $k \to 0$ we then obtain from (4.46) an expression for $\mathring{\Sigma}_{p\tilde{j}}$ in terms of $\mathring{R}_\psi$, which, according to (4.10), can be expressed as a relation between $\mathring{\Sigma}(k,\omega)$ and $\mathring{R}_\psi(k,\omega)$. This relation reads

$$\mathring{\Sigma}(t,\omega) = \lim_{k\to 0} \frac{\mathring{R}_\psi(k,\omega)}{1+\Lambda(k,\omega)\mathring{R}_\psi(k,\omega)} \tag{4.47}$$

with

$$\Lambda(k,\omega) = \mathring{\gamma}_p^2 \mathring{\chi}_p + \frac{\mathring{\gamma}_m^2 \mathring{\lambda}_m k^2}{-i\omega + \mathring{\lambda}_m \mathring{\chi}_m^{-1} k^2} . \tag{4.48}$$

After substitution into (4.44) and (4.43) this leads to the correspondence

$$C(T,\omega)_{\rm FB} \hat{=} {\rm const} \times \lim_{k\to 0} \lim_{c_0\to\infty} \left[1 + \mathring{\gamma}^2 \frac{\mathring{R}_\psi(k,\omega)}{1+\overline{\Lambda}(k,\omega)\mathring{R}_\psi(k,\omega)} \right] \tag{4.49}$$

with

$$\overline{\Lambda}(k,\omega) = \frac{-i\omega \mathring{\gamma}_m^2 \mathring{\chi}_m}{-i\omega + \mathring{\lambda}_m \mathring{\chi}_m^{-1} k^2} . \tag{4.50}$$

For $\omega = 0$ we recover from (4.49) and I (4.31) the static relation $C_{\rm FB} \hat{=} {\rm const}(1+\mathring{\gamma}^2 \mathring{C}_\psi)$, as expected. For $\omega \neq 0$ the result (4.49) differs from that in the simpler case of the liquid-gas critical point where, up to a constant, $C_{\rm FB} - B \hat{=} \mathring{R}_\psi$.[47] This difference originates from the different role played by the order parameter. In ^{4}He it is a separate quantity coupled to the heat and sound modes by two static couplings $\mathring{\gamma}_m$ and $\mathring{\gamma}_p$, whereas in ordinary liquid-gas systems there exists only one static coupling between the heat variable (order parameter) and the pressure fluctuations.

We note that at finite k the possibility of interpreting the critical attenuation in terms of a frequency-dependent specific heat is not at all obvious. As indicated by the existence of nonstatic contributions even at $\omega=0$ [see (4.14)] a more general examination of the dynamic structure factor at finite k seems necessary in order to properly define k-dependent corrections to D_1 and c_1.

We conclude this section with some general comments. We consider the phenomenological dynamic-scaling approach by FB based on the idea of a frequency-dependent specific heat[23–26,50] as an important achievement which provides considerable insight into the leading physical mechanism of critical sound attenuation and dispersion. On the other hand, we find definite disadvantages in this approach as far as the actual computation of the critical t and ω dependence is concerned. We mention the following points.

(i) Critical statics: Our theory of first sound is devised such that the complete previous knowledge on static properties[27–30] can be readily exploited. This includes the results of the Borel resummation method applied to the nonasymptotic region[30] which is particularly relevant at higher pressure.[27] The static treatment by FB (Ref. 23) is restricted only to a qualitative or empirical description.

(ii) Critical dynamics: The dynamic RG theory provides a proper treatment of both static (dissipative) and dynamic (reversible) couplings. This is not possible within a mode-coupling approach of the type used previously by FB.[52] A more definite qualification does not seem possible at the present time since in none of the previous publications have FB specified the underlying equations of motion for sound propagation in ^{4}He.

(iii) Explicit results: We have presented the two-loop expressions for the sound velocity and damping in a framework which permits the incorporation of the two-loop model-F flow equations[34,37] (see also Sec. V following.) Information of equivalent accuracy is not available in the FB approach.

(iv) Corrections due to finite k and c_0: Our complete stochastic model provides the possibility of calculating corrections due to the finite values of k and c_0 in real sound propagation.

A quantitative evaluation of the two-loop integrals in (4.21) and a comparison with FB are planned to be carried out in paper III of this work after we have presented the field-theoretic renormalization-group treatment of the critical behavior in the subsequent section.

V. STATIC AND DYNAMIC RENORMALIZATIONS

The main purpose of this section is (1) to derive algebraic relations between the *static* renormalizations of the Hamiltonian I (2.79) and those of model F,[29] and (2) to show that also the *dynamic* renormalizations of the equations of motions I (2.72)–(2.78) are related to those of model F (Ref. 34) provided that the limit $c_0 \to \infty$ has been performed.

We shall use the minimal renormalization scheme[53] as it permits to perform the static and dynamic renormalizations separately[32] and because detailed knowledge of the renormalization-group functions of model F is available within this renormalization scheme.

A. Static renormalizations

First we remind the reader of the standard renormalizations related to the Landau-Ginzburg-Wilson Hamiltonian H_ψ, I (3.4). The multiplicative renormalizations are introduced as

$$\psi = Z_\psi^{-1/2}\psi_0 , \tag{5.1}$$

$$r = Z_r^{-1}(r_0 - r_{0c}) , \tag{5.2}$$

$$u = \mu^{-1} Z_u^{-1} Z_\psi^2 A_d u_0 , \tag{5.3}$$

with $\epsilon = 4-d$. μ is a reference wave number, and A_d a convenient geometric factor.[29] Furthermore an additive renormalization function $A(u)$ is necessary for the renormalization of $\mathring{C}_\psi$, I (3.3),

$$C_\psi = \mu^\epsilon A_d^{-1} Z_r^2 \mathring{C}_\psi + A(u) , \tag{5.4}$$

where $A(u)$ contains only pole terms and C_ψ is finite for $\epsilon \to 0$. Next we introduce the renormalizations of the complete Hamiltonian I (2.79). (The purely Gaussian field $j_0(x)$ will be omitted in this subsection.) We shall verify that an appropriate introduction of renormalized fields m and p is provided by the relation

$$\begin{pmatrix} m(x) \\ p(x) \end{pmatrix} = \begin{pmatrix} Z_m^{-1/2} & A_m \\ A_p & Z_p^{-1/2} \end{pmatrix} \begin{pmatrix} m_0(x)\mathring{\chi}_m^{-1/2} \\ p_0(x)\mathring{\chi}_p^{-1/2} \end{pmatrix} , \tag{5.5}$$

where we require

$$Z_m^{(0)} = Z_p^{(0)} = 1, \quad A_m^{(0)} = A_p^{(0)} = 0 \tag{5.6}$$

in the lowest order of renormalized perturbation theory. Further requirements will be imposed upon A_m and A_p below. Substituting (5.5) and (5.4) into I (3.1) and I (3.2) yields the renormalized correlation functions

$$C_{\alpha\beta}(k) = \int_V d^d x e^{-ikx} [\langle \alpha(x)\beta(0)\rangle - \langle\alpha\rangle\langle\beta\rangle] \tag{5.7}$$

in the form

$$C_{mm} = Z_m^{-1} + A_m^2 + \gamma_m^2 (C_\psi - A) , \tag{5.8}$$

$$C_{pp} = Z_p^{-1} + A_p^2 + \gamma_p^2 (C_\psi - A) , \tag{5.9}$$

$$C_{mp} = A_m Z_p^{-1/2} + A_p Z_m^{-1/2} + \gamma_m \gamma_p (C_\psi - A) , \tag{5.10}$$

provided that we define the renormalized couplings by

$$\begin{pmatrix} \gamma_m \\ \gamma_p \end{pmatrix} = \mu^{-\epsilon/2} A_d^{1/2} Z_r^{-1} \begin{pmatrix} Z_m^{-1/2} & A_m \\ A_p & Z_p^{-1/2} \end{pmatrix} \begin{pmatrix} \mathring{\gamma}_m \mathring{\chi}_m^{1/2} \\ \mathring{\gamma}_p \mathring{\chi}_p^{1/2} \end{pmatrix} . \tag{5.11}$$

A cancellation of all pole terms in (5.8), (5.9), and (5.10) is obviously possible by requiring

$$Z_m^{-1} + A_m^2 = 1 + \gamma_m^2 A(u) , \tag{5.12}$$

$$Z_p^{-1}+A_p^2=1+\gamma_p^2 A(u)\ , \qquad (5.13)$$

$$A_m Z_p^{-1/2}+A_p Z_m^{-1/2}=\gamma_m\gamma_p A(u)\ . \qquad (5.14)$$

This implies

$$C_{mm}=1+\gamma_m^2 C_\psi\ , \qquad (5.15)$$

$$C_{pp}=1+\gamma_p^2 C_\psi\ , \qquad (5.16)$$

$$C_{mp}=\gamma_m\gamma_p C_\psi\ . \qquad (5.17)$$

Finally we introduce the renormalizations related to the complete Hamiltonian H, I (2.79), in the simplified representation I (3.11). We shall need only

$$q=Z_q^{-1/2}q_0\ , \qquad (5.18)$$

$$C_{qq}=Z_q^{-1}\mathring{C}_{qq}\ , \qquad (5.19)$$

$$\gamma=\mu^{-\epsilon/2}Z_\gamma^{-1}Z_\psi Z_q^{1/2}A_d^{1/2}\mathring{\gamma}\ , \qquad (5.20)$$

where[29]

$$Z_q^{-1}=1+\gamma^2 A(u)\ , \qquad (5.21)$$

$$Z_\gamma=Z_q Z_r Z_\psi\ . \qquad (5.22)$$

From I (3.6), (5.11), and (5.12) we find

$$\gamma^2=\gamma_m^2+\gamma_p^2 \qquad (5.23)$$

and

$$C_{qq}=C_{mm}+C_{pp}-1=C_{mm}C_{pp}-C_{mp}^2\ , \qquad (5.24)$$

which are the renormalized versions of I (3.16) and I (3.17).

The three relations (5.12)–(5.14) do not yet determine the four renormalization factors Z_m, Z_p, A_m, A_p, uniquely [in terms of $\gamma_m, \gamma_p, A(u)$] and therefore allow for a simplifying choice of A_m or A_p. We shall make the symmetric choice

$$A_m=A_p\ . \qquad (5.25)$$

As an important consequence we find from (5.11)–(5.14)

$$y_0=\frac{\mathring{\gamma}_p\mathring{\chi}_p^{1/2}}{\mathring{\gamma}_m\mathring{\chi}_m^{1/2}}=\frac{\gamma_p}{\gamma_m}=y\ , \qquad (5.26)$$

thus no renormalization factor needs to be introduced for this dimensionless ratio of bare parameters. The same property follows for the orthogonal matrix I (3.9)

$$R(y_0)=R(y)=(1+y^2)^{-1/2}\begin{bmatrix}1 & y\\ -y & 1\end{bmatrix}\ , \qquad (5.27)$$

which describes the relation between $C_{\alpha\beta}$ and C_{qq} according to

$$\begin{bmatrix}C_{mm} & C_{mp}\\ C_{pm} & C_{pp}\end{bmatrix}=R^T\begin{bmatrix}C_{qq} & 0\\ 0 & 1\end{bmatrix}R \qquad (5.28)$$

with

$$C_{qq}=1+\gamma^2 C_\psi\ . \qquad (5.29)$$

The representation (5.28) of the renormalized correlation functions $C_{\alpha\beta}$ provides a basic simplification in that the algebraic dependence on y described by $R^T(y)$ and $R(y)$ *remains independent of temperature.* The critical temperature dependence is entirely contained in C_{qq} or C_ψ and can be taken from Ref. 29 as will be summarized in the following.

C_{qq} has the structure

$$C_{qq}(u,\gamma,r,\mu)=1+\gamma^2F_+[u,r/\mu^2]\ , \qquad (5.30)$$

and satisfies the RG equation

$$(\mu\partial_\mu+\beta_u\partial_u+\beta_\gamma\partial_\gamma+\zeta_r r\partial_r-\zeta_q)C_{qq}=0\ . \qquad (5.31)$$

The structure of the RG functions is

$$\beta_u(u,\epsilon)=(\mu\partial_\mu u)_0=-\epsilon u+\tilde{\beta}(u)\ , \qquad (5.32)$$

$$\zeta_r(u)=(\mu\partial_\mu \ln Z_r^{-1})_0\ , \qquad (5.33)$$

$$\zeta_q(u,\gamma^2)=(\mu\partial_\mu \ln Z_q^{-1})_0=4\gamma^2 B(u)\ , \qquad (5.34)$$

$$\beta_\gamma(\gamma,u,\epsilon)=(\mu\partial_\mu\gamma)_0=\tfrac{1}{2}\gamma(-\epsilon+2\zeta_r+\zeta_q)\ . \qquad (5.35)$$

These functions are now known to very good accuracy,[30] and the function F_+ is known in two-loop order.[29] The formal solution of the RG equation reads

$$C_{qq}=\{1+\gamma(l)^2F_+[u(l),r(l)/\mu^2l^2]\}\exp\int_l^1\zeta_q\frac{dl'}{l'}\ , \qquad (5.36)$$

where the effective parameters are determined by

$$l\frac{du(l)}{dl}=\beta_u,\quad l\frac{d\gamma(l)}{dl}=\beta_\gamma \qquad (5.37)$$

and

$$r(l)=r(1)\exp\int_1^l\zeta_r\frac{dl'}{l'}\ , \qquad (5.38)$$

with nonuniversal initial values $u(1)$, $\gamma(1)$, and

$$r(1)\equiv r=a(P)t+O(t^2)\ .$$

In dynamics we shall also need the effective static couplings $\gamma_m(l)$ and $\gamma_p(l)$ separately. Instead of introducing the corresponding RG flow equations we use (5.23) and (5.26) to obtain

$$\begin{aligned}\gamma_m(l)&=(1+y^2)^{-1/2}\gamma(l)\ ,\\ \gamma_p(l)&=(1+y^{-2})^{-1/2}\gamma(l)\ ,\end{aligned} \qquad (5.39)$$

with $\gamma(l)$ determined by (5.37). For completeness we also define the renormalized parameters

$$\hat{\tau}=\hat{Z}_\tau^{-1}\hat{\tau}_0,\quad \hat{u}=\mu^{-\epsilon}\hat{Z}_u^{-1}Z_\psi^2A_d\hat{u}_0\ . \qquad (5.40)$$

An application of the results of this subsection to ^{3}He-^{4}He mixtures and an extension to more than two secondary variables is straightforward. For a somewhat different treatment, restricted to asymptotic static critical properties, see also Onuki.[54]

B. Dynamic renormalizations

The dynamic renormalizations are most easily performed after an orthogonal transformation of the dynamic functional I (A1)–(A8) analogous to that of the Hamiltonian in Sec. III of (I). We include $\mathbf{j}_0(x,t)$ and $\tilde{\mathbf{j}}_0(x,t)$ in this transformation since they are dynamically coupled to $p_0(x,t)$ and $\tilde{p}_0(x,t)$ via the matrix of transport coefficients I (4.6) [see also (5.47) later]. Thus we use (5.27) as a submatrix of the orthogonal 3×3 matrix

$$[R(y_0)]=[R(y)]=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}, \qquad (5.41)$$

where

$$\cos\varphi=(1+y_0^2)^{-1/2},\quad \sin\varphi=y_0(1+y_0^2)^{-1/2}, \qquad (5.42)$$

with $y_0=y$ given by (5.26). In the notation of Appendix A of (I) we introduce the transformed fields

$$[R]\boldsymbol{\alpha}_0(x,t)=\boldsymbol{\alpha}'(x,t)\equiv\begin{bmatrix}q_0(x,t)\\ n_0(x,t)\\ \mathbf{j}_0(x,t)\mathring{\chi}_j^{-1/2}\end{bmatrix}, \qquad (5.43)$$

$$[R]\tilde{\boldsymbol{\alpha}}_0(x,t)=\tilde{\boldsymbol{\alpha}}'(x,t)\begin{bmatrix}\tilde{q}_0(x,t)\\ \tilde{n}_0(x,t)\\ \tilde{\mathbf{j}}_0(x,t)\mathring{\chi}_j^{1/2}\end{bmatrix}, \qquad (5.44)$$

the transformed couplings

$$[R]\mathring{\gamma}=\mathring{\gamma}'\equiv\begin{bmatrix}\mathring{\gamma}\\ 0\\ 0\end{bmatrix}, \qquad (5.45)$$

$$[R]\mathring{\mathbf{g}}=\mathring{\mathbf{g}}'\equiv\begin{bmatrix}\mathring{g}_q\\ \mathring{g}_n\\ 0\end{bmatrix}, \qquad (5.46)$$

and the matrix of transformed transport coefficients

$$[R][\mathring{L}(k)][R]^{-1}=[\mathring{L}'(k)]\equiv\begin{bmatrix}\mathring{L}_{qq} & \mathring{L}_{qn} & i\mathring{\mathcal{E}}_q k\\ \mathring{L}_{nq} & \mathring{L}_{nn} & i\mathring{\mathcal{E}}_n k\\ i\mathring{\mathcal{E}}_q k & i\mathring{\mathcal{E}}_n k & \mathring{L}_{jj}\end{bmatrix}. \qquad (5.47)$$

Correspondingly the matrix of vertex functions (3.6) is transformed as

$$[R][\mathring{\Gamma}(k,\omega)][R]^{-1}=[\mathring{\Gamma}'(k,\omega)]. \qquad (5.48)$$

Due to the conservation property of the variables m_0, p_0, and $\mathbf{j}_0$, all perturbation contributions to (5.48) vanish for $k\to 0$. Therefore we have simply

$$[\mathring{\Gamma}(0,\omega)]=[\mathring{\Gamma}'(0,\omega)]=-i\omega[1]. \qquad (5.49)$$

Within statics, the basic simplification achieved by this transformation consists in the decoupling of q_0, n_0, and $\mathbf{j}_0$, which implies that the renormalized fields q, n, and $\mathbf{j}$ can be introduced by separate multiplicative renormalizations (rather than via a matrix of Z factors), see (5.18). This has the following simplifying consequences for the dynamic renormalizations:

(i) Since (5.49) has no pole terms, the matrix of $\tilde{Z}$ factors that renormalize $\tilde{q}_0$, $\tilde{n}_0$, and $\tilde{\mathbf{j}}_0$ must be the inverse of the corresponding matrix of Z factors that renormalize q_0, n_0, and $\mathbf{j}_0$. Since the latter matrix is diagonal, the former matrix is diagonal as well. Hence we may introduce the separately renormalized response fields

$$\tilde{q}=\tilde{Z}_q^{-1/2}\tilde{q}_0,\quad \tilde{n}=\tilde{Z}_n^{-1/2}\tilde{n}_0,\quad \tilde{\mathbf{j}}=\tilde{Z}_j^{-1/2}\tilde{\mathbf{j}}_0\mathring{\chi}_j^{1/2}, \qquad (5.50)$$

with

$$\tilde{Z}_q=Z_q^{-1},\quad \tilde{Z}_n=Z_n^{-1}=1,\quad \tilde{Z}_j=Z_j^{-1}=1. \qquad (5.51)$$

This implies

$$[\mathring{\Gamma}'(k,\omega)]=\begin{bmatrix}\Gamma_{q\tilde{q}} & \Gamma_{q\tilde{n}}Z_q^{-1/2} & \Gamma_{q\tilde{j}}Z_q^{-1/2}\\ \Gamma_{n\tilde{q}}Z_q^{1/2} & \Gamma_{n\tilde{n}} & \Gamma_{n\tilde{j}}\\ \Gamma_{j\tilde{q}}Z_q^{1/2} & \Gamma_{j\tilde{n}} & \Gamma_{j\tilde{j}}\end{bmatrix}, \qquad (5.52)$$

where the renormalized vertex functions $\Gamma_{\alpha\tilde{\beta}}(k,\omega)$ on the right-hand side of (5.52) are considered as functions of the renormalized static and dynamic parameters (the latter will be defined later).

(ii) As a consequence of a Ward identity[32] the renormalizations of $\mathring{g}_q$ and $\mathring{g}_n$ are of purely static nature. Because of $\mathring{\gamma}_n\equiv 0$ [see (5.45)], separate multiplicative renormalizations

$$g_p=\mu^{-\epsilon/2}A_d^{1/2}Z_{g_q}^{-1}\mathring{g}_q \qquad (5.53)$$

and

$$g_n=\mu^{-\epsilon/2}A_d^{1/2}Z_{g_n}^{-1}\mathring{g}_n \qquad (5.54)$$

are sufficient where

$$Z_{g_q}=Z_q^{1/2},\quad Z_{g_n}=Z_n^{1/2}=1. \qquad (5.55)$$

The dynamic renormalizations of the order-parameter response field $\tilde{\psi}_0(x,t)$ and of the complex kinetic coefficient Γ_0, I (2.61), are introduced as

$$\tilde{\psi}(x,t)=\tilde{Z}_\psi^{-1/2}\tilde{\psi}_0(x,t), \qquad (5.56)$$

$$\Gamma=Z_\Gamma\Gamma_0, \qquad (5.57)$$

where both $\tilde{Z}_\psi$ and Z_Γ are complex. They are determined as usual by requiring that the pole terms of the renormalized vertex function

$$\Gamma_{\psi\tilde{\psi}^*}(k,\omega)=(Z_\psi\tilde{Z}_\psi^*)^{1/2}\mathring{\Gamma}_{\psi\tilde{\psi}^*}(k,\omega) \qquad (5.58)$$

vanish. In the limit $c_0\to\infty$ this vertex function becomes identical with that of model F, as shown in Sec. IV C of I. Consequently, for $c_0\to\infty$, both $\tilde{Z}_\psi$ and Z_Γ can be calculated within model F and are known explicitly up to two-loop order.[34,37]

Finally we have to renormalize the matrix elements of (5.47). For this purpose we have computed the various vertex functions $\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)$ at $\omega=0$ and to leading order in

k. In Appendix B it is shown (up to two-loop order) that for $c_0 \to \infty$ the matrix of these vertex functions can be written as

$$[\mathring{\Gamma}'(k,0)] = \{[\mathring{L}(k)] + k^2[\mathring{g},\mathring{g}[\mathring{P}(t,0) + O(ik^3,k^4)\} \times \{[1] + [\mathring{\gamma},\mathring{\gamma}]\mathring{F}_+(t,0)\}^{-1} , \quad (5.59)$$

where $\mathring{F}_+(t,0)$ is given by (4.29). The function $\mathring{P}(t,0)$ is given in (B31) up to two-loop order and is identical with the corresponding model F function calculated previously.[37] The next step is to transform (5.59) into the matrix $[\mathring{\Gamma}'(k,0)]$ according to (5.48) and then to express the latter matrix in terms of renormalized vertex functions by means of (5.52). Thereby the bare transformed parameters must be expressed in terms of renormalized ones according to

$$[\mathring{g}',\mathring{g}'] = \mu^\epsilon A_d^{-1} \begin{bmatrix} Z_q g_q^2 & Z_q^{1/2} g_q g_n & 0 \\ Z_q^{1/2} g_n g_p & g_n^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad (5.60)$$

$$[\mathring{\gamma},\mathring{\gamma}'] = \mu^\epsilon A_d^{-1} Z_\gamma^2 Z_\psi^{-2} Z_q^{-1} \gamma^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad (5.61)$$

and

$$[\mathring{L}'(k)] = \begin{bmatrix} Z_{qq}^{-1} L_{qq} & Z_L^{-1} L_{qn} & iZ_{c_q}^{-1} c_q k \\ Z_L^{-1} L_{nq} & Z_{nn}^{-1} L_{nn} & iZ_{c_n}^{-1} c_n k \\ iZ_{c_q}^{-1} c_q k & iZ_{c_n}^{-1} c_n k & Z_{jj}^{-1} L_{jj} \end{bmatrix} . \quad (5.62)$$

with $L_{qn} = L_{nq}$. In (5.62) we have already anticipated the symmetric form of the renormalization of the symmetric bare matrix $[\mathring{L}'(k)]$ as implied by the symmetric form of (5.60) and (5.61). The requirement that all pole terms of the rhs of (5.52) are canceled determines the Z factors in (5.62) uniquely as

$$Z_{c_q} = Z_q^{-1/2}, \quad Z_{c_n} = Z_n^{-1/2} = 1 , \quad (5.63)$$

and

$$Z_{jj} = 1 , \quad (5.64)$$

$$Z_{qq}^{-1} = Z_q(1 + g_q^2 \hat{P}/L_{qq}) , \quad (5.65)$$

$$Z_L^{-1} = Z_q^{1/2}(1 + g_q g_n \hat{P}/L_{qn}) , \quad (5.66)$$

$$Z_{nn}^{-1} = 1 + g_n^2 \hat{P}/L_{nn} . \quad (5.67)$$

The dimensionless function $\hat{P}$ contains only pole terms and is identical with the corresponding renormalization function that determines Z_{λ_m} of model F. The two-loop expression for $\hat{P}$ is given in (B31).

The remaining steps are a standard application of the field-theoretic renormalization-group formalism. The various renormalized vertex functions on the right-hand side of (5.52) can be considered as a function of the renormalized parameters u, γ, y, r, g_q, g_n, c_q, c_n, Γ, $L_{\alpha\beta}$ (with $\alpha,\beta = n,q$) and of the reference wave number μ. These vertex functions satisfy the RG equation

$$\left[\mu\partial_\mu + \sum_\sigma \zeta_\sigma \sigma \partial_\sigma + \sum_i \beta_i \partial_i + \tfrac{1}{2}\zeta_\alpha + \tfrac{1}{2}\zeta_{\tilde{\beta}}\right] \Gamma_{\alpha\tilde{\beta}} = 0 \quad (5.68)$$

with $i = u$, γ and $\sigma = r$, g_q, g_n, c_q, c_n, Γ, $L_{\alpha\beta}$ and $\alpha,\beta = n,q$. The RG functions ζ_ρ, $\rho = \sigma$, α, $\tilde{\beta}$, are defined as usual

$$\zeta_\rho = (\mu\partial_\mu \ln Z_\rho^{-1})_0 , \quad (5.69)$$

where Z_ρ denotes the corresponding Z factor. The formal solution of (5.68) is

$$\Gamma_{\alpha\tilde{\beta}}(i,\sigma,\mu,k,\omega) = \Gamma_{\alpha\tilde{\beta}}(i(l),\sigma(l),\mu l,k,\omega) \exp\left[\int_1^l \tfrac{1}{2}(\zeta_\alpha + \zeta_{\tilde{\beta}}) \frac{dl'}{l'}\right] , \quad (5.70)$$

with the effective parameters

$$\sigma(l) = \sigma(1) \exp\left[\int_1^l \zeta_\sigma \frac{dl'}{l'}\right] \quad (5.71)$$

and $\sigma(1) \equiv \sigma$, where σ stands for the various parameters indicated above [for $i(l)$, see (5.37)].

An application to the sound attenuation and dispersion is straightforward. We start from the bare vertex functions in (3.18) which are transformed according to (5.48). This yields, instead of (3.18),

$$y(t,\omega) = -\frac{\partial}{\partial k^2}(\mathring{\Gamma}_{j\tilde{q}} \mathring{\Gamma}_{q\tilde{j}} + \mathring{\Gamma}_{j\tilde{n}} \mathring{\Gamma}_{n\tilde{j}})_{k=0} . \quad (5.72)$$

Next we express the right-hand side of (5.72) in terms of the renormalized vertex function $\Gamma_{\alpha\tilde{\beta}}$ according to (5.52). Because of the special combination of the vertex functions in (5.72) all Z factors drop out, thus we obtain

$$y(t,\omega) = -\frac{\partial}{\partial k^2}(\Gamma_{j\tilde{q}} \Gamma_{q\tilde{j}} + \Gamma_{j\tilde{n}} \Gamma_{n\tilde{j}})_{k=0} . \quad (5.73)$$

The final step is to substitute (5.70) into (5.73) for each vertex function $\Gamma_{\alpha\tilde{\beta}}$. Again all exponential factors drop out [because there is no Z factor in (5.73)]. This means that $y(t,\omega)$ is simply given by the rhs of (5.73) where all $\Gamma_{\alpha\tilde{\beta}}$ are considered as a function of the *effective* parameters; thus it suffices to substitute

$$\Gamma_{q\tilde{j}} = \Gamma_{q\tilde{j}}(i(l),\sigma(l),\mu l,k,\omega) , \quad (5.74)$$

and similarly for $\Gamma_{j\tilde{q}}$, $\Gamma_{j\tilde{n}}$, and $\Gamma_{n\tilde{j}}$. Analogous steps can be performed to express $z_1(t,\omega)$, (3.19), in terms of the corresponding vertex functions. It turns out that also for

z_1 all Z factors (and exponential factors) drop out. The result reads

$$z_1(t,\omega)=\frac{\partial}{\partial k^2}\left[\Gamma_{j\tilde{j}}+\frac{N_{j\tilde{j}}}{M_{j\tilde{j}}}\right]_{k=0} \qquad (5.75)$$

with

$$N_{j\tilde{j}}=\Gamma_{q\tilde{q}}\Gamma_{q\tilde{j}}\Gamma_{j\tilde{q}}+\Gamma_{n\tilde{n}}\Gamma_{n\tilde{j}}\Gamma_{j\tilde{n}}+\Gamma_{q\tilde{n}}\Gamma_{n\tilde{j}}\Gamma_{j\tilde{q}}+\Gamma_{n\tilde{q}}\Gamma_{q\tilde{j}}\Gamma_{j\tilde{n}} \qquad (5.76)$$

and

$$M_{j\tilde{j}}=\Gamma_{q\tilde{j}}\Gamma_{j\tilde{q}}+\Gamma_{n\tilde{j}}\Gamma_{j\tilde{n}} , \qquad (5.77)$$

where all $\Gamma_{\alpha\tilde{\beta}}$ are considered as functions of the effective parameters [see (5.74)]. For completeness we also present the result for $z_2(t,\omega)$, (3.20), in the form

$$z_2(t,\omega)=-z_1(t,\omega)+\frac{\partial}{\partial k^2}(\Gamma_{q\tilde{q}}+\Gamma_{n\tilde{n}}+\Gamma_{j\tilde{j}})_{k=0} . \qquad (5.78)$$

A quantitative application of these expressions to ^{4}He requires an identification of the initial values $i(1)$ and $\sigma(1)$ of the various parameters and an appropriate choice of the flow parameter $l(t,\omega)$. These points are deferred to a planned paper III of this work.

VI. SUMMARY

On the basis of a new stochastic model for the critical dynamics of ^{4}He we have presented a systematic field-theoretic treatment of the critical behavior of first sound. Our theory is applicable to the entire critical region above and at T_λ and to the complete frequency range that is experimentally accessible.

The connection between measurable thermodynamic quantities and static correlation functions of our model has been established in (2.6)–(2.13), which can serve as the starting point for the quantitative identification of the nonuniversal static parameters. The definitions (3.23) and (3.24) for the sound velocity c_1 and damping D_1 and the corresponding definition of the attenuation coefficient α (4.35) are based on the dynamic structure factor for $k\to 0$. A two-loop calculation has been carried out for the vertex function that determines the dominant critical contribution to c_1 and D_1. Exact results derived from dissipation-fluctuation theorems have been employed in order to properly separate static from dynamic contributions. Extensive use has been made of the relation between our model and model F. The results for c_1, D_1, and α are given in (4.31), (4.32), and (4.35) in terms of unrenormalized quantities. A precise statistical-dynamical definition of the phenomenological frequency-dependent specific heat introduced by Ferrell and Bhattacharjee[22] has been presented in (4.43)–(4.45) and in (4.49). The static and dynamic renormalizations of the various model parameters have been carried out in Sec. V. The final results for c_1 and D_1 in terms of renormalized parameters are obtained from (3.23) and (3.24) together with (5.73) and (5.75) after multiple substitutions of the various transformations and explicit computational expressions. These results will lead to detailed quantitative predictions for the sound attenuation and dispersion for all pressures along the λ line and over many decades of frequencies without new adjustments of parameters.[21]

APPENDIX A: STATICS

We start from the distribution I (2.4) with I (2.14)–(2.17). Since $\Delta\rho_0$ and Δs_0 appear in $\hat{H}_{ps}$ only up to second order the following exact relations hold:

$$\langle\rho_0\rangle=\bar{\rho}_0-a_\rho\langle|\psi_0|^2\rangle , \qquad (A1)$$

$$\langle s_0\rangle=\bar{s}_0-a_s\langle|\psi_0|^2\rangle , \qquad (A2)$$

with

$$\begin{bmatrix} a_\rho \\ a_s \end{bmatrix}=\underline{X}\begin{bmatrix} \mathring{\gamma}_\rho \\ \mathring{\gamma}_s \end{bmatrix} , \qquad (A3)$$

where

$$\underline{X}^{-1}=\begin{bmatrix} \mathring{\chi}_s^{-1} & \mathring{\chi}_{\rho s}^{-1} \\ \mathring{\chi}_{\rho s}^{-1} & \mathring{\chi}_\rho^{-1} \end{bmatrix} . \qquad (A4)$$

Comparison with (2.4) and (2.5) yields

$$\bar{\rho}_0=-(\partial\Omega^{(0)}/\partial\mu)_T, \quad \bar{s}_0=-(\partial\Omega^{(0)}/\partial T)_\mu \qquad (A5)$$

and

$$a_\rho=-\tfrac{1}{2}k_BT\dot{r}_0, \quad a_s=-\tfrac{1}{2}k_BTr_0' . \qquad (A6)$$

Furthermore,

$$\begin{bmatrix} \mathring{C}_{\rho\rho} & \mathring{C}_{\rho s} \\ \mathring{C}_{s\rho} & \mathring{C}_{ss} \end{bmatrix}=\underline{X}+\mathring{C}_\psi\begin{bmatrix} a_\rho^2 & a_\rho a_s \\ a_s a_\rho & a_s^2 \end{bmatrix} . \qquad (A7)$$

Alternatively we obtain from (2.2)–(2.5)

$$(k_BT)^{-1}\mathring{C}_{\rho\rho}=-\ddot{\Omega}=(\partial\langle\rho_0\rangle/\partial\mu)_T = -\ddot{\Omega}^{(0)}+\tfrac{1}{4}k_BT\dot{r}_0^2\mathring{C}_\psi , \qquad (A8)$$

$$(k_BT)^{-1}\mathring{C}_{s\rho}=-\dot{\Omega}'=(\partial\langle\rho_0\rangle/\partial T)_\mu = -\dot{\Omega}'^{(0)}+\tfrac{1}{4}k_BT\dot{r}_0r_0'\mathring{C}_\psi , \qquad (A9)$$

$$(k_BT)^{-1}\mathring{C}_{ss}=-\Omega''=(\partial\langle s_0\rangle/\partial T)_\mu = -\Omega''^{(0)}+\tfrac{1}{4}k_BTr_0'^2\mathring{C}_\psi . \qquad (A10)$$

In the following equations (A11)–(A19) we use the abbreviation $\rho\equiv\langle\rho_0\rangle$ and $\sigma\equiv\langle\sigma_0\rangle=\langle s_0\rangle/\langle\rho_0\rangle$ (not to be confused with renormalized quantities) and turn to thermodynamic derivatives with respect to P instead of μ. From $d\mu=-\sigma\,dT+\rho^{-1}dP$ we have

$$\left[\frac{\partial}{\partial\mu}\right]_T=\rho\left[\frac{\partial}{\partial P}\right]_T , \qquad (A11)$$

$$\left[\frac{\partial}{\partial T}\right]_\mu-\left[\sigma\frac{\partial}{\partial\mu}\right]_T=\left[\frac{\partial}{\partial T}\right]_P . \qquad (A12)$$

Accordingly we obtain

$$-\ddot{\Omega}=-\rho(\partial\dot{\Omega}/\partial P)_T=\rho(\partial\rho/\partial P)_T\equiv\kappa_T\rho^2 , \qquad (A13)$$

$$-(\dot{\Omega}'-\sigma\ddot{\Omega})=-(\partial\dot{\Omega}/\partial T)_P=(\partial\rho/\partial T)_P\ , \quad (A14)$$

$$-(\Omega''-2\sigma\dot{\Omega}'+\sigma^2\ddot{\Omega})=-(\partial\Omega'/\partial T)_P+\sigma(\partial\dot{\Omega}/\partial T)_P =\rho(\partial\sigma/\partial T)_P\equiv C_p/T\ . \quad (A15)$$

Finally we substitute (2.2). This leads to

$$\rho\kappa_T=\left[\frac{\partial\bar{\rho}_0}{\partial P}\right]_T+\tfrac{1}{4}\rho k_B T\left[\frac{\partial r_0}{\partial P}\right]_T^2\mathring{C}_\psi\ , \quad (A16)$$

$$\left[\frac{\partial\rho}{\partial T}\right]_P=\left[\frac{\partial\bar{\rho}_0}{\partial T}\right]_P+\tfrac{1}{4}\rho k_B T\left[\frac{\partial r_0}{\partial T}\right]_P\left[\frac{\partial r_0}{\partial P}\right]_T\mathring{C}_\psi\ , \quad (A17)$$

$$C_p/T=\bar{\rho}_0\left[\frac{\partial\bar{\sigma}_0}{\partial T}\right]_P-(\sigma-\bar{\sigma}_0)\left[\frac{\partial\bar{\rho}_0}{\partial T}\right]_P +\tfrac{1}{4}k_B T\left[\frac{\partial r_0}{\partial T}\right]_P^2\mathring{C}_\psi \quad (A18)$$

(A18) yields (2.8) if the approximation $(\partial/\partial T)_P \approx(\partial/\partial T)_{\bar{P}_0}$ is made and the relations I (2.83) and I (2.85) are used. Finally we give the result for the adiabatic compressibility

$$\kappa_s=\rho^{-1}\left[\frac{\partial\rho}{\partial P}\right]_\sigma=(k_B T)^{-1}(\mathring{C}_{\rho\rho}-\mathring{C}^2_{m\rho}\mathring{C}^{-1}_{mm}) =k_B T\bar{\rho}_0^2\mathring{\chi}_p^{-2}(\mathring{C}_{pp}-\mathring{C}^2_{mp}\mathring{C}^{-1}_{mm})\ . \quad (A19)$$

In the last equation we have used I (2.65).

APPENDIX B: DYNAMIC PERTURBATION CALCULATION

In this Appendix we derive perturbative expressions for the two-point vertex functions $\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)$ appearing in the matrix (3.6), as well as for the composite-field vertex function $\mathring{\Gamma}_{\tilde{\beta}\tilde{\varphi}}(k,\omega)$ defined by I (4.22)–(4.24). Ordinary dynamic perturbation theory with the dynamic functional I (A1)–(A8) yields

$$\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)=-i\omega\delta_{\alpha\beta}+\mathring{L}_{\alpha\beta}(k)-\mathring{\Sigma}_{\alpha\tilde{\beta}}(k,\omega)\ , \quad (B1)$$

where $\mathring{L}_{\alpha\beta}(k)$ are the matrix elements of $[\mathring{L}(k)]$ given in I (4.6). The self-energies $\mathring{\Sigma}_{\alpha\tilde{\beta}}(k,\omega)$ consist of the sum of all one-particle irreducible diagrams with two external (truncated) legs α_0 and $\tilde{\beta}_0$. The one-loop contribution to $\mathring{\Sigma}_{\alpha\tilde{\beta}}$ reads

$$\mathring{\Sigma}_{\alpha\tilde{\beta}}(k,\omega)^{(1)}=\int_p\frac{4b^*_\alpha a_\beta(\mathbf{k},\mathbf{p})}{p^-(\Gamma_0^*p^++\Gamma_0p^--i\omega)} +\int_p\frac{4b_\alpha a_\beta(\mathbf{k},\mathbf{p})^*}{p^-(\Gamma_0p^++\Gamma_0^*p^--i\omega)}\ , \quad (B2)$$

where b_α and a_α are the components of the three-component vectors

$$\mathbf{b}=\Gamma_0\mathring{\boldsymbol{\gamma}}-\frac{i}{2}\mathring{\mathbf{g}}\ , \quad (B3)$$

$$\mathbf{a}(\mathbf{k},\mathbf{p})=[\mathring{L}(k)]\mathring{\boldsymbol{\gamma}}+i\mathring{\mathbf{g}}(\mathbf{p}\cdot\mathbf{k})\ , \quad (B4)$$

with $\mathring{\boldsymbol{\gamma}}$ and $\mathring{\mathbf{g}}$ given in I (A6). In (B2) we have used the abbreviation

$$p^+\equiv(\mathbf{p}+\tfrac{1}{2}\mathbf{k})^2+\mathring{\tau}_0,\quad p^-\equiv(\mathbf{p}-\tfrac{1}{2}\mathbf{k})^2+\mathring{\tau}_0\ . \quad (B5)$$

To leading order in k, (B1) and (B2) yield (4.1)–(4.5).

In two-loop order, four types of diagrams contribute to $\mathring{\Sigma}_{\alpha\tilde{\beta}}$ as shown in Fig. 1 of Ref. 34. (Again we have eliminated tadpole diagrams.) In the present context the dashed lines represent the various response and correlation propagators related to all secondary fields $m_0, p_0, j_0, \tilde{m}_0, \tilde{p}_0, \tilde{j}_0$. The expressions for these propagators are identical with the matrix elements of $\underline{\mathring{G}}$ defined in I (4.4)–(4.6). Clearly, a complete two-loop calculation would be extremely lengthy. A basic simplification arises, however, in the limit $c_0\to\infty$ in which the only nonvanishing propagators of the secondary fields are those of model F,

$$\mathring{G}_{m\tilde{m}}(k,\omega)=(-i\omega+\mathring{\lambda}_m\mathring{\chi}_m^{-1}k^2)^{-1}+O(c_0^{-1}) \quad (B6)$$

and

$$\mathring{G}_{mm}(k,\omega)=2\mathring{\lambda}_m k^2|-i\omega+\mathring{\lambda}_m\mathring{\chi}_m^{-1}k^2|^{-2}+O(c_0^{-1})\ . \quad (B7)$$

In addition we have to keep the leading part of the propagators

$$\mathring{G}_{m\tilde{j}}(k,\omega)=\frac{L_0k^2}{-ic_0k}(-i\omega+\mathring{\lambda}_m\chi_m^{-1}k^2)^{-1}+O(c_0^{-2}) \quad (B8)$$

and

$$\mathring{G}_{p\tilde{j}}(k,\omega)=\frac{\mathring{\chi}_p}{ic_0k}+O(c_0^{-2}) \quad (B9)$$

because they appear in combination with the coupling $-ic_0\mathring{\gamma}_pk$ associated with the $\tilde{\mathbf{j}}_0\nabla(\psi_0\psi_0^*)$ vertex. The m_0 field couples to the order parameter in the same way as in model F, whereas the $\tilde{m}_0$ field has an additional coupling $\sim L_0\mathring{\gamma}_p$. Since the (internal) propagators $\mathring{G}_{m\tilde{m}}$ and $\mathring{G}_{m\tilde{j}}$ correspond to topologically equivalent lines they appear, after multiplication with the appropriate couplings, only in the form of the sum

$$(\mathring{\gamma}_m\mathring{\lambda}_m+\mathring{\gamma}_pL_0)k^2\mathring{G}_{m\tilde{m}}+(-ic_0\mathring{\gamma}_pk)\mathring{G}_{m\tilde{j}}=\mathring{\gamma}_m\mathring{\lambda}_mk^2\mathring{G}_{m\tilde{m}}\ . \quad (B10)$$

Thus the net effect of $\mathring{G}_{m\tilde{j}}$ is to cancel the coupling $\mathring{\gamma}_p$. The propagator $\mathring{G}_{p\tilde{j}}$ is most conveniently incorporated in the diagrammatic treatment by the formal replacement

$$\mathring{G}_{p\tilde{j}}(k,\omega)\to\frac{\mathring{\chi}_p}{ic_0k}\frac{\mu k^2}{-i\omega+\mu k^2} \quad (B11)$$

with the auxiliary parameter μ for which the limit $\mu\to\infty$

is taken in the final integral expressions. The remaining diagrammatics is parallel to that carried out previously for model F.[34,55] The corresponding two-loop contributions to $\mathring{\Sigma}_{\alpha\beta}(k,\omega)$ will be denoted by $D^{(\nu)}_{\alpha\beta}(k,\omega)$, where $\nu=1,2,3,4$ refers to the corresponding type (ν) of diagrams. We shall use the notation

$$p_1^+ \equiv (\mathbf{p}_1+\tfrac{1}{2}\mathbf{k})^2+\hat{r}_0,\quad p_1^- \equiv (\mathbf{p}_1-\tfrac{1}{2}\mathbf{k})^2+\hat{r}_0\,,$$

$$p_2^+ \equiv (\mathbf{p}_2+\tfrac{1}{2}\mathbf{k})^2+\hat{r}_0,\quad p_2^- \equiv (\mathbf{p}_2-\tfrac{1}{2}\mathbf{k})^2+\hat{r}_0\,,$$

$$p_{12}^+ \equiv (\mathbf{p}_1+\mathbf{p}_2+\tfrac{1}{2}\mathbf{k})^2+\hat{r}_0,\quad p_{12}^- \equiv (\mathbf{p}_1+\mathbf{p}_2-\tfrac{1}{2}\mathbf{k})^2+\hat{r}_0\,.$$

The analytic expressions for $D^{(\nu)}_{\alpha\beta}$ can be written as

$$D^{(\nu)}_{\alpha\beta}(k,\omega)=\int_{p_1}\int_{p_2} B^{(\nu)}_{\alpha}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},i\omega)\frac{16a_\beta(\mathbf{k},\mathbf{p}_2)}{\Gamma_0^* p_2^+ +\Gamma_0 p_2^- -i\omega}+\text{c.c.}\,, \tag{B12}$$

where +c.c. means that the same expression must be added with all parameters replaced by the complex conjugate parameters, but with the sign of $i\omega$ left unaltered. The integrands $B^{(\nu)}_{\alpha}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},i\omega)$ can be considered as components of the three-component vectors $\mathbf{B}^{(\nu)}(\mathbf{p}_1,\mathbf{p}_2,\mathbf{k},i\omega)$. In the limit $c_0\to\infty$ they are given by

$$\mathbf{B}^{(1)}=-\frac{4\hat{u}_0\mathbf{b}^*}{(p_1^2+\hat{r}_0)p_2^-}\left[\frac{2\Gamma'}{\Gamma^* p_2^+ +\Gamma p_2^- -i\omega}+\frac{1}{p_2^-}\right], \tag{B13}$$

$$\mathbf{B}^{(2)}=-\frac{\mathbf{b}^*}{p_1^-(\Gamma^* p_1^+ +\Gamma p_1^- -i\omega)}\left[\frac{2b_4^*}{p_2^-}+\frac{2b_4}{p_2^+}\right], \tag{B14}$$

$$\mathbf{B}^{(3)}=\frac{(\mathring{\boldsymbol{\gamma}}\cdot\mathring{\boldsymbol{\gamma}})\mathbf{b}}{p_{12}^+(p_2^+)^2}+\frac{(\mathring{\boldsymbol{\gamma}}\cdot\mathbf{b}^*)}{p_{12}^+(\Gamma^* p_2+\Gamma p_2^- -i\omega)}\left[\frac{\mathbf{b}^*}{p_2^-}+\frac{\mathbf{b}}{p_2^+}\right]$$
$$+\frac{b_m}{p_{12}^+(\Gamma^* p_{12}^+ +\Gamma p_2^- +\hat{\lambda}_m p_1^2-i\omega)}\left[\left[\frac{\mathbf{b}^*}{p_2^-}+\frac{\mathbf{b}}{p_2^+}\right]\frac{b_m^* p_2^+}{\Gamma^* p_2^+ +\Gamma p_2^- -i\omega}-\frac{\hat{\gamma}_m\mathbf{b}^*}{p_2^-}\right] \tag{B15}$$

and

$$\mathbf{B}^{(4)}=\frac{\mathbf{b}^*}{p_{12}^-(\Gamma^* p_{12}^+ +\Gamma p_{12}^- -i\omega)}\left[\frac{(\mathring{\boldsymbol{\gamma}}\cdot\mathbf{b}^*)}{p_2^-}+\frac{(\mathring{\boldsymbol{\gamma}}\cdot\mathbf{b})}{p_2^+}\right]+\frac{b_m\mathbf{b}^*}{p_2^+p_{12}^-(\Gamma^* p_2^+ +\Gamma p_{12}^- +\hat{\lambda}_m p_1^2-i\omega)}\left[\frac{b_m^* p_{12}^+}{\Gamma^* p_{12}^+ +\Gamma p_{12}^- -i\omega}-\hat{\gamma}_m\right]$$
$$+\frac{|b_m|^2\mathbf{b}^*}{p_2^-(\Gamma^* p_{12}^+ +\Gamma p_2^- +\hat{\lambda}_m p_1^2-i\omega)(\Gamma^* p_{12}^+ +\Gamma p_{12}^- -i\omega)}\,. \tag{B16}$$

In (B13)–(B16) we have used the abbreviations

$$b_m=\Gamma_0\mathring{\gamma}_m\mathring{\chi}_m^{1/2}-\frac{i}{2}\mathring{g}_m\mathring{\chi}_m^{-1/2}\,, \tag{B17}$$

$$b_4=2\Gamma_0\hat{u}_0-\frac{i}{2}(\mathbf{g}\cdot\mathring{\boldsymbol{\gamma}})\,, \tag{B18}$$

$$\hat{\lambda}_m=\mathring{\lambda}_m\mathring{\chi}_m^{-1/2},\quad \hat{\gamma}_m=\mathring{\gamma}_m\mathring{\chi}_m^{1/2}\,, \tag{B19}$$

and have dropped, for simplicity, the index 0 of Γ_0 and Γ_0^*. Comparison with (2)–(5) of Ref. 55 shows that the external model-F vertex (connected with the external m leg) has been replaced in (B13)–(B16) simply by $\mathbf{b}$ or $\mathbf{b}^*$, whereas the appropriate modification of the internal vertices has required a more detailed inspection of the individual two-loop diagrams [three of type (1), two of type (2), seven of type (3), and four of type (4)].

The composite-field vertex function $\mathring{\Gamma}_{\beta\tilde{\varphi}}(k,\omega)$ can be defined diagrammatically by $\mathring{\Sigma}_{\alpha\tilde{\beta}}(k,\omega)$ with the additional prescription that the external vertex b_α is replaced by $-i/4$ and the external vertex b_α^* by $+i/4$. For example in one-loop order we have, according to (B2),

$$\mathring{\Gamma}_{\beta\tilde{\varphi}}(k,\omega)=-i\int_p\frac{a_\beta(\mathbf{k},\mathbf{p})}{p^-(\Gamma_0^* p^+ +\Gamma_0 p^- -i\omega)}+i\int_p\frac{a_\beta(\mathbf{k},\mathbf{p})^*}{p^-(\Gamma_0 p^+ +\Gamma_0^* p^- -i\omega)}\,, \tag{B20}$$

and the two-loop contributions are obtained in a similar fashion from $\mathbf{D}^{(\nu)}$.

From (B1)–(B5) and (B12)–(B19) we derive the leading expression for $\mathring{\Gamma}_{p\tilde{j}}(k,\omega)$ in the limit $k\to 0$. A straightforward calculation yields

$$\mathring{\Gamma}_{p\tilde{j}}(k,\omega)=-ic_0\mathring{\chi}_p^{-1}k\left[1-\int_{p_1}\frac{4\mathring{\gamma}_p^2\mathring{\chi}_p}{\hat{\Pi}_1(\hat{\Pi}_1-i\Omega_0)}-\bar{D}(\omega)\right] \tag{B21}$$

with the two-loop contribution

$$\overline{D}(\omega)=-16\mathring{\gamma}_p^2\mathring{\chi}_p[4\hat{u}_0-(\mathring{\gamma}\cdot\mathring{\gamma})]\left[\left[\int_{p_1}\frac{1}{\hat{\Pi}_1(\hat{\Pi}_1-i\Omega_0)}\right]^2+\int_{p_1}\frac{1}{\hat{\Pi}_1}\int_{p_2}\frac{1}{\hat{\Pi}_2(\hat{\Pi}_2-i\Omega_0)}\left[\frac{1}{\hat{\Pi}_2}+\frac{1}{\hat{\Pi}_2-i\Omega_0}\right]\right]$$
$$+8\mathring{\gamma}_p^2\mathring{\chi}_p(\Gamma_0')^{-1}\int_{p_1}\int_{p_2}\frac{Q(\mathbf{p}_1,\mathbf{p}_2,i\Omega_0)}{\hat{\Pi}_2-i\Omega_0}\ , \tag{B22}$$

where

$$\hat{\Pi}_1\equiv p_1^2+\hat{r}_0,\ \hat{\Pi}_2=p_2^2+\hat{r}_0 \tag{B23}$$

and $\Omega_0\equiv\omega/2\Gamma_0'$. The function Q is given in (4.22). The omission of tadpole diagrams is equivalent to a special choice of the parameter $\hat{r}_0$ according to

$$\hat{r}_0=r_0-4(\mathring{\gamma}\cdot\mathring{\gamma})\int_p\frac{1}{p^2+r_0}\ , \tag{B24}$$

apart from higher-order terms, compare (A7) of Ref. 29. Thus in the two-loop terms (B22) it is consistent to replace $\hat{r}_0$ by r_0. In the one-loop term of (B21) we substitute (B24) and expand with respect to $(\mathring{\gamma}\cdot\mathring{\gamma})$. One can verify that the resulting expression for $\mathring{\Gamma}_{p\bar{j}}$ can indeed be rewritten in the form (4.10), (4.8), (4.21). At $\omega=0$ this was to be expected on the basis of (3.27). At $\omega\neq0$ this is a nontrivial feature which was not obvious *a priori*; it will not remain valid at finite k.

Finally we derive the expression for $\mathring{\Gamma}_{\alpha\bar{\beta}}(k,0)$ given in (5.59). To leading order in k and for $\omega=0$ we obtain from (B1)–(B5) and (B12)–(B19),

$$[\mathring{\Gamma}(k,0)]=[\mathring{L}(k)]-4[\mathring{L}(k)][\mathring{\gamma},\mathring{\gamma}]J_2(\hat{r}_0)+\frac{2}{\Gamma_0'd}[\mathring{g}\cdot\mathring{g}][J_2(\hat{r}_0)-\hat{r}_0J_3(\hat{r}_0)]k^2-(\overline{D}^{(1)}+\overline{D}^{(2)}+\overline{D}^{(3)}+\overline{D}^{(4)})\ , \tag{B25}$$

with J_n defined in (4.30). The elements $\overline{D}_{\alpha\beta}^{(\nu)}$ of the matrices $\overline{D}^{(\nu)}$ are the $k\to0$ parts of the matrix elements (B12) according to

$$D_{\alpha\beta}^{(\nu)}(k,0)=\overline{D}_{\alpha\beta}^{(\nu)}+O(k^3,k^4)\ . \tag{B26}$$

The expressions for $\overline{D}^{(\nu)}$ read (for simplicity we drop the index 0 in the following)

$$\overline{D}^{(1)}=-128\hat{u}[L(k)][\gamma,\gamma]J_1J_3+\frac{96\hat{u}}{\Gamma'd}[\mathbf{g},\mathbf{g}]J_1(J_3-\hat{r}J_4)k^2\ , \tag{B27}$$

$$\overline{D}^{(2)}=-64\hat{u}[\gamma,\gamma]J_2^2k^2+\frac{16}{\Gamma'd}(\gamma\cdot\mathbf{g})[\mathbf{g},\gamma]J_2(J_2-\hat{r}J_3)k^2\ , \tag{B28}$$

$$\overline{D}^{(3)}+\overline{D}^{(4)}=32\gamma^2[\gamma,\gamma]J_1J_3k^2-\frac{24}{\Gamma'd}\gamma^2[\mathbf{g},\mathbf{g}]J_1(J_3-\hat{r}J_4)k^2-\frac{8}{(\Gamma')^2\hat{\lambda}_md}[\mathbf{g},\mathbf{g}]\mathrm{Re}[b_m^2(J_{2,1}-\hat{r}J_{3,1})]k^2$$
$$+\left[16\gamma^2[\gamma,\gamma]J_2^2-\frac{8}{\Gamma'd}(\gamma\cdot\mathbf{g})[\mathbf{g},\gamma]J_2(J_2-\hat{r}J_3)\right]k^2-\frac{4b_mb_m^*}{(\Gamma')^2\hat{\lambda}_md}[\mathbf{g},\mathbf{g}]\{2\,\mathrm{Re}[(1+w)J_{2,1}-\hat{r}J_{2,2}]-J_2^2\}k^2\ . \tag{B29}$$

Equations (B25)–(B29) are the generalizations of (A14) and (A21)–(A26) of Ref. 34. After using the same algebra it can be shown that $[\mathring{\Gamma}(k,0)]$ can be written in the form (5.59) with

$$\mathring{P}(t,0)=\frac{2}{\Gamma_0'd}[J_2(r_0)-r_0J_3(r_0)]-\frac{96u_0}{\Gamma_0'd}J_1(r_0)[J_3(r_0)-r_0J_4(r_0)]+\frac{8}{(\Gamma_0')^2\hat{\lambda}_md}\mathrm{Re}[b_m^2(J_{2,1}-r_0J_{3,1})]$$
$$+\frac{4b_mb_m^*}{(\Gamma_0')^2\hat{\lambda}_md}\{2\,\mathrm{Re}[(1+w_0)J_{2,1}-r_0J_{2,2}]-J_2^2\}\ . \tag{B30}$$

This function $\mathring{P}$ is identical with the corresponding function of model F and therefore yields model-F pole terms. The latter determine the Z factors (5.65)–(5.67) and are contained in the quantity [compare (B10) of Ref. 34]

$$\hat{P}=-\frac{1}{2\epsilon}-\frac{1}{2\epsilon}G+\frac{1}{w'\epsilon^2}\mathrm{Re}\left[\frac{D^2}{1+w}\right]\ , \tag{B31}$$

where G and D are given by (3.24) and (3.25) of Ref. 34.

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