Theory of critical first sound near the λ transition of ^{4}He. I. Model and correlation functions

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A detailed foundation and general discussion are given for a new model that describes the critical dynamics of ^{4}He near T_λ, including the thermal-diffusion and first-sound modes. All parameters are identified in terms of thermodynamic and hydrodynamic quantities. The relation between static and dynamic correlation functions of our model and those of the simpler model F is derived. This provides the basis for a quantitative theory of critical first sound.

I. INTRODUCTION

The possibility of testing the renormalization-group theory of bulk dynamic critical phenomena depends crucially on the availability of accurate experimental data. Most suitable for this purpose are the critical dynamics of strain-free fluids (rather than solids), and the most sensitive physical quantities are those transport coefficients that develop a divergence as criticality is approached. In principle this divergence can be studied as a function of temperature, wave number k, and frequency ω. In practice quantitative comparisons between renormalization-group studies and experiments on transport coefficients have been restricted mainly to the temperature dependence at a few values of k and at low (or zero) frequencies.[1-4]

The reason for this restriction is of both experimental and theoretical origin. As far as the k dependence is concerned experimental data exist only over a rather limited k range due to difficulties related to light- and neutron-scattering techniques. As far as the ω dependence is concerned the study of sound propagation offers the possibility of probing the entire critical frequency regime, but no detailed theoretical predictions have been made in the early stages of the development of the renormalization-group theory.[1] Also more recent renormalization-group studies on sound propagation in fluids[5] have not been satisfactory, as will be discussed in a subsequent paper.[6] On the other hand, a wealth of experimental data on critical sound attenuation in fluids over an enormous range of frequencies is available, in particular near the superfluid transition of ^{4}He and ^{3}He-^{4}He mixtures. The main motivation of the present work is to fill the gap on the theoretical side by extending the renormalization-group approach to sound propagation in ^{4}He. Our theory will be applicable to the entire experimentally accessible regime and therefore will make possible a test of the renormalization-group theory *as a function of frequency.*

A phenomenological theory of critical sound attenuation based on a frequency-dependent generalization of the specific heat has been developed already by Ferrell and Bhattacharjee (FB).[7] Because of the remarkable success of this theory in its application to experimental data it is of interest to clarify to what extent the FB approach can be justified within a renormalization-group treatment.

A proper statistical-dynamical treatment has been initiated recently[8-10] where we have introduced stochastic equations for a complete set of slow variables near the superfluid transition. They are well suited for a systematic study of both the thermal-diffusion and first-sound mode coupled to the critical fluctuations of the order parameter.[11]

This paper is devoted to a detailed foundation and general discussion of these equations; quantitative applications above and below T_λ as well as an analysis of the FB theory on the basis of our model will be deferred to subsequent parts of this work. For a short summary of some results see Refs. 8 and 9.

In constructing our dynamic model we employ known concepts of statistical dynamics.[1,12] We start from Poisson-bracket relations[13] for the slow variables, the fluctuations of the order parameter, the entropy density, the mass density, and the momentum density. Invoking renormalization-group arguments, we eliminate irrelevant terms that are present in previous equations of motion.[14-19] The fundamental difference between our model and the well-known models $A-J$ (Ref. 1) is that our model includes secondary dynamic variables (essentially the pressure and the moment density) whose fluctuations do not affect the asymptotic critical dynamics of the order parameter because of the finiteness of the sound velocity. Nevertheless these secondary variables are of primary importance in describing the critical behavior of the first-sound mode itself.

For the purpose of making quantitative predictions of the nonuniversal critical behavior along the λ line[6] we identify all static and dynamic parameters in terms of noncritical thermodynamic and hydrodynamic quantities. Particular attention will be paid to the connection of our model with model F introduced previously[1,20] for the description of the thermal-diffusion mode. This will enable us to incorporate into our renormalized theory[6] of the first-sound mode the detailed knowledge on critical statics[21,22] and low-frequency dynamics,[3,23-26] in a consistent fashion. Conversely, our model will also permit a study of the effect of the first-sound mode on the thermal-diffusion mode above T_λ and the second-sound mode below T_λ.

II. THE MODEL

In this section we present the detailed arguments that lead to the model[8] appropriate for the description of the critical behavior of both the thermal-diffusion (or second-sound) mode and the first-sound mode of ^{4}He near T_λ. In particular we derive the connection of various static and dynamic parameters with thermodynamic and hydrodynamic quantities, since these identifications are important for quantitative applications of the model.

From hydrodynamics[27,28] it is known that a complete set of slow variables consists of the long-wavelength fluctuations of the mass density $\rho_0(x)$, the entropy density $s_0(x)$, the momentum density $\mathbf{j}_0(x)$, and, near T_λ, of the complex order parameter $\hat{\psi}_0(x)$, a coarse-grained wave function of the Bose condensate.[29,30] We shall consider these fluctuations in an equilibrium state of liquid ^{4}He in a given total volume V, at given equilibrium temperature T, and equilibrium chemical potential per unit mass μ, i.e., with a fluctuating number N of ^{4}He atoms. This circumvents the conceptual inconvenience of a fluctuating total volume, which arises in the equilibrium ensemble at given pressure P.[16,31] For the purpose of a comparison with experiments we shall of course make contact with the common thermodynamics in the T-P plane.

A. Static probability distribution

After integrating over the short-wavelength fluctuations of the microscopic variables, the partition function can be written as a functional integral

$$Z(T,\mu,V)=\int D[\rho_0,s_0,\hat{\psi}_0,\mathbf{j}_0]\exp\left[-\int_V d^dx\ \Omega_{\text{eff}}(x)/k_BT\right] \tag{2.1}$$

with the effective local potential

$$\Omega_{\text{eff}}(x)=\Omega_{\text{eff}}(\rho_0(x),s_0(x),\hat{\psi}_0(x),\nabla\hat{\psi}_0(x),\mathbf{j}_0(x)) \ . \tag{2.2}$$

In (2.1) and (2.2) the variables have only long-wavelength fluctuations. The corresponding probability distribution is

$$w[\rho_0,s_0,\hat{\psi}_0,\mathbf{j}_0]=Z^{-1}\exp\left[-\int_V d^dx\ \Omega_{\text{eff}}(x)/k_BT\right] \ . \tag{2.3}$$

This can be identified with the probability distribution constructed within the usual theory of thermodynamic fluctuations.[31] Starting from the probability distribution for small subsystems of a given volume, and at given T and μ, and summing over these subsystems one arrives at the probability distribution for the total system

$$w\sim\exp\left[-\int_V d^dx\ \Delta\Omega(x)/k_BT\right] \tag{2.4}$$

with

$$\Delta\Omega(x)=\Delta E_0(x)-\mu\Delta\rho_0(x)-T\Delta s_0(x) \ , \tag{2.5}$$

where

$$\Delta E_0(x)\equiv E_0(x)-\bar{E}_0 \ ,$$

$$\Delta\rho_0(x)\equiv\rho_0(x)-\bar{\rho}_0 \ ,$$

$$\Delta s_0(x)\equiv s_0(x)-\bar{s}_0$$

denote local fluctuations of the energy, mass, and entropy densities. An interaction between subsystems is taken into account only via the $\nabla\hat{\psi}_0$ dependence of the energy density

$$E_0(x)=E_0(\rho_0(x),s_0(x),\hat{\psi}_0(x),\nabla\hat{\psi}_0(x),\mathbf{j}_0(x)) \ , \tag{2.6}$$

which includes the local kinetic energy. The phenomenological reference values $\bar{\rho}_0=\bar{\rho}_0(T,\mu)$, $\bar{s}_0=\bar{s}_0(T,\mu)$, and

$$\bar{E}_0=E_0(\bar{\rho}_0,\bar{s}_0,0,0,0)$$

with $\bar{\hat{\psi}}_0=0$ and $\bar{\mathbf{j}}_0=0$ may be interpreted as equilibrium values of noninteracting finite subsystems and can be determined approximately from experimental data of ^{4}He well above T_λ. The energy density $E_0(x)$ serves as a local potential whose derivatives yield the local temperature

$$T_0(x)=\partial E_0(x)/\partial s_0(x)$$

and chemical potential

$$\mu_0(x)=\partial E_0(x)/\partial\rho_0(x) \ .$$

Their reference values

$$\left[\frac{\partial\bar{E}_0}{\partial\bar{s}_0}\right]_{\bar{\rho}_0}=\bar{T}_0=T \ , \quad \left[\frac{\partial\bar{E}_0}{\partial\bar{\rho}_0}\right]_{\bar{s}_0}=\bar{\mu}_0=\mu \tag{2.7}$$

coincide with the global equilibrium parameters T and μ. Since $\bar{E}_0$ is a smooth function of $\bar{s}_0$ and $\bar{\rho}_0$, the latter can be considered as smooth analytic functions of T and μ (even if the thermodynamic limit $V\to\infty$ is eventually taken for the total system). They must be distinguished from the full equilibrium values

$$\langle s_0\rangle=-\left[\frac{\partial\Omega}{\partial T}\right]_\mu , \quad \langle\rho_0\rangle=-\left[\frac{\partial\Omega}{\partial\mu}\right]_T \tag{2.8}$$

to be calculated from the thermodynamic potential

$$\Omega(T,\mu)=-\frac{k_BT}{V}\ln Z \ , \tag{2.9}$$

i.e., from the distribution (2.3). The connection between (2.3) and (2.5) is made by

$$\Omega_{\text{eff}}(x)=\Omega^{(0)}(T,\mu)+\Delta\Omega(x) \tag{2.10}$$

with a smooth "background" part $\Omega^{(0)}$. From $E_0(x)$, (2.6), we define the normal velocity

$$\mathbf{v}_{n0}(x)=\partial E_0(x)/\partial \mathbf{j}_0(x)$$

and the superfluid momentum density

$$\mathbf{j}_{s0}(x)=i\frac{m_4}{\hbar}\left[\hat{\psi}_0(x)\frac{\partial E_0(x)}{\partial \nabla\hat{\psi}_0}-\hat{\psi}_0(x)^*\frac{\partial E_0(x)}{\partial\nabla\hat{\psi}_0^*}\right] \tag{2.11}$$

with $2\pi\hbar$ being Planck's constant and m_4 being the mass of a ^{4}He atom. From Galilean invariance it follows that

$$\mathbf{j}_0=\mathbf{j}_{s0}+\rho_0\mathbf{v}_{n0}=\mathbf{j}_{s0}+\mathbf{j}_{n0}\;, \tag{2.12}$$

which defines the normal momentum density $\mathbf{j}_{n0}=\rho_0\mathbf{v}_{n0}$. The superfluid velocity is given by

$$\mathbf{v}_{s0}(x)=\frac{\hbar}{m_4}\nabla\varphi_0(x)\;, \tag{2.13}$$

where $\varphi_0(x)$ is the phase of

$$\hat{\psi}_0(x)=|\hat{\psi}_0(x)|\exp[i\varphi_0(x)]\;.$$

We expand $\Delta\Omega(x)$ with respect to $\mathbf{j}_0$, $\hat{\psi}_0$, $\nabla\hat{\psi}_0$, $\Delta\rho_0$, Δs_0, and obtain

$$\frac{\Delta\Omega}{k_BT}=\hat{H}_\psi+\hat{H}_{\rho s}+\hat{H}_j\;, \tag{2.14}$$

where

$$\hat{H}_\psi=\tfrac{1}{2}r_0|\psi_0|^2+\tfrac{1}{2}|\nabla\psi_0|^2+\hat{u}_0|\psi_0|^4\;, \tag{2.15}$$

$$\hat{H}_{\rho s}=\tfrac{1}{2}[\mathring{\chi}_s^{-1}(\Delta s_0)^2+2\mathring{\chi}_{\rho s}^{-1}\Delta s_0\Delta\rho_0+\mathring{\chi}_\rho^{-1}(\Delta\rho_0)^2]+(\mathring{\gamma}_s\Delta s_0+\mathring{\gamma}_\rho\Delta\rho_0)|\psi_0|^2\;, \tag{2.16}$$

$$\hat{H}_j=\tfrac{1}{2}\mathring{\chi}_j^{-1}\mathbf{j}_0^2\left[1+\frac{k_BTm_4^2}{\hbar^2\bar{\rho}_0}|\psi_0|^2\right]-i\hat{b}_0\mathbf{j}_0(\psi_0\nabla\psi_0^*-\psi_0^*\nabla\psi_0)\;, \tag{2.17}$$

apart from higher-order terms. As seen from the $|\nabla\psi_0|^2$ term in (2.15), we have used the order-parameter field in the standard normalization, $\psi_0(x)=\hat{c}_0\hat{\psi}_0(x)$; we shall not need an explicit specification of the normalization constant $\hat{c}_0$. From the two-fluid model[16,27,28] and from Galilean invariance, the coefficients in (2.17) can be identified as

$$\mathring{\chi}_j=k_BT\bar{\rho}_0,\quad \hat{b}_0=\frac{m_4}{2\hbar\bar{\rho}_0}\;. \tag{2.18}$$

In $\hat{H}_\psi$ and $\hat{H}_{\rho s}$ we have retained only terms that are relevant in the renormalization-group (RG) sense, whereas in $\hat{H}_j$, (2.17), the last two terms are irrelevant in the RG sense and will be dropped in the final formulation of the model. Nevertheless it is useful to discuss these terms as they represent leading contributions to the kinetic energy density $\frac{1}{2}\rho_{n0}v_{n0}^2$ of the normal part of the two-fluid kinetic energy density (divided by k_BT) and will play an important role in the calculation of the critical shear viscosity.[32,33] An alternative representation of $\hat{H}_j$ is obtained by considering $\mathbf{j}_{n0}$ and ψ_0 as the fundamental variables instead of $\mathbf{j}_0$ and ψ_0. From (2.11), (2.15), (2.17), and (2.18) we find

$$\mathbf{j}_{s0}=ik_BT\frac{m_4}{2\hbar}\left[\psi_0\nabla\psi_0^*-\psi_0^*\nabla\psi_0+2i\frac{m_4}{\hbar\bar{\rho}_0}|\psi_0|^2\mathbf{j}_0\right]\;. \tag{2.19}$$

Thus the right-hand side (rhs) of (2.17) can be written in terms of $\mathbf{j}_{n0}=\mathbf{j}_0-\mathbf{j}_{s0}$ and ψ_0 as

$$\hat{H}_j=\tfrac{1}{2}\mathring{\chi}_j^{-1}\mathbf{j}_{n0}^2\left[1-\frac{k_BTm_4^2}{\hbar^2\bar{\rho}_0}|\psi_0|^2\right]\;, \tag{2.20}$$

apart from irrelevant terms of $O(|\psi_0|^2|\nabla\psi_0|^2)$ and

$$O(\mathbf{j}_{n0}|\psi_0|^2(\psi_0\nabla\psi_0^*-\psi_0^*\nabla\psi_0))\;.$$

In (2.20) there exists no correction term of the type $\mathbf{j}_{n0}(\psi_0\nabla\psi_0^*-\psi_0^*\nabla\psi_0)$, therefore the Gaussian term $\sim\mathbf{j}_{n0}^2$ of (2.20) provides a more appropriate description of the normal part of the kinetic energy density than the Gaussian part $\sim\mathbf{j}_0^2$ of (2.17). The superfluid part $\frac{1}{2}\rho_{s0}v_{s0}^2/k_BT$ of the two-fluid kinetic energy density is approximately described by $\frac{1}{2}|\nabla\psi_0|^2$ in (2.15). If the irrelevant $|\psi_0|^2$ correction in (2.20) is dropped we obtain

$$\hat{H}_j=\tfrac{1}{2}\mathring{\chi}_j^{-1}\mathbf{j}_{n0}^2\;. \tag{2.21}$$

These different formulations of $\hat{H}_j$ will be taken up in the discussion of the equations of motion later.

All parameters of $\hat{H}_\psi$ and $\hat{H}_{\rho s}$ can be expressed as derivatives of the energy density

$$e_0[\rho_0(x),s_0(x),|\psi_0(x)|^2]\equiv E_0(\rho_0(x),s_0(x),\hat{\psi}_0(x),0,0) \tag{2.22}$$

taken at $\bar{\rho}_0$, $\bar{s}_0$, and $\hat{\psi}_0=0$. We have

$$r_0=\frac{2}{k_BT}\left[\frac{\partial e_0[\bar{\rho}_0,\bar{s}_0,|\psi_0|^2]}{\partial|\psi_0|^2}\right]_{\psi_0=0}, \tag{2.23}$$

$$\hat{u}_0=\frac{1}{2k_BT}\left[\frac{\partial^2 e_0[\bar{\rho}_0,\bar{s}_0,|\psi_0|^2]}{\partial|\psi_0|^4}\right]_{\psi_0=0}, \tag{2.24}$$

$$\mathring{\gamma}_\rho=\tfrac{1}{2}\frac{\partial r_0}{\partial\bar{\rho}_0},\quad \mathring{\gamma}_s=\tfrac{1}{2}\frac{\partial r_0}{\partial\bar{s}_0}\;, \tag{2.25}$$

$$\mathring{\chi}_s=k_BT\left[\frac{\partial^2\bar{e}_0}{\partial\bar{s}_0^2}\right]^{-1},\quad \mathring{\chi}_\rho=k_BT\left[\frac{\partial^2\bar{e}_0}{\partial\bar{\rho}_0^2}\right]^{-1},\quad \mathring{\chi}_{\rho s}=k_BT\left[\frac{\partial^2\bar{e}_0}{\partial\bar{\rho}_0\partial\bar{s}_0}\right]^{-1}, \tag{2.26}$$

with $\bar{e}_0\equiv e_0[\bar{\rho}_0,\bar{s}_0,0]$. By elementary thermodynamic transformations one obtains from (2.25) and (2.26)

$$\frac{\mathring{\gamma}_\rho}{\mathring{\gamma}_s}=-\left[\frac{\partial\bar{s}_0}{\partial\bar{\rho}_0}\right]_{r_0}, \tag{2.27}$$

$$\mathring{\chi}_s^{-1}-\mathring{\chi}_{\rho s}^{-1}\frac{\mathring{\gamma}_s}{\mathring{\gamma}_\rho}=\frac{1}{k_BT}\left[\frac{\partial\bar{s}_0}{\partial T}\right]_{r_0}^{-1}, \tag{2.28}$$

$$\mathring{\chi}_\rho^{-1}-\mathring{\chi}_{\rho s}^{-1}\frac{\mathring{\gamma}_\rho}{\gamma_s}=\frac{1}{k_BT}\left[\frac{\partial\bar{\rho}_0}{\partial\mu}\right]_{r_0}^{-1}, \quad (2.29)$$

where the derivatives are taken along a path $r_0=\text{const}$. Relations equivalent to (2.27)–(2.29) have been derived already by Onuki[34] on the basis of Pippard-Buckingham-Fairbank relations.

All parameters (2.23)–(2.26) are smooth functions of T and μ. In $r_0[T,\mu]$ we shall keep only a linear temperature dependence around the transition temperature $T_\lambda[\mu]$,

$$r_0[T,\mu]=r_{0c}[\mu]+a_0[\mu](T-T_\lambda[\mu]), \quad (2.30)$$

where

$$a_0[\mu]=2\mathring{\gamma}_\rho(\partial\bar{\rho}_0/\partial T)_\mu+2\mathring{\gamma}_s(\partial\bar{s}_0/\partial T)_\mu>0 .$$

At given T and μ, with the total system at rest, the equilibrium pressure $P=P(T,\mu)$ is given by

$$P=-\Omega(T,\mu). \quad (2.31)$$

This can be considered as an implicit relation for $\mu=\mu(T,P)$. In a comparison of our model with experiments at given T and P we may therefore substitute $\mu=\mu(T,P)$ and consider all static parameters as smooth functions of T and P. In particular the transition temperature $T_\lambda(P)$ at given P can be defined as the solution of $T_\lambda=T_\lambda[\mu(T_\lambda,P)]$. Correspondingly we may turn from the temperature variable $T-T_\lambda[\mu]$ to the usual variable $T-T_\lambda(P)$ by means of well known thermodynamic relations.[35] Hence we eventually take r_0 as a function of T and P, linearized around $T_\lambda(P)$,

$$r_0(T,P)=r_{0c}(P)+a_0(P)\frac{T-T_\lambda(P)}{T_\lambda(P)}, \quad (2.32)$$

instead of (2.30), with

$$a_0(P)=T_\lambda(P)(\partial r_0/\partial T)_P>0 .$$

Within the linear approximation (2.32), it is justified to linearize (2.31) around $T_\lambda[\mu]$ and hence to neglect the singular (but higher-order) contribution $\sim(T-T_\lambda[\mu])^{2-\alpha}$ in this context. This amounts to replacing Ω by $\Omega^{(0)}$ in (2.31).

In summary we have the static probability distribution $w\sim\exp(-\tilde{H}/k_BT)$ with

$$\frac{\tilde{H}}{k_BT}=\frac{V\Omega^{(0)}}{k_BT}+\int_V d^dx(\hat{H}_\psi+\hat{H}_{\rho s}+\hat{H}_j), \quad (2.33)$$

where $\hat{H}_\psi$, $\hat{H}_{\rho s}$, and $\hat{H}_j$ are given by (2.15), (2.16), and (2.17) or (2.21).

B. Equations of motion

The physical basis for constructing the appropriate equations of motion for the fluctuating variables above T_λ is provided by the hydrodynamic equations of superfluid ^{4}He. Although essential parts of these equations have been known for a long time[14] the detailed interpretation of some terms is to some extent controversial.[16–19] This controversy, however, does not touch on the derivation of our model equations. In order to indicate clearly the basis of our model we start from the well-known general form of stochastic equations of motion for the slow variables[12]

$$\frac{\partial}{\partial t}\varphi_i=-\sum_j\left[\frac{\mathring{\lambda}_{ij}}{k_BT}\frac{\partial\tilde{H}}{\partial\varphi_j}\right]+V_i+\theta_i(t), \quad (2.34)$$

where $\tilde{H}$ is defined by Eq. (2.33) and where $\varphi_i(t)$ represent the spatial Fourier components of $\rho_0(x,t)$, $s_0(x,t)$, $\psi_0(x,t)$, and $\mathbf{j}_0(x,t)$. The Gaussian Langevin forces satisfy

$$\langle\theta_i(t)\theta_j(t')\rangle=2\mathring{\lambda}_{ij}\delta(t-t') \quad (2.35)$$

with the symmetric matrix $\mathring{\lambda}_{ij}=\mathring{\lambda}_{ji}$. The reversible part

$$V_i=\sum_j\left[\frac{\partial Q_{ij}}{\partial\varphi_i}-Q_{ij}\frac{\partial\tilde{H}}{\partial\varphi_j}\right] \quad (2.36)$$

contains the antisymmetric matrix $Q_{ij}=-Q_{ji}$, which is determined by Poisson brackets. The nonvanishing Poisson brackets between $\rho_0(x)$, $s_0(x)$, and the components $j_{k0}(x)$ of $\mathbf{j}_0(x)$ read[13]

$$\{j_{k0}(x),j_{i0}(y)\}=j_{i0}(x)\nabla_k\delta(x-y)+j_{k0}(y)\nabla_i\delta(x-y), \quad (2.37)$$

$$\{\mathbf{j}_0(x),\rho_0(y)\}=\rho_0(x)\nabla\delta(x-y), \quad (2.38)$$

$$\{\mathbf{j}_0(x),s_0(y)\}=s_0(x)\nabla\delta(x-y), \quad (2.39)$$

with ∇ and ∇_k acting on x only. Nonvanishing Poisson brackets exist also between $\mathbf{j}_0$, ρ_0, and the phase $\varphi_0(x)$ of the order parameter[13]

$$\{\mathbf{j}_0(x),\varphi_0(y)\}=-\delta(x-y)\nabla\varphi_0(x), \quad (2.40)$$

$$\{\rho_0(x),\varphi_0(y)\}=-\delta(x-y)m_4/\hbar . \quad (2.41)$$

These relations determine the reversible parts V_i, (2.36), except for additional nondissipative terms related to the time dependence of $|\psi_0|$. Since exact Poisson bracket relations with respect to $|\psi_0|$ are not known we argue on the basis of the equations of motion for interacting Bose systems[30,36] that a nondissipative contribution to $\partial|\psi_0|^2/\partial t$ must exist which in the noninteracting case represents the kinetic energy term of the Schrödinger equation. This contribution corresponds to the model F term $\sim\Gamma_0''$ discussed previously.[37] Incuding this term the reversible parts of the equations of motion as derived from (2.36)–(2.41) become

$$\frac{\partial\psi_0}{\partial t}=-i\frac{m_4}{\hbar}\psi_0\frac{\delta\tilde{H}}{\delta\rho_0}-\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}\nabla\psi_0-2i\frac{\Gamma_0''}{k_BT}\frac{\delta\tilde{H}}{\delta\psi_0^*}, \quad (2.42)$$

$$\frac{\partial\rho_0}{\partial t}=i\frac{m_4}{\hbar}\left[\psi_0\frac{\delta\tilde{H}}{\delta\psi_0}-\psi_0^*\frac{\delta\tilde{H}}{\delta\psi_0^*}\right]-\nabla\left[\rho_0\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}\right], \quad (2.43)$$

$$\frac{\partial s_0}{\partial t}=-\nabla\left[s_0\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}\right], \quad (2.44)$$

$$\frac{\partial \mathbf{j}_0}{\partial t}=-\rho_0\nabla\frac{\delta\tilde{H}}{\delta\rho_0}-s_0\nabla\frac{\delta\tilde{H}}{\delta s_0}-\sum_k\left[j_{k0}\nabla\frac{\delta\tilde{H}}{\delta j_{k0}}+\nabla_k\mathbf{j}_0\frac{\delta\tilde{H}}{\delta j_{k0}}\right]+\frac{\delta\tilde{H}}{\delta\psi_0}\nabla\psi_0+\frac{\delta\tilde{H}}{\delta\psi_0^*}\nabla\psi_0^* , \qquad (2.45)$$

where, at this stage, $\tilde{H}$ contains $\hat{H}_j$ in the form (2.17). If one substitutes $\Delta\Omega(x)$, in the original form of Eq. (2.5), into $\tilde{H}$, it can be seen that Eqs. (2.42)–(2.45) are equivalent to the reversible parts of the hydrodynamic equations derived by Khalatnikov and Lebedev[17] except for two minor differences: (i) The first term of Eq. (2.42) becomes

$$-i(m_4/\hbar)[\mu_0(x,t)-\mu]\psi_0 ,$$

whereas in the corresponding Eq. (28) of Ref. 17 the term $\sim\mu$ is missing. This is equivalent to a different, μ-dependent, gauge factor $\exp(-i\mu m_4 t/\hbar)$ for the order parameter used in Ref. 17. (ii) In Eqs. (2.42) and (2.45) we have not included the nondissipative terms $\sim\zeta_3'$ and $\sim\zeta_4'$ of Eqs. (27) and (28) of Ref. 17 since they are irrelevant in the RG sense. In the notation of Khalatnikov and Lebedev, Γ_0'' in (2.42) corresponds to $\frac{1}{2}\zeta_1''$.

Equations (2.42)–(2.45) still contain irrelevant terms. From a closer inspection of the various contributions it can be seen (as far as the leading critical behavior is concerned) that these equations may be reduced to

$$\frac{\partial\psi_0}{\partial t}=-2i\frac{\Gamma_0''}{k_BT}\frac{\delta\tilde{H}}{\delta\psi_0^*}-i\frac{m_4}{\hbar}\psi_0\frac{\delta\tilde{H}}{\delta\rho_0} , \qquad (2.46)$$

$$\frac{\partial\rho_0}{\partial t}=i\frac{m_4}{\hbar}\left[\psi_0\frac{\delta\tilde{H}}{\delta\psi_0}-\psi_0^*\frac{\delta\tilde{H}}{\delta\psi_0}\right]-\bar{\rho}_0\nabla\frac{\delta\tilde{H}}{\delta\mathbf{j}_0} , \qquad (2.47)$$

$$\frac{\partial s_0}{\partial t}=-\bar{s}_0\nabla\frac{\delta\tilde{H}}{\delta\mathbf{j}_0} , \qquad (2.48)$$

$$\frac{\partial\mathbf{j}_0}{\partial t}=-\bar{\rho}_0\nabla\frac{\delta\tilde{H}}{\delta\rho_0}-\bar{s}_0\nabla\frac{\delta\tilde{H}}{\delta s_0} . \qquad (2.49)$$

It should be kept in mind that the approximations $\rho_0\approx\bar{\rho}_0$ and $s_0\approx\bar{s}_0$ on the rhs of (2.47)–(2.49) will imply correspondingly approximate results, e.g., $\langle\rho_0\rangle\approx\bar{\rho}_0$ in the prefactor of the dynamic structure factor and $\langle s_0\rangle\approx\bar{s}_0$ in the expression for the second-sound velocity which should be corrected in a more refined theory.

$\tilde{H}$ contains $\hat{H}_j$ in the form (2.17) without the (irrelevant) ψ_0 dependent terms. The omission of these terms, as dictated by relevance arguments, implies that (2.47) differs from the correct continuity equation

$$\frac{\partial\rho_0}{\partial t}=-\nabla\mathbf{j}_0 \qquad (2.50)$$

by the first term of (2.47), which would otherwise be canceled by the $\hat{b}_0$ contributions arising from the last term in (2.17). A similar defect appears in (2.48), which should be compared with the continuity equation for the entropy density

$$\frac{\partial s_0}{\partial t}=-\nabla(s_0\mathbf{v}_{n0}) \qquad (2.51)$$

(dissipative terms are neglected here). Part of these shortcomings can be avoided by employing the variable $\mathbf{j}_{n0}$ instead of $\mathbf{j}_0$. Using (2.19) one can show that the difference between $\mathbf{j}_0$ and $\mathbf{j}_{n0}$ produces only irrelevant contributions and that the equations of motions for $(\psi_0, \rho_0, s_0, \mathbf{j}_{n0})$ can be simply obtained from (2.46)–(2.49) by the formal substitution $\mathbf{j}_0\rightarrow\mathbf{j}_{n0}$. Then (2.47) and (2.48) become

$$\frac{\partial\rho_0}{\partial t}=-ik_BT\frac{m_4}{2\hbar}(\psi_0\nabla^2\psi_0^*-\psi_0^*\nabla^2\psi_0)-k_BT\bar{\rho}_0\nabla\frac{\delta\hat{H}_j}{\delta\mathbf{j}_{n0}} \qquad (2.52)$$

and

$$\frac{\partial s_0}{\partial t}=-k_BT\bar{s}_0\nabla\frac{\delta\hat{H}_j}{\delta\mathbf{j}_{n0}} , \qquad (2.53)$$

which leads to improved agreement with (2.50) and (2.51) after substituting $\hat{H}_j$ in the form (2.21). In any case, at our level of treating the critical behavior of correlation functions these differences between the $\mathbf{j}_0$ or $\mathbf{j}_{n0}$ formulations are unimportant (but they come into play when the theory is matched with the noncritical background behavior) and therefore we shall not make a further distinction between $\mathbf{j}_0$ and $\mathbf{j}_{n0}$ in the following.

Next we turn to the dissipative terms. We take into account

$$\frac{\partial\psi_0}{\partial t}=-2\frac{\Gamma_0'}{k_BT}\frac{\delta\tilde{H}}{\delta\psi_0^*} , \qquad (2.54)$$

$$\frac{\partial s_0}{\partial t}=\frac{\kappa_0}{T}\nabla^2\frac{\delta\tilde{H}}{\delta s_0} , \qquad (2.55)$$

and (with summation over k)

$$\frac{\partial j_{i0}}{\partial t}=(\zeta_0-\tfrac{2}{3}\eta_0)\delta_{ij}\nabla_k\nabla\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}+\eta_0\nabla_k\left[\nabla_i\frac{\delta\tilde{H}}{\delta j_{k0}}+\nabla_k\frac{\delta\tilde{H}}{\delta j_{i0}}\right] , \qquad (2.56)$$

in agreement with previous equations,[14–19] apart from irrelevant terms. The kinetic coefficients Γ_0', κ_0, ζ_0, and η_0 are real positive quantities and represent noncritical contributions to the relaxation coefficient of the order parameter, to the thermal conductivity, and to the bulk and shear viscosities, respectively, well away from T_λ. In the notation of Khalatnikov and Lebedev[17] they correspond to $\frac{1}{2}\zeta_1'$, $T\zeta_6$, ζ_5, and ζ_7, respectively.

Inspection of the terms related to $\mathbf{j}_0$ or $\mathbf{j}_{n0}$ shows that the equations for the transverse components of $\mathbf{j}_0$ are decoupled from the remaining equations. This implies that the transverse viscosity modes do not affect the leading critical dynamics of the thermal-diffusion and first-sound modes. Therefore we shall consider only the longitudinal component of Eqs. (2.49) and (2.56).

In summary, on the basis of (2.46)–(2.56) we arrive at the following set of equations

$$\frac{\partial\psi_0}{\partial t}=-\frac{2\Gamma_0}{k_BT}\frac{\delta\tilde{H}}{\delta\psi_0^*}-i\frac{m_4}{\hbar}\psi_0\frac{\delta\tilde{H}}{\delta\rho_0}\,, \tag{2.57}$$

$$\frac{\partial\rho_0}{\partial t}=i\frac{m_4}{\hbar}\left[\psi_0\frac{\delta\tilde{H}}{\delta\psi_0}-\psi_0^*\frac{\delta\tilde{H}}{\delta\psi_0^*}\right]-\bar{\rho}_0\nabla\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}\,, \tag{2.58}$$

$$\frac{\partial s_0}{\partial t}=\frac{\kappa_0}{T}\nabla^2\frac{\delta\tilde{H}}{\delta s_0}-\bar{s}_0\nabla\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}\,, \tag{2.59}$$

$$\frac{\partial\mathbf{j}_0}{\partial t}=\frac{\mathring{\lambda}_j}{k_BT}\nabla^2\frac{\delta\tilde{H}}{\delta\mathbf{j}_0}-\bar{\rho}_0\nabla\frac{\delta\tilde{H}}{\delta\rho_0}-\bar{s}_0\nabla\frac{\delta\tilde{H}}{\delta s_0}\,, \tag{2.60}$$

where

$$\Gamma_0=\Gamma_0'+i\Gamma_0'' \tag{2.61}$$

and

$$\mathring{\lambda}_j=k_BT(\zeta_0+\tfrac{4}{3}\eta_0)\,. \tag{2.62}$$

These equations, together with $\tilde{H}$ introduced in Sec. II A, constitute the basis of our dynamic model.

C. Model equations

In the final formulation of our model we shall use, instead of s_0 and ρ_0, more convenient variables $m_0(x)$ and $p_0(x)$, which will be introduced in accord with the following requirements: (1) The static fluctuations of m_0 and p_0 should be independent within the Gaussian approximation, with $\bar{m}_0=0$ and $\bar{p}_0=0$; (2) the time dependence of $m_0(x,t)$ and $p_0(x,t)$ should be adapted to the separate modes of linearized hydrodynamics; (3) the mass density variable $\rho_0(x,t)-\bar{\rho}_0$ should be a linear combination of $m_0(x,t)$ and $p_0(x,t)$ in order to simplify the calculation of the dynamic structure factor. These requirements do not yet determine $m_0(x,t)$ and $p_0(x,t)$ uniquely.

As a simple choice we take $m_0(x,t)$ to be proportional to the fluctuation $\Delta\sigma_0=\sigma_0(x,t)-\bar{\sigma}_0$ of the entropy per unit mass,

$$m_0(x,t)=k_B^{-1}\bar{\rho}_0\Delta\sigma_0(x,t)\,, \tag{2.63}$$

where

$$\sigma_0(x,t)=s_0(x,t)/\rho_0(x,t)\,, \tag{2.64}$$

with $\bar{\sigma}_0=\bar{s}_0/\bar{\rho}_0$. Then the requirements (3) and (1) determine $p_0(x,t)$ uniquely as

$$p_0(x,t)=b_\rho[\rho_0(x,t)-\bar{\rho}_0]+b_m m_0(x,t)\,, \tag{2.65}$$

where

$$b_\rho=\left[\frac{\partial\bar{P}_0}{\partial\bar{\rho}_0}\right]_{\bar{\sigma}_0}, \tag{2.66}$$

$$b_m=\frac{k_B}{\bar{\rho}_0}\left[\frac{\partial\bar{P}_0}{\partial\bar{\sigma}_0}\right]_{\bar{\rho}_0}. \tag{2.67}$$

Here $\bar{P}_0$ is the reference value of a local pressure variable (at $\mathbf{j}_0=0$, $\nabla\psi_0=0$)

$$P_0(x,t)=-e_0(x,t)+Ts_0(x,t)+\mu\rho_0(x,t)\,, \tag{2.68}$$

$$\bar{P}_0=-\bar{e}_0+T\bar{s}_0+\mu\bar{\rho}_0\neq P\,. \tag{2.69}$$

The physical meaning of the variable $p_0(x,t)$ is seen by comparison with the fluctuation

$$\Delta P_0(x,t)=P_0(x,t)-\bar{P}_0\,.$$

Expanding p_0 with respect to ΔP_0, $\Delta\sigma_0$, $|\psi_0|^2$, we find

$$p_0(x,t)=\Delta P_0(x,t)+O((\Delta P_0)^2,\Delta P_0\Delta\sigma_0,(\Delta\sigma_0)^2,|\psi_0|^2)\,, \tag{2.70}$$

which justifies to interpret $p_0(x,t)$ as a pressure variable.

An alternative choice of the variable $m_0(x,t)$ would be the linear combination of s_0 and ρ_0

$$\hat{m}_0(x,t)=k_B^{-1}[s_0(x,t)-\bar{\sigma}_0\rho_0(x,t)]\,. \tag{2.71}$$

This choice would imply slightly modified thermodynamic derivatives in the definition of the static parameters (2.66), (2.67), and (2.83)–(2.86), but would lead to the same form of the equations of motion since the difference between m_0 and $\hat{m}_0$ shows up only in higher-order contributions that are irrelevant in the RG sense. Similarly $\Delta q_0=q_0(x,t)-\bar{q}_0$ would be a possible choice where

$$q_0(x,t)=e_0(x,t)-\bar{h}_0\rho_0(x,t) \tag{2.72}$$

is the heat variable of Kadanoff and Martin.[38] (h_0 denotes the enthalpy per unit mass.) Because of

$$\Delta q_0-m_0\sim O(r_0|\psi_0|^2,m_0^2,(\Delta p_0)^2)$$

the form of the equations of motion would again be the same.

From the definitions (2.63) and (2.65) we obtain

$$\frac{\partial m_0}{\partial t}=k_B^{-1}\bar{\rho}_0\left[\frac{1}{\rho_0}\frac{\partial s_0}{\partial t}-\frac{s_0}{\rho_0^2}\frac{\partial\rho_0}{\partial t}\right] \tag{2.73}$$

and

$$\frac{\partial p_0}{\partial t}=\left[b_\rho-b_mk_B^{-1}\frac{\bar{\rho}_0 s_0}{\rho_0^2}\right]\frac{\partial\rho_0}{\partial t}+b_mk_B^{-1}\frac{\bar{\rho}_0}{\rho_0}\frac{\partial s_0}{\partial t}\,, \tag{2.74}$$

hence it is straightforward to rewrite our equations of motion (2.57)–(2.60) in terms of m_0 and p_0. Thereby we may set $\rho_0\approx\bar{\rho}_0$ and $s_0\approx\bar{s}_0$ in the coefficients of (2.73) and (2.74). Including stochastic forces, we arrive at our model equations in the final form[8]

$$\frac{\partial}{\partial t}\psi_0=-2\Gamma_0\frac{\delta H}{\delta\psi_0^*}+i\mathring{g}_m\psi_0\frac{\delta H}{\delta m_0}-i\mathring{g}_p\psi_0\frac{\delta H}{\delta p_0}+\theta_\psi\,, \tag{2.75}$$

$$\frac{\partial}{\partial t}m_0=\mathring{\lambda}_m\nabla^2\frac{\delta H}{\delta m_0}+L_0\nabla^2\frac{\delta H}{\delta p_0}-2\mathring{g}_m\,\mathrm{Im}\left[\psi_0^*\frac{\delta H}{\delta\psi_0^*}\right]+\theta_m\,, \tag{2.76}$$

$$\frac{\partial}{\partial t}p_0 = L_0\nabla^2\frac{\delta H}{\delta m_0} + \mathring{\lambda}_p\nabla^2\frac{\delta H}{\delta p_0} + 2\mathring{g}_p \mathrm{Im}\left[\psi_0^*\frac{\delta H}{\delta\psi_0^*}\right] - c_0\nabla\frac{\delta H}{\delta \mathbf{j}_0} + \theta_p \ , \tag{2.77}$$

$$\frac{\partial}{\partial t}\mathbf{j}_0 = \mathring{\lambda}_j\nabla^2\frac{\delta H}{\delta \mathbf{j}_0} - c_0\nabla\frac{\delta H}{\delta p_0} + \theta_j \ , \tag{2.78}$$

$$H = \int d^dx\,[\tfrac{1}{2}(\mathring{r}_0|\psi_0|^2 + |\nabla\psi_0|^2 + \mathring{\chi}_m^{-1}m_0^2 + \mathring{\chi}_p^{-1}p_0^2 + \mathring{\chi}_j^{-1}\mathbf{j}_0^2) + \hat{u}_0|\psi_0|^4 + (\mathring{\gamma}_m m_0 + \mathring{\gamma}_p p_0)|\psi_0|^2 - \mathring{h}_m m_0 - \mathring{h}_p p_0] \ . \tag{2.79}$$

(In Eq. (3) of Ref. 8 the dynamic coupling $\mathring{G}_p$ should read $\mathring{g}_p$.)

The Gaussian Langevin forces have the nonvanishing correlations

$$\langle \theta_\psi(x,t)\theta_\psi^*(x',t')\rangle = 4\Gamma_0'\delta(x-x')\delta(t-t') \ , \tag{2.80}$$

$$\langle \theta_k(x,t)\theta_k(x',t')\rangle = -2\mathring{\lambda}_k\nabla^2\delta(x-x')\delta(t-t') \ , \tag{2.81}$$

where k stands for m, p, or j, and

$$\langle \theta_m(x,t)\theta_p(x',t')\rangle = -2L_0\nabla^2\delta(x-x')\delta(t-t') \ . \tag{2.82}$$

From Eqs. (2.15), (2.16), (2.25), (2.26), (2.65), and (2.66) we obtain the static parameters

$$\mathring{\chi}_m = \frac{\bar{\rho}_0 T}{k_B}\left[\frac{\partial\bar{\sigma}_0}{\partial T}\right]_{\bar{P}_0} \ , \tag{2.83}$$

$$\mathring{\chi}_p = k_B T\bar{\rho}_0\left[\frac{\partial\bar{\rho}_0}{\partial\bar{P}_0}\right]_{\bar{\sigma}_0}^{-1} \ , \tag{2.84}$$

$$\mathring{\gamma}_m = \frac{k_B}{2\bar{\rho}_0}\left[\frac{\partial r_0}{\partial\bar{\sigma}_0}\right]_{\bar{P}_0} \ , \tag{2.85}$$

$$\mathring{\gamma}_p = \frac{1}{2}\left[\frac{\partial r_0}{\partial\bar{P}_0}\right]_{\bar{\sigma}_0} \ , \tag{2.86}$$

$$\mathring{r}_0 = r_0 - 2\mathring{\gamma}_m\mathring{h}_m\mathring{\chi}_m - 2\mathring{\gamma}_p\mathring{h}_p\mathring{\chi}_p \ , \tag{2.87}$$

$$\mathring{h}_m = 0 \ , \tag{2.88}$$

$$\mathring{h}_p = 0 \ . \tag{2.89}$$

[Although $\mathring{h}_m$ and $\mathring{h}_p$ vanish, we have included formally the terms $\mathring{h}_m m_0$ and $\mathring{h}_p p_0$ in (2.79), since they may serve to perform a constant shift of the m_0 and p_0 variables such that all tadpole diagrams vanish.] According to Eqs. (2.83) and (2.84), $\mathring{\chi}_m$ and $\mathring{\chi}_p$ are noncritical contributions to the constant-pressure specific heat per unit volume (divided by k_B) and to the inverse adiabatic compressibility (divided by k_BT), respectively. From (2.85) and (2.86) we obtain

$$\frac{\mathring{\gamma}_p}{\mathring{\gamma}_m} = -k_B^{-1}\bar{\rho}_0\left[\frac{\partial\bar{\sigma}_0}{\partial\bar{P}_0}\right]_{r_0} \ . \tag{2.90}$$

Since both r_0 and $\bar{\sigma}_0$ are monotonic-increasing functions of T, we conclude from (2.85)

$$\mathring{\gamma}_m > 0 \tag{2.91}$$

in agreement with Onuki.[39]

The only independent kinetic coefficient in Eqs. (2.76) and (2.77) is

$$\mathring{\lambda}_m = \kappa_0 k_B^{-1} \ , \tag{2.92}$$

whereas L_0 and $\mathring{\lambda}_p$ are determined by

$$L_0^2 = \mathring{\lambda}_m\mathring{\lambda}_p \tag{2.93}$$

and

$$\mathring{\lambda}_p = \bar{\rho}_0^2 k_B\left[\frac{\partial T}{\partial\bar{\rho}_0}\right]_{\bar{\sigma}_0}^2\kappa_0 = k_B T\kappa_0\left[\frac{\partial\bar{\rho}_0}{\partial\bar{P}_0}\right]_{\bar{\sigma}_0}^{-1}\left[\frac{1}{C_v^{(0)}} - \frac{1}{C_p^{(0)}}\right] \tag{2.94}$$

with

$$C_v^{(0)} = T\left[\frac{\partial\bar{\sigma}_0}{\partial T}\right]_{\bar{\rho}_0}, \quad C_p^{(0)} = T\left[\frac{\partial\bar{\sigma}_0}{\partial T}\right]_{\bar{P}_0} \ . \tag{2.95}$$

These relations are consistent with the absence of a kinetic coefficient in the continuity equation (2.50) for ρ_0. Correspondingly the Langevin forces θ_m and θ_p are not independent of one another.

Equations (2.75)–(2.79) have a gauge-invariance property analogous to that of model F.[24] The signs in Eqs. (2.75)–(2.78) are chosen such that all dynamic couplings Γ_0'', $\mathring{g}_m$, $\mathring{g}_p$, and c_0 are positive. With the exception of Γ_0'',[37] they can be expressed in terms of static quantities according to

$$\mathring{g}_m = \frac{m_4}{\hbar}\bar{\sigma}_0 T \ , \tag{2.96}$$

$$\mathring{g}_p = \frac{m_4}{\hbar}k_B T\left[\frac{\partial\bar{P}_0}{\partial\bar{\rho}_0}\right]_{\bar{\sigma}_0}\left[1 + \frac{\bar{\sigma}_0}{\bar{\rho}_0}\left[\frac{\partial\bar{\rho}_0}{\partial\bar{\sigma}_0}\right]_{\bar{P}_0}\right] \ , \tag{2.97}$$

$$c_0 = \bar{\rho}_0\left[\frac{\partial\bar{\rho}_0}{\partial\bar{P}_0}\right]_{\bar{\sigma}_0}^{-1} k_B T \equiv \mathring{\chi}_P \ . \tag{2.98}$$

These relations will be used in the quantitative application to the attenuation and dispersion of first sound.[6] For $c_0 = 0$ our model is appropriate for the study of the critical statics and low-frequency dynamics of ^{3}He-^{4}He mixtures if the variable p_0 is reinterpreted as the ^{3}He concentration. In this case, however, $\mathring{\lambda}_m$, L_0, and $\mathring{\lambda}_p$ are independent kinetic coefficients.[40] The connection of our model with model F (Ref. 20) will be discussed in Secs. III and IV.

III. STATIC CORRELATION FUNCTIONS

In this Section we discuss the structure of the static two-point correlation functions,

$$\mathring{C}_{\alpha\beta}(k) = \int_V d^dx\, e^{-ikx}[\langle\alpha_0(x)\beta_0(0)\rangle - \langle\alpha_0\rangle\langle\beta_0\rangle] \ , \tag{3.1}$$

with α_0 and β_0 being one of the variables $m_0\mathring{\chi}_m^{-1/2}$ or $p_0\mathring{\chi}_p^{-1/2}$. Here the brackets $\langle \cdots \rangle$ denote averages with the distribution $\sim\exp\text{-}H$, where H is given by (2.79), with the couplings $\hat{u}_0$, $\mathring{\gamma}_m$, and $\mathring{\gamma}_p$. Since m_0 and p_0 enter H only up to the second order we have the exact relations

$$\mathring{C}_{\alpha\beta}(k)=\delta_{\alpha\beta}+\mathring{\gamma}_\alpha\mathring{\gamma}_\beta\mathring{C}_\psi(k)\ , \tag{3.2}$$

with $\mathring{\gamma}_\alpha$ and $\mathring{\gamma}_\beta$ being $\mathring{\gamma}_m\mathring{\chi}_m^{1/2}$ or $\mathring{\gamma}_p\mathring{\chi}_p^{1/2}$, and

$$\mathring{C}_\psi(k)=\int_V d^dx\, e^{-ikx}[\langle|\psi_0(x)|^2|\psi_0(0)|^2\rangle-\langle|\psi_0|^2\rangle^2]\ . \tag{3.3}$$

The average in (3.3) can be performed with the statistical weight $\sim\exp(-H_\psi)$, with the usual Ginzburg-Landau Hamiltonian

$$H_\psi=\int_V d^dx(\tfrac{1}{2}r_0|\psi_0|^2+\tfrac{1}{2}|\nabla\psi_0|^2+u_0|\psi_0|^4)\ , \tag{3.4}$$

where

$$u_0=\hat{u}_0-\tfrac{1}{2}\gamma_0^2\ , \tag{3.5}$$

$$\gamma_0^2=\mathring{\gamma}_m^2\mathring{\chi}_m+\mathring{\gamma}_p^2\mathring{\chi}_p\ . \tag{3.6}$$

We shall show that the critical behavior of $\mathring{C}_{\alpha\beta}(k)$ is most easily analyzed in terms of the couplings u_0, γ_0^2, and the dimensionless ratio

$$y_0=\frac{\mathring{\gamma}_p\mathring{\chi}_p^{1/2}}{\mathring{\gamma}_m\mathring{\chi}_m^{1/2}}\ . \tag{3.7}$$

For a similar discussion see Onuki.[34] We introduce new variables $n_0(x)$ and $q_0(x)$ defined by

$$\begin{bmatrix} q_0\\ n_0\end{bmatrix}=\mathbf{R}(y_0)\begin{bmatrix} m_0\mathring{\chi}_m^{-1/2}\\ p_0\mathring{\chi}_p^{-1/2}\end{bmatrix} \tag{3.8}$$

with the orthogonal matrix

$$\mathbf{R}(y_0)=(1+y_0^2)^{-1/2}\begin{bmatrix} 1 & y_0\\ -y_0 & 1\end{bmatrix}\ . \tag{3.9}$$

Rewriting H in terms of n_0 and q_0 yields

$$H=H_q+\int_V d^dx(\tfrac{1}{2}n_0^2+\tfrac{1}{2}\mathring{\chi}_j^{-1}\mathbf{j}_0^2-\mathring{h}_n n_0) \tag{3.10}$$

with the model-F-type Hamiltonian[21]

$$H_q=\int_V d^dx(\tfrac{1}{2}\hat{r}_0|\psi_0|^2+\tfrac{1}{2}|\nabla\psi_0|^2+\hat{u}_0|\psi_0|^4+\tfrac{1}{2}q_0^2+\gamma_0 q_0|\psi_0|^2-\mathring{h}_q q_0)\ , \tag{3.11}$$

where

$$\begin{bmatrix} \mathring{h}_q\\ \mathring{h}_n\end{bmatrix}=\mathbf{R}(y_0)\begin{bmatrix} \mathring{h}_m\mathring{\chi}_m^{1/2}\\ \mathring{h}_p\mathring{\chi}_p^{1/2}\end{bmatrix}\ . \tag{3.12}$$

Thus $n_0(x)$ has only Gaussian fluctuations, while $q_0(x)$ is coupled to the order parameter through (3.6). This implies

$$\mathring{C}_{nn}=1,\ \mathring{C}_{nq}=\mathring{C}_{qn}=0 \tag{3.13}$$

and

$$\mathring{C}_{qq}(k)=1+\gamma_0^2\mathring{C}_\psi(k)\ . \tag{3.14}$$

$\mathring{C}_{\alpha\beta}$ can be expressed in terms of $\mathring{C}_{qq}$ via the matrix relation

$$\begin{bmatrix} \mathring{\chi}_m^{-1}\mathring{C}_{mm} & (\mathring{\chi}_m\mathring{\chi}_p)^{-1/2}\mathring{C}_{mp}\\ (\mathring{\chi}_p\mathring{\chi}_m)^{-1/2}\mathring{C}_{pm} & \mathring{\chi}_p^{-1}\mathring{C}_{pp}\end{bmatrix}=\mathbf{R}^T(y_0)\begin{bmatrix} \mathring{C}_{qq} & 0\\ 0 & 1\end{bmatrix}\mathbf{R}(y_0)\ . \tag{3.15}$$

(In (3.15)–(3.18) the coefficients $\mathring{\chi}_m$ and $\mathring{\chi}_p$ are given explicitly, i.e., $\mathring{C}_{mm}=\langle m_0m_0\rangle-\langle m_0\rangle^2$, etc.)

After substitution of (3.14) and (3.9), Eq. (3.15) clearly exhibits the separation between the simple algebraic dependence on y_0 and γ_0^2 and the u_0 dependence via $\mathring{C}_\psi$. The crucial point of the representation (3.15) is that the ratio and the matrix $\mathbf{R}(y_0)$ constitute noncritical quantities (even in the renormalized theory, see Ref. 6) determined by the thermodynamic derivative (2.90) parallel to the λ line. Consequently the static critical behavior of $\mathring{C}_{\alpha\beta}$ is reduced to that of a single model-F-type correlation function $\mathring{C}_{qq}$. Their asymptotic and nonasymptotic critical properties at $k=0$ are well known.[21,22]

For completeness we mention the invariance of the trace and determinant of the matrix $(\mathring{C}_{\alpha\beta})$ under the orthogonal transformation (3.15),

$$\mathring{C}_{qq}+1=\mathring{\chi}_m^{-1}\mathring{C}_{mm}+\mathring{\chi}_p^{-1}\mathring{C}_{pp}\ , \tag{3.16}$$

$$\mathring{C}_{qq}=\mathring{\chi}_m^{-1}\mathring{\chi}_p^{-1}(\mathring{C}_{mm}\mathring{C}_{pp}-\mathring{C}_{mp}^2)\ . \tag{3.17}$$

The relations derived above can be applied to the static structure factor

$$\mathring{C}_{\rho\rho}(k)=b_\rho^{-2}[\mathring{C}_{pp}(k)+2b_m\mathring{C}_{pm}(k)+b_m^2\mathring{C}_{mm}(k)] \tag{3.18}$$

[compare (2.65)] and, in the limit $k\to 0$, to various thermodynamic quantities (see Ref. 6).

IV. DYNAMIC CORRELATION FUNCTIONS ABOVE T_λ

Because of the complexity of our model it is worthwhile first to discuss the general formalism and to defer explicit calculations to subsequent parts of this work. The main application will be the calculation of dynamic correlation functions

$$\mathring{C}_{\alpha\beta}(k,\omega)=\int d^dx\int dt\, e^{-i(kx-\omega t)}\langle[\alpha_0(x,t)-\langle\alpha_0\rangle][\beta_0(0,0)-\langle\beta_0\rangle]\rangle \tag{4.1}$$

with α_0 and β_0 being one of the hydrodynamic variables $m_0\mathring{\chi}_m^{-1/2}$, $p_0\mathring{\chi}_p^{-1/2}$ or $\mathbf{j}_0\mathring{\chi}_j^{-1/2}$. These correlation functions suffice to determine the dynamic structure factor according to (3.18). For $T>T_\lambda$, mixed two-point correlation functions between ψ_0 (or ψ_0^*) and the variables m_0, p_0, and $\mathbf{j}_0$ vanish because of gauge symmetry. The average in (4.1) will be performed with the statistical weight $\sim\exp J$, where J is the dynamic functional[41–44] that is equivalent to the Langevin equations (2.75)–(2.79). It is given explicitly in Appendix A. The definition of response functions $C_{\tilde{\alpha}\beta}(k,\omega)$ is analogous to (4.1) with $\tilde{\alpha}_0$ denoting one of the response fields[45] $\tilde{m}_0\mathring{\chi}_m^{1/2}$, $\tilde{p}_0\mathring{\chi}_p^{1/2}$, or $\tilde{\mathrm{j}}_0\mathring{\chi}_j^{1/2}$.

A. General form

We shall use the matrix notation

$$\underline{\mathring{C}}(k,\omega)=\begin{bmatrix}[\mathring{C}_{\alpha\beta}(k,\omega)] & [\mathring{C}_{\tilde{\alpha}\beta}(k,\omega)]\\ [\mathring{C}_{\alpha\tilde{\beta}}(k,\omega)] & 0\end{bmatrix}, \tag{4.2}$$

where $[\mathring{C}_{\alpha\beta}]$ and $\underline{\mathring{C}}$ are 3×3 and 6×6 matrices, respectively. From (4.1) we have

$$\mathring{C}_{\beta\tilde{\alpha}}(k,\omega)=\mathring{C}_{\tilde{\alpha}\beta}(-k,-\omega)=\mathring{C}^{\,*}_{\tilde{\alpha}\beta}(k,\omega)\ .$$

In the spirit of perturbation theory we split $\underline{\mathring{C}}$ as

$$\underline{\mathring{C}}(k,\omega)=\underline{\mathring{G}}(k,\omega)+\underline{\mathring{Y}}(k,\omega)\ , \tag{4.3}$$

where $\underline{\mathring{Y}}$ represents the contribution due to the nonlinear couplings $\hat{u}_0$, $\mathring{\gamma}_m$, $\mathring{\gamma}_p$, $\mathring{g}_m$, and $\mathring{g}_p$. The zeroth-order part $\underline{\mathring{G}}$ is well known from linearized hydrodynamics of ordinary fluids and is given by

$$\underline{\mathring{G}}(k,\omega)=-\underline{\mathring{M}}^{-1}(-k,-\omega) \tag{4.4}$$

with

$$\underline{\mathring{M}}(k,\omega)=\begin{bmatrix}0 & -i\omega-[\mathring{L}(k)]^{\dagger}\\ i\omega-[\mathring{L}(k)] & [2\mathring{\lambda}(k)]\end{bmatrix}, \tag{4.5}$$

and

$$[\mathring{L}(k)]=\begin{bmatrix}\mathring{\lambda}_m\mathring{\chi}_m^{-1}k^2 & L_0(\mathring{\chi}_m\mathring{\chi}_p)^{-1/2}k^2 & 0\\ L_0(\mathring{\chi}_m\mathring{\chi}_p)^{-1/2}k^2 & \mathring{\lambda}_p\mathring{\chi}_p^{-1}k^2 & ic_0(\mathring{\chi}_p\mathring{\chi}_j)^{-1/2}k\\ 0 & ic_0(\mathring{\chi}_j\mathring{\chi}_p)^{-1/2}k & \mathring{\lambda}_j\mathring{\chi}_j^{-1}k^2\end{bmatrix}. \tag{4.6}$$

In (4.5) $[\mathring{\lambda}(k)]$ denotes a 3×3 matrix that is identical with (4.6) for $c_0\equiv0$, i.e., $[\mathring{\lambda}(k)]$ is purely dissipative, whereas the matrix elements $\sim ic_0k$ of $[\mathring{L}(k)]$ reflect the two propagating sound modes. The transverse viscosity modes have been omitted. In Appendix A it is shown that the perturbation part of (4.3) can be written as

$$\underline{\mathring{Y}}(k,\omega)=\underline{\mathring{G}}(k,\omega)\underline{\mathring{\Pi}}(k,\omega)\underline{\mathring{G}}(k,\omega) \tag{4.7}$$

with

$$\underline{\mathring{\Pi}}(k,\omega)=\begin{bmatrix}0 & [\mathring{C}_{\mu\tilde{\nu}}(k,\gamma)]\\ [\mathring{C}_{\tilde{\mu}\nu}(k,\gamma)] & [\mathring{C}_{\mu\nu}(k,\omega)]\end{bmatrix}. \tag{4.8}$$

Here $\mathring{C}_{\mu\nu}$ represent correlation functions whose definition is analogous to (4.1) but with μ and ν denoting one of the composite fields

$$\mathring{s}_m(x,t)=\mathring{\chi}_m^{-1/2}[(\mathring{\lambda}_m\mathring{\gamma}_m+L_0\mathring{\gamma}_p)\nabla^2|\psi_0|^2+\mathring{g}_m\,\mathrm{Im}(\psi_0^*\nabla^2\psi_0)]\ , \tag{4.9}$$

$$\mathring{s}_p(x,t)=\mathring{\chi}_p^{-1/2}[(L_0\mathring{\gamma}_m+\mathring{\lambda}_p\mathring{\gamma}_p)\nabla^2|\psi_0|^2-\mathring{g}_p\,\mathrm{Im}(\psi_0^*\nabla^2\psi_0)]\ , \tag{4.10}$$

$$\mathring{\mathbf{s}}_j(x,t)=-\mathring{\chi}_j^{-1/2}c_0\mathring{\gamma}_p\nabla|\psi_0|^2\ . \tag{4.11}$$

Similarly $\mathring{C}_{\tilde{\mu}\nu}$ and $\mathring{C}_{\mu\tilde{\nu}}$ are response functions where $\tilde{\mu}$, $\tilde{\nu}$ denote the composite fields

$$\tilde{s}_m(x,t)=2\mathring{\chi}_m^{1/2}\mathrm{Re}[-(\mathring{\gamma}_m\Gamma_0^*+\frac{i}{2}\mathring{g}_m\mathring{\chi}_m^{-1})\tilde{\psi}_0\psi_0^*]\ , \tag{4.12}$$

$$\tilde{s}_p(x,t)=2\mathring{\chi}_p^{1/2}\mathrm{Re}[-(\mathring{\gamma}_p\Gamma_0^*-\frac{i}{2}\mathring{g}_p\mathring{\chi}_p^{-1})\tilde{\psi}_0\psi_0^*]\ , \tag{4.13}$$

$$\tilde{\mathbf{s}}_j=0\ . \tag{4.14}$$

In conclusion, according to (4.3)–(4.8) the problem of calculating $\mathring{C}_{\alpha\beta}$ is transferred to the calculation of $\mathring{C}_{\mu\nu}$ and $\mathring{C}_{\tilde{\mu}\nu}$. As seen from (4.9)–(4.14) the composite fields do not involve the hydrodynamic variables m_0, p_0, j_0, $\tilde{m}_0$, $\tilde{p}_0$, and $\tilde{\mathrm{j}}_0$. In this respect the structure of (4.3) and (4.7) is parallel to the representation of the static correlation functions $\mathring{C}_{\alpha\beta}$ in terms of the static four-point order-parameter correlation function $\mathring{C}_{\psi}$, compare (3.2) and (3.15) of Sec. III and (2.10) of Ref. 21. The advantage of this representation is that it exhibits the structure of the correlation functions as far as the leading dependence on the nonlinear coupling constants $\mathring{\gamma}_m$, $\mathring{\gamma}_p$, $\mathring{g}_m$, $\mathring{g}_p$ is concerned.

For the purpose of explicit perturbation calculations it is convenient to introduce the matrix of two-point vertex functions

$$\underline{\mathring{\Gamma}}(k,\omega)=\underline{\mathring{C}}(-k,-\omega)^{-1}=\begin{bmatrix}0 & [\mathring{\Gamma}_{\tilde{\alpha}\beta}(k,\omega)]\\ [\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)] & [\mathring{\Gamma}_{\tilde{\alpha}\tilde{\beta}}(k,\omega)]\end{bmatrix}, \tag{4.15}$$

which in zeroth order coincides with $-\underline{\mathring{M}}(k,\omega)$, (4.5), thus

$$\underline{\mathring{\Gamma}}(k,\omega)=-\underline{\mathring{M}}(k,\omega)-\underline{\mathring{\Sigma}}(k,\omega)\ . \tag{4.16}$$

The contribution $\underline{\mathring{\Sigma}}$ is given by all one-particle irreducible diagrams with two external (truncated) legs. According to (4.2) and (4.15) the 3×3 matrix of correlation functions $\mathring{C}_{\alpha\beta}$ can be expressed as

$$[\mathring{C}_{\alpha\beta}(k,\omega)]=-[\mathring{\Gamma}_{\alpha\tilde{\beta}}(k,\omega)]^{-1}[\mathring{\Gamma}_{\tilde{\alpha}\tilde{\beta}}(k,\omega)][\mathring{\Gamma}_{\tilde{\alpha}\beta}(k,\omega)]^{-1}\ . \tag{4.17}$$

For the application to the dynamic structure factor it is more convenient to rewrite the matrix elements of (4.17) as

$$\mathring{C}_{\alpha\beta}(k,\omega)=-\frac{N_{\alpha\beta}(k,\omega)}{|\mathring{\Delta}(k,\omega)|^2}\,, \tag{4.18}$$

where $\mathring{\Delta}(k,\omega)$ is the determinant of the 3×3 matrix $[\mathring{\Gamma}_{\alpha\beta}(k,\omega)]$.

B. Fluctuation-dissipation relations

We shall apply well-known fluctuation-dissipation relations in order to express the correlation functions (4.1) in terms of response functions. One relation reads[46,47]

$$\mathring{R}_{\alpha\beta}(x-x',t-t')=-\theta(t-t')\frac{d}{dt}\mathring{C}_{\alpha\beta}(x-x',t-t')\,, \tag{4.19}$$

where $\mathring{R}_{\alpha\beta}(x-x',t-t')$ describes the linear response of $\langle\alpha_0(x,t)\rangle$ to an external field $\mathring{h}_\beta(x',t')$ coupled linearly to $\beta_0(x',t')$ via the additive term

$$-\int d^dx'\mathring{h}_\beta(x',t')\beta_0(x',t')$$

in the Hamiltonian (2.79). In terms of Fourier transforms (4.19) becomes

$$\mathring{C}_{\alpha\beta}(k,\omega)=\frac{1}{i\omega}[\mathring{R}_{\alpha\beta}(k,\omega)-\mathring{R}_{\beta\alpha}(-k,-\omega)]\,. \tag{4.20}$$

Note that in the present model

$$\mathring{R}_{\alpha\beta}(k,\omega)\neq\mathring{R}_{\beta\alpha}(k,\omega)$$

due to the reversible couplings $\mathring{g}_m$, $\mathring{g}_p$. Therefore $\mathring{C}_{\alpha\beta}(k,\omega)$ is complex for $\alpha\neq\beta$, whereas

$$\mathring{C}_{\alpha\alpha}(k,\omega)=\mathring{C}_{\alpha\alpha}(k,-\omega)=\frac{2}{\omega}\mathrm{Im}\mathring{R}_{\alpha\alpha}(k,\omega) \tag{4.21}$$

is real. A further relation can be given in terms of response functions $\mathring{C}_{\alpha\gamma}(k,\omega)$ that describe the linear response of $\langle\alpha_0\rangle$ to an external force added to the equation of motion for $\gamma=m_0\mathring{\chi}_m^{-1/2}$, $p_0\mathring{\chi}_p^{-1/2}$, $j_0\mathring{\chi}_j^{-1/2}$ (Refs. 43, 44, and 47). In the present model we obtain

$$\mathring{R}_{\alpha\beta}(k,\omega)=\mathring{C}_{\alpha\tilde{\gamma}}(k,\omega)\mathring{L}_{\gamma\beta}(k)+2\mathring{g}_\beta\mathring{C}_{\alpha\tilde{\phi}}(k,\omega)\,, \tag{4.22}$$

where summation over γ is implied. The matri elements $\mathring{L}_{\gamma\beta}(k)$ are given in (4.6). In the last term the index $\tilde{\phi}$ stands for the composite field

$$\tilde{\phi}_0(x,t)=\mathrm{Im}[\psi_0(x,t)\tilde{\psi}_0(x,t)^*]\,, \tag{4.23}$$

which arises from the terms $\sim\mathring{g}_\beta\mathring{h}_\beta$ in J_ψ, (A 3). $\mathring{C}_{\alpha\tilde{\phi}}$ can be decomposed as

$$\mathring{C}_{\alpha\tilde{\phi}}(k,\omega)=\mathring{C}_{\alpha\tilde{\gamma}}(k,\omega)\mathring{\Gamma}_{\tilde{\gamma}\tilde{\phi}}(k,\omega)\,, \tag{4.24}$$

where again summation over $\tilde{\gamma}$ is implied. The vertex functions $\mathring{\Gamma}_{\tilde{\gamma}\tilde{\phi}}(k,\omega)$ will be specified diagrammatically in Ref. 6.

For the application in Ref. 6 we shall also consider

$$\mathring{C}_\psi(k,\omega)=\int d^dx\int dt\, e^{-i(kx-\omega t)}\langle\mathring{\Delta}_\psi(x,t)\rangle$$
$$=\frac{2}{\omega}\mathrm{Im}\mathring{R}_\psi(k,\omega)\,, \tag{4.25}$$

where

$$\mathring{\Delta}_\psi(x,t)\equiv|\psi_0(x,t)|^2|\psi_0(0,0)|^2-\langle|\psi_0|^2\rangle^2 \tag{4.26}$$

and

$$\mathring{R}_\psi(k,\omega)=-\int d^dx\int dt\, e^{-i(kx-\omega t)}\theta(t)\frac{d}{dt}\langle\mathring{\Delta}_\psi(x,t)\rangle\,. \tag{4.27}$$

From a dissipation-fluctuation theorem applied to $\mathring{R}_\psi$ a relation corresponding to (4.22) can be obtained. It reads

$$\mathring{R}_\psi(k,\omega)=\int d^dx\int dt\, e^{-i(kx-\omega t)}\langle|\psi_0(x,t)|^2\tilde{\pi}_0(0,0)\rangle \tag{4.28}$$

with

$$\tilde{\pi}_0(x,t)=\Gamma_0\psi_0(x,t)\tilde{\psi}_0(x,t)^*+\Gamma_0^*\psi_0(x,t)^*\tilde{\psi}_0(x,t)\,. \tag{4.29}$$

The connection with the static correlation functions (3.1) and (3.3) is given by the sum rules

$$\mathring{C}_{\alpha\beta}(k)=(2\pi)^{-1}\int_{-\infty}^{\infty}d\omega\, C_{\alpha\beta}(k,\omega)=\mathring{R}_{\alpha\beta}(k,0)\,, \tag{4.30}$$

$$\mathring{C}_\psi(k)=(2\pi)^{-1}\int_{-\infty}^{\infty}d\omega\,\mathring{C}_\psi(k,\omega)=\mathring{R}_\psi(k,0)\,, \tag{4.31}$$

thus, according to (3.2),

$$\mathring{C}_{\alpha\beta}(k)=\delta_{\alpha\beta}+\mathring{\gamma}_\alpha\mathring{\gamma}_\beta\mathring{R}_\psi(k,0)\,. \tag{4.32}$$

Therefore one may regard the response function $\mathring{R}_\psi(k,\omega)$ as a k and ω dependent generalization of the specific heat. This will be further discussed in Ref. 6. In the limit $\omega\to 0$ we obtain from (4.30), (4.22), and (4.24)

$$\mathring{C}_{\alpha\beta}(k)=\mathring{C}_{\alpha\tilde{\gamma}}(k,0)[\mathring{L}_{\gamma\beta}(k)+2\mathring{g}_\beta\mathring{\Gamma}_{\tilde{\gamma}\tilde{\phi}}(k,0)]\,. \tag{4.33}$$

C. Relation to model F

It is obvious that our model equations (2.75)–(2.79) contain model F (Ref. 20) as a special case. In a formal fashion this is seen by dropping p_0 and j_0 or by setting $\mathring{g}_p=\mathring{\gamma}_p=L_0=0$. A more physical way to describe the relation to model F is the following.

The basic time scale of the propagating modes is set by the inverse frequency $[c_0(\mathring{\chi}_j\mathring{\chi}_p)^{-1/2}k]^{-1}$, where $c_0(\mathring{\chi}_j\mathring{\chi}_p)^{-1/2}$ is essentially the velocity of sound. This time scale is short compared to the characteristic time scale of heat diffusion, $(\mathring{\lambda}_m\mathring{\chi}_m^{-1}k^2)^{-1}$, for all wave numbers k well below the cutoff $\Lambda\sim 10^8$ cm^{-1}. The crucial point is that the sound velocity remains infinite at T_λ, as noted previously.[48] Therefore the sound modes can be considered as fast modes even at T_λ, which in the limit of small k do not influence the asymptotic critical behavior of the heat mode. The effect of the sound modes consists essentially of a modification of the background values of the couplings and the transport coefficients appearing in model F. Our model is an appropriate basis for proving the correctness of these qualitative remarks.

Here we do not attempt to provide such a proof but confine ourselves to specifying the relation between model F and our model in the limit $c_0 \to \infty$.[10,11] According to Eqs. (4.3) and (4.7), the effect of the critical fluctuations on the correlation functions (4.1) is described by the quantity $\mathring{\underline{\Pi}}$ defined in Eq. (4.8). As noted after Eq. (4.14), $\underline{\Pi}$ does not explicitly involve the hydrodynamic variables, except via the couplings to ψ_0 and ψ_0^* in the statistical weight $\exp J$ of the averages $\langle \cdots \rangle$. A simple way of eliminating the sound modes consists of a partial average in $\mathring{C}_{\mu\nu}$ and $\mathring{C}_{\mu\tilde{\nu}}$ over the variables p_0, $\tilde{p}_0$, $\mathbf{j}_0$, $\tilde{\mathbf{j}}_0$, and then performing the limit $c_0 \to \infty$. These steps can be carried out explicitly (see Appendix B).[49] In this limit p_0, $\tilde{p}_0$, $\mathbf{j}_0$, and $\tilde{\mathbf{j}}_0$ become nonhydrodynamic fast variables. This procedure leaves the resulting effective dynamic functional J_{eff} free of hydrodynamic singularities (which would otherwise arise if the integration is carried out at finite c_0). The reduced functional J_{eff} depends only on the slow model F variables m_0, $\tilde{m}_0$ and ψ_0, ψ_0^*, $\tilde{\psi}_0$, $\tilde{\psi}_0^*$. As shown in Appendix B, J_{eff} has indeed the structure of model F (with $\mathring{h}_m = 0$)

$$J_{\text{eff}} = \int dt \int d^dx \, (-\mathring{\lambda}_m \tilde{m}_0 \nabla^2 \tilde{m}_0 + \Gamma_0' \tilde{\psi}_0 \tilde{\psi}_0^* - \tilde{m}_0 [(\partial_t - \mathring{\lambda}_m \mathring{\chi}_m^{-1} \nabla^2) m_0 - \mathring{g}_m \operatorname{Im}(\psi_0^* \nabla^2 \psi_0)]$$
$$- \operatorname{Re}\{ \tilde{\psi}_0^* [\partial_t + \Gamma_0 (r_0 - \nabla^2 + 4\tilde{u}_0 |\psi_0|^2 + 2\mathring{\gamma}_m m_0)] \psi_0 - i\mathring{g}_m \tilde{\psi}_0^* \psi_0 (m_0 \mathring{\chi}_m^{-1} + \mathring{\gamma}_m |\psi_0|^2) \}) \,. \tag{4.34}$$

Here $\mathring{g}_p$, $\mathring{\lambda}_p$, and L_0 have dropped out because of $c_0 \to \infty$, whereas $\mathring{\gamma}_p$ has been absorbed in the effective four-point coupling

$$\tilde{u}_0 = \hat{u}_0 - \tfrac{1}{2} \mathring{\gamma}_p^2 \mathring{\chi}_p \,. \tag{4.35}$$

This connection with model F will be exploited in the subsequent paper.[6]

On the other hand the preceding discussion clearly shows that there must exist corrections to model F calculations due to the finiteness of the sound velocity. These corrections will affect the precritical dynamic behavior and may be of relevance in a quantitative comparison of second-sound damping with experiments. A systematic study of such corrections to model F can be carried out on the basis of our model.

V. OUTLOOK

The main result of this paper is the derivation of the model equations (2.75)–(2.79), which we believe are appropriate for the description of the critical thermal-diffusion and first-sound modes in the vicinity of the λ transition in ^{4}He. This opens the possibility of treating a number of problems:

(1) Our principal aim is to provide a systematic and quantitative renormalization-group analysis of the sound mode. This will be the subject of subsequent papers.[6]

(2) A study of the relation between our complete model and model F enables us to investigate the effects of the sound mode on the heat mode above T_λ and the second-sound mode below T_λ. This may be of relevance in the context of a previous controversy[50] that is still unresolved. In addition the unexpectedly large background value of the transport coefficient Γ_0'' (Refs. 25 and 37) could be modified if the sound mode is correctly included in the analysis.

(3) The complete light-scattering spectrum can be calculated. As noted recently[3] this will provide the basis for a *dynamic* calculation of the Landau-Placzek ratio, which seems necessary in view of the discrepancy between light-scattering experiments[51] and *static* renormalization-group calculations at finite k (Refs. 3, 26, and 52).

(4) Our model constitutes the appropriate basis for an analysis of the approximations and the range of validity of the phenomenological approach by Ferrell and Bhattacharjee.[7]

(5) From a purely formal point of view, our model contains (for the case $c_0 = 0$) the model-F-type generalization of the Siggia-Nelson model for ^{3}He-^{4}He mixtures.[40,53] Therefore part of the results of the present and subsequent papers[6] are directly applicable to the statics and low-frequency dynamics of ^{3}He-^{4}He mixtures.

APPENDIX A

In this Appendix we work with the dynamic functional J (Refs. 41–44) that is equivalent to the Langevin equations (2.75)–(2.79). The functional reads

$$J = J_\alpha + J_\psi \,, \tag{A1}$$

where

$$J_\alpha = \int d^dx \int dt \left[\frac{1}{2} \begin{pmatrix} \boldsymbol{\alpha}_0 \\ \tilde{\boldsymbol{\alpha}}_0 \end{pmatrix} \underline{\mathring{M}} \begin{pmatrix} \boldsymbol{\alpha}_0 \\ \tilde{\boldsymbol{\alpha}}_0 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\alpha}_0 \\ \tilde{\boldsymbol{\alpha}}_0 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{s}}_0 \\ \mathbf{s}_0 \end{pmatrix} \right] , \tag{A2}$$

and

$$J_\psi = \int d^dx \int dt \, \operatorname{Re}\{ \Gamma_0' \tilde{\psi}_0 \tilde{\psi}_0^* - \tilde{\psi}_0 [\partial_t + \Gamma_0^* (r_0 - \nabla^2 + 4\hat{u}_0 |\psi_0|^2)] \psi_0^* + \tilde{\psi}_0 [i \mathring{\mathbf{g}} \cdot \mathring{\mathbf{h}} - i \mathring{\mathbf{g}} \cdot \mathring{\boldsymbol{\gamma}} |\psi_0|^2] \psi_0^* \} \,, \tag{A3}$$

with the three-component vectors

$$\boldsymbol{\alpha}_0 = \begin{pmatrix} m_0 \mathring{\chi}_m^{-1/2} \\ p_0 \mathring{\chi}_p^{-1/2} \\ \mathbf{j}_0 \mathring{\chi}_j^{-1/2} \end{pmatrix} ; \quad \tilde{\boldsymbol{\alpha}}_0 = \begin{pmatrix} \tilde{m}_0 \mathring{\chi}_m^{1/2} \\ \tilde{p}_0 \mathring{\chi}_p^{1/2} \\ \tilde{\mathbf{j}}_0 \mathring{\chi}_j^{1/2} \end{pmatrix} , \tag{A4}$$

$$\mathbf{s}_0=\begin{bmatrix}\mathring{s}_m\\ \mathring{s}_p\\ \mathring{\mathbf{s}}_j\end{bmatrix};\quad \tilde{\mathbf{s}}_0=\begin{bmatrix}\mathring{\tilde{s}}_m\\ \mathring{\tilde{s}}_p\\ \mathring{\tilde{\mathbf{s}}}_j\end{bmatrix}, \tag{A5}$$

$$\mathring{\gamma}=\begin{bmatrix}\mathring{\gamma}_m\mathring{\chi}_m^{1/2}\\ \mathring{\gamma}_p\mathring{\chi}_p^{1/2}\\ 0\end{bmatrix},\quad \mathring{\mathbf{g}}=\begin{bmatrix}\mathring{g}_m\mathring{\chi}_m^{-1/2}\\ -\mathring{g}_p\mathring{\chi}_p^{-1/2}\\ 0\end{bmatrix},\quad \mathring{\mathbf{h}}=\begin{bmatrix}\mathring{h}_m\mathring{\chi}_m^{1/2}\\ \mathring{h}_p\mathring{\chi}_p^{1/2}\\ 0\end{bmatrix}, \tag{A6}$$

and the 6×6 matrix $\underline{\mathring{M}}$ and 3×3 matrix $[\mathring{L}]$, respectively

$$\underline{\mathring{M}}=\begin{bmatrix}0 & (-\partial_t-[\mathring{L}])^\dagger\\ -\partial_t-[\mathring{L}] & [2\mathring{\lambda}]\end{bmatrix}, \tag{A7}$$

$$[\mathring{L}]=\begin{bmatrix}-\mathring{\lambda}_m\mathring{\chi}_m^{-1}\nabla^2 & -L_0(\mathring{\chi}_m\mathring{\chi}_p)^{-1/2}\nabla^2 & 0\\ -L_0(\mathring{\chi}_m\mathring{\chi}_p)^{-1/2}\nabla^2 & -\mathring{\lambda}_p\mathring{\chi}_p^{-1}\nabla^2 & c_0(\mathring{\chi}_j\mathring{\chi}_p)^{-1/2}\nabla\\ 0 & c_0(\mathring{\chi}_j\mathring{\chi}_p)^{-1/2}\nabla & -\mathring{\lambda}_j\mathring{\chi}_j^{-1}\nabla^2\end{bmatrix}. \tag{A8}$$

The components of (A5) are given in (4.9)–(4.14). In (A7) the differential operator $(-\partial_t-[\mathring{L}])^\dagger$ is the adjoint operator of $(-\partial_t-[\mathring{L}])$, i.e., it acts on the vector on its left. $[\mathring{\lambda}]$ in (A7) is given by (A8) with $c_0\equiv0$.

Since the fields $\boldsymbol{\alpha}_0,\tilde{\boldsymbol{\alpha}}_0$ enter J only up to second order, some exact relations can be derived by an appropriate shift of the variables. We need these relations in order to derive (4.7)–(4.14). In terms of the six-component vectors

$$\mathbf{T}_0=\begin{bmatrix}\tilde{\mathbf{s}}_0\\ \mathbf{s}_0\end{bmatrix},\quad \mathbf{q}_0=\begin{bmatrix}\boldsymbol{\alpha}_0\\ \tilde{\boldsymbol{\alpha}}_0\end{bmatrix}+\underline{\mathring{M}}^{-1}\mathbf{T}_0\ . \tag{A9}$$

Equation (A2) can be rewritten as

$$J_\alpha=\int d^dx\int dt(\tfrac{1}{2}\mathbf{q}_0\underline{\mathring{M}}\mathbf{q}_0-\tfrac{1}{2}\mathbf{T}_0\underline{\mathring{M}}\mathbf{T}_0)\ . \tag{A10}$$

This implies

$$\langle\mathbf{q}_0\rangle=0\ , \tag{A11}$$

$$\langle q_{0i}T_{0j}\rangle=0\ , \tag{A12}$$

$$\int d^dx\int dt\, e^{-i(kx-\omega t)}\langle q_{0i}(x,t)q_{0j}(0,0)\rangle =-\mathring{M}_{ij}^{-1}(k,\omega)^*=-\mathring{M}_{ij}^{-1}(-k,-\omega)\ . \tag{A13}$$

In (A13) $\underline{\mathring{M}}(k,\omega)$ is the Fourier transform of (A7), as given in (4.5) and (4.6). From (A11)–(A13) one obtains (4.3)–(4.14).

APPENDIX B

In this Appendix we derive the model-F dynamic functional from the complete functional J, (A1), by integrating over the pressure and momentum fields p_0, $\tilde{p}_0$, $\mathbf{j}_0$, $\tilde{\mathbf{j}}_0$, and passing to the limit $c_0\rightarrow\infty$. We consider only the case $\mathring{h}_m=\mathring{h}_p=0$. The functional J can be split as

$$J=J_\psi+J_m+J_y\ , \tag{B1}$$

with J_ψ given in (A3), and

$$\begin{aligned}J_m=\int d^dx\int dt\{&[-\tilde{m}_0\lambda_m\nabla^2\tilde{m}_0-\tilde{m}_0(\partial_t-\mathring{\lambda}_m\mathring{\chi}_m^{-1}\nabla^2)m_0\\ &+\tilde{m}_0[\nabla^2(\mathring{\gamma}_m\mathring{\lambda}_m+\mathring{\gamma}_pL_0)|\psi_0|^2-\frac{i}{2}\mathring{g}_m(\psi_0^*\nabla^2\psi_0-\psi_0\nabla^2\psi_0^*)]\\ &-m_0[\mathring{\gamma}_m\Gamma_0^*\tilde{\psi}_0\psi_0^*+\mathring{\gamma}_m\Gamma_0\tilde{\psi}_0^*\psi_0+\frac{i}{2}\mathring{g}_m\mathring{\chi}_m^{-1}(\tilde{\psi}_0\psi_0^*-\tilde{\psi}_0^*\psi_0)]\}\ ,\end{aligned} \tag{B2}$$

$$J_y=\int d^dx\int dt\tfrac{1}{2}(\mathbf{y}_0\underline{A}_0\mathbf{y}_0-\mathbf{x}_0\underline{A}_0^{-1}\mathbf{x}_0)\ . \tag{B3}$$

Here $\mathbf{x}_0$ and $\mathbf{y}_0$ denote four-component vectors

$$\mathbf{x}_0=\begin{bmatrix}(L_0\mathring{\gamma}_m+\mathring{\lambda}_p\mathring{\gamma}_p)\nabla^2|\psi_0|^2-\mathring{g}_p\mathrm{Im}(\psi_0^*\nabla^2\psi_0)+L_0\nabla^2(\mathring{\chi}_m^{-1}m_0-2\tilde{m}_0)\\ c_0\mathring{\gamma}_p\nabla|\psi_0|^2\\ -\mathrm{Re}[(\mathring{\gamma}_p\Gamma_0^*-\frac{i}{2}\mathring{g}_p\mathring{\chi}_p^{-1})\tilde{\psi}_0\psi_0^*]+L_0\nabla^2\tilde{m}_0\mathring{\chi}_p^{-1}\\ 0\end{bmatrix}, \tag{B4}$$

$$\mathbf{y}_0 = \begin{bmatrix} \tilde{p}_0 \\ \tilde{\mathbf{j}}_0 \\ p_0 \\ \mathbf{j}_0 \end{bmatrix} + \underline{A}_0^{-1} \mathbf{x}_0 \; . \tag{B5}$$

The 4×4 matrices $\underline{A}_0$ and $\underline{A}_0^{-1}$ have the form

$$\underline{A}_0 = \begin{bmatrix} \underline{\beta} & \underline{B} \\ \underline{B}^\dagger & 0 \end{bmatrix}, \quad \underline{A}_0^{-1} = \begin{bmatrix} 0 & (\underline{B}^\dagger)^{-1} \\ \underline{B}^{-1} & -\underline{B}^{-1} \beta (\underline{B}^\dagger)^{-1} \end{bmatrix} , \tag{B6}$$

where $\underline{\beta}$ and $\underline{B}$ represent the 2×2 matrices

$$\underline{\beta} = \begin{bmatrix} -2\mathring{\lambda}_p \nabla^2 & 0 \\ 0 & -2\mathring{\lambda}_j \nabla^2 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} -\partial_t + \mathring{\lambda}_p \mathring{\chi}_p^{-1} \nabla^2 & c_0 \mathring{\chi}_j^{-1} \nabla \\ c_0 \mathring{\chi}_p^{-1} \nabla & -\partial_t + \mathring{\lambda}_j \mathring{\chi}_j^{-1} \nabla^2 \end{bmatrix} . \tag{B7}$$

The functional integration over p_0, $\mathbf{j}_0$, $\tilde{p}_0$, $\tilde{\mathbf{j}}_0$ in the distribution $\sim \exp J$ is equivalent to the integration over $\mathbf{y}_0$. Thus the first term on the rhs of (B3) does not contribute to the correlation and response functions in $\mathring{\text{II}}$, Eq. (4.8). In the limit $c_0 \to \infty$ the second part of (B3) reads

$$\lim_{c_0 \to \infty} J_y = \int d^d x \int dt \, \mathring{\gamma}_p |\psi_0|^2 \{ \mathrm{Re}[(\mathring{\gamma}_p \mathring{\chi}_p \Gamma_0^* - \tfrac{i}{2} \mathring{g}_p) \tilde{\psi}_0 \psi_0^*] - L_0 \nabla^2 \tilde{m}_0 \} \; . \tag{B8}$$

This leads to the model-F functional (4.34).

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