[20]C. Blake, C. E. Chase, and E. Maxwell, Rev. Sci. Instr. 29, 715 (1958); C. Blake and C. E. Chase, *ibid.* 34, 984 (1963).
[21]Wallace and Tiernan, Belleville, N. J.
[22]MKS Instruments, Inc., Burlington, Mass.
[23]Keithley Instruments, Cleveland, Ohio.
[24]L. Onsager, Phys. Rev. Letters 7, 50 (1961).
[25]C. N. Yang, Rev. Mod. Phys. 34, 694 (1962).
[26]J. R. Schrieffer, *Superconductivity* (Benjamin, New York, 1964), p. 256.
[27]W. E. Masker, S. Macelja, and R. D. Parks, Phys. Rev. 188, 745 (1969); W. E. Masker and R. D. Parks, Phys. Rev. B 1, 2164 (1970).
[28]P. M. Tedrow, R. Meservey, and B. B. Schwartz, Phys. Rev. Letters 24, 1004 (1970).
[29]J. E. Crow, R. S. Thompson, M. A. Klenin, and A. K. Bhatnagar, Phys. Rev. Letters 24, 371 (1970).
[30]P. M. Tedrow and R. Meservey (unpublished).

PHYSICAL REVIEW B VOLUME 4, NUMBER 3 1 AUGUST 1971

Thermal Conductivity and Ultrasonic Attenuation in Clean Type-II Superconductors

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We calculate the thermal conductivity and low-frequency ultrasonic attenuation in clean type-II superconductors in the high-field region. Use is made of a Green's function due to Brandt, Pesch, and Tewordt which enables accurate calculation of the transport properties in the vicinity of the upper critical field. It is shown that in the low-temperature limit the transport coefficients are simple functions of a single parameter $\mu = 2\sqrt{\pi}\,(\Delta/k_c v_F)^2 k_c l$, where Δ^2 is the spatial average of the square of the order parameter, k_c is the reciprocal-lattice vector of the vortex lattice, v_F is the Fermi velocity, and l is the electronic mean free path.

I. INTRODUCTION

It is now recognized that there is a remarkable difference between the dynamical properties of dirty ($l \ll \xi_0$, where $l = v_F\tau$ is the electronic mean free path and ξ_0 is the pure-superconductor coherence distance) type-II superconductors in high magnetic fields and those of clean ($l \gg \xi_0$) type-II superconductors in high magnetic fields. In the dirty case, it is found experimentally that the change in the transport coefficients (for example, the ultrasonic attenuation and thermal conductivity) near H_{c2} is proportional to $H_{c2} - B$. This behavior is readily understood in terms of gapless superconductivity.[1] On the other hand, in clean type-II superconductors the change appears[2] to scale as $(H_{c2} - B)^{1/2}$; further, even though $l \gg \xi_0$, the transport coefficients are strongly dependent on mean free path.

In this paper, we determine the transport coefficients of clean type-II superconductors near H_{c2} by a method which makes use of the single-particle propagator recently derived by Brandt *et al.*[3] The method has so far only been used to determine high-frequency response functions; for example, the ultrasonic attenuation at high frequency has been calculated by Cerdeira and Houghton,[4] and the complex conductivity in the extreme anomalous limit was determined by Hibler and Cyrot.[5] In both these cases, calculation is considerably simplified by noting that the energy integral over the product of Green's functions can be well approximated by essentially a product of the densities of initial and final states. However, as yet, there has been no experimental work under these conditions, and, therefore, no direct check on the validity of the calculations. In this paper, we show that it is possible to derive simple analytical expressions for the transport coefficients in the low-frequency limit. We calculate both the thermal conductivity and the ultrasonic attenuation and show that the predictions of the theory are consistent with the main features of the experimental results.

II. GENERAL FORMULATION

The response functions for clean type-II superconductors near H_{c2} can be expressed in terms of Brandt[3] Green's functions as

$$Q_{AB}(\omega_m) = T\sum_n \int \frac{d^3p}{(2\pi)^3}\, A_{p,p+q} B_{p,p+q}$$

$$\times\left(G^B(\vec{p},\omega_n)G^B(\vec{p}+\vec{q},\omega_n+\omega_m)\pm\Delta^2\int du\,\rho(u,\Omega)\times F(\vec{p},u,\omega_n)F^\dagger(\vec{p}+\vec{q},u,\omega_n+\omega_m)\right), \quad (1)$$

where

$$G^B(\vec{p},\omega_n)=\left(i\omega_n-\xi_p-\Delta^2\int_{-\infty}^{\infty}\frac{\rho(u)\,du}{(i\omega_n+\xi_p-u)}\right)^{-1}, \quad (2)$$

$$F(\vec{p},u,\omega_n)=G^B(\vec{p},\omega_n)(i\omega_n+\xi_p-u)^{-1}. \quad (3)$$

In Eq. (1), $A_{p,p+q}$ and $B_{p,p+q}$ are current operators and might be, for example, the heat current

$$\vec{j}^h_\mu=(2p_\mu/m)(2\omega_n+\omega_m) \quad (4)$$

in the case of thermal conductivity, or in the case of longitudinal ultrasonic attenuation

$$\tilde{T}_{zz}=(\tilde{\tau}_{zz}-\tfrac{1}{3}p_Fv_F\tilde{n})=(p_zv_z-\tfrac{1}{3}p_Fv_F)\tilde{n}. \quad (5)$$

The ± sign in the correlation function depends on whether or not the external perturbation breaks the time-reversal symmetry of the electronic system.[1] In particular, in the cases of thermal conductivity and sound attenuation, we have the minus sign, whereas in the cases of electromagnetic conductivity and spin correlation, we have the plus sign. In Eqs. (1)–(3), $\omega_m=2m\pi T$ and $\omega_n=(2n+1)\pi T$ are the even and odd Matsubara frequencies, respectively; $\xi_p=p^2/2m-\mu$; and the spectral function $\rho(u,\Omega)$ is given by

$$\rho(u,\Omega)=\frac{1}{\sqrt{\pi}\,k_cv_F\sin\theta}\exp\left[-\left(\frac{\alpha}{k_cv_F\sin\theta}\right)^2\right], \quad (6)$$

where $k_c=(2eB/c)^{1/2}$ is the reciprocal-lattice vector of the flux-line lattice, and θ is the angle between $\vec{p}$ and the dc magnetic field.

As we are only interested in the absorptive part of the response functions, it is convenient to recast Eq. (1) into an integral over real frequencies. Making use of standard techniques, we obtain

$$\mathrm{Im}Q_{AB}(\omega)=\mathrm{Re}\left\{\int\frac{d\omega'}{2\pi i}\int\frac{d^3p'}{(2\pi)^3}A_{p,p+q}B_{p,p+q}\times\left[\tanh\left(\frac{\omega+\omega'}{2T}\right)-\tanh\left(\frac{\omega'}{2T}\right)\right]I(\omega',\omega'+\omega)\right\}, \quad (7)$$

where

$$I(\omega',\omega+\omega)=G^B(\vec{p},\omega'-i\delta)\times[G^B(\vec{p},\omega+\omega'+i\delta)-G^B(\vec{p},\omega+\omega'-i\delta)]$$
$$\pm\Delta^2\int du\,\rho(u)F(\vec{p},u,\omega'-i\delta)\times[F^\dagger(\vec{p},u,\omega+\omega'+i\delta)-F^\dagger(\vec{p},u,\omega+\omega'-i\delta)]. \quad (8)$$

In the low-frequency limit, Eq. (6) can be further reduced to

$$\mathrm{Im}Q_{AB}(\omega)=\frac{\omega}{2T}\int d\omega'\cosh^{-2}\left(\frac{\omega'}{2T}\right)\times\int\frac{d^3p}{(2\pi)^3}A_{p,p+q}B_{p,p+q}I(\omega',\omega'). \quad (9)$$

Further, in the clean limit $l/\xi_0\gg1$, it is a trivial matter to generalize these results to include the effects of impurity scattering; we simply let $\delta\to1/2\tau$ in Eqs. (7)–(9). In general, impurity scattering also modifies the spectral function defined in Eq. (6); however, this modification only leads to corrections of higher order in ξ_0/l which can, therefore, be neglected in this treatment.

We will evaluate the correlation function by first carrying out the integral over $|p|$. The volume element in momentum space is replaced by $d^3p=N(0)(d\Omega/4\pi)d\xi_p$, where $N(0)$ is the density of states at the Fermi surface. We note that $G^B(p,\omega+i/2\tau)=G^B_I$ has a simple pole in the upper half-plane at

$$\xi_0=\omega'+\frac{i}{2\tau}+i\sqrt{\pi}\left(\frac{\Delta}{k_cv_F\sin\theta}\right)^2W(z_0), \quad (10)$$

where

$$W(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}dt, \quad (11)$$

$$z_0=\frac{\omega'+i/2\tau\pm\xi_0}{k_cv_F\sin\theta}=\frac{2(\omega'+i/2\tau)}{k_cv_F\sin\theta}+i\sqrt{\pi}\left(\frac{\Delta}{k_cv_F\sin\theta}\right)^2W(z_0). \quad (12)$$

Further, as can be seen from Eq. (11), G^B_I has a branch cut in the lower half-plane displaced by an amount $i/2\tau$ from the real axis. On the other hand, $G^B(p,\omega+i\delta)=G^B_2$ has a pole in the lower half-plane at z_0^* and a branch cut in the upper half-plane.

Let us first consider the G^BG^B term in the correlation function. The integral over the magnitude of p is evaluated by deforming the path of integration in such a way as to enclose the pole of G^B_I. In doing so, it is necessary to remember that we pass over the branch cut of G^B_2 resulting in a new G^B_2 in which $W(z)$ is replaced by $W_1(z)=W(z)-2e^{-z^2}$, which is the value of $W(z)$ on the second Riemann sheet. The integral is now given by the residue at the pole of G^B_I, plus a contribution coming from the integration around the branch cut of G^B_2, which is easily shown to be negligible if $l/\xi\gg1$. Thus, we have

$$I^{GG}_{(\omega',\omega')}=\frac{2\pi iA(z_0)}{k_cv_F\sin\theta[z_0-2\omega'-i\sqrt{\pi}\,\Delta^2W_1(z_0-i/\tau)]}, \quad (13)$$

where

$$A(z_0)=[1-i\sqrt{\pi}\,\Delta^2W'(z_0)]^{-1} \quad (14)$$

is the residue of G_I^B at the pole. In Eqs. (13) and (14), we have expressed the energies Δ, ω', and $1/\tau$ in units of $k_c v_F \sin\theta$.

In a similar way, we evaluate the FF term and obtain

$$I^{FF}_{(\omega',\omega')} = 2\mathrm{Re}\left(\frac{2\pi i \Delta^2 \tau\sqrt{\pi}\, W(z_0) A(z_0)}{k_c v_F \sin\theta[z_0 - 2\omega' - i\sqrt{\pi}\,\Delta^2 W_1(z_0 - i/\tau)]}\right), \tag{15}$$

where again all energies are measured in units of $k_c v_F \sin\theta$. Thus, the final expression for $\mathrm{Im}Q_{AB}(\omega)$ is given by

$$\mathrm{Im}Q_{AB}(\omega) = \omega N(0)\,\mathrm{Re}\left[\int \frac{d\omega'}{2T}\cosh^{-2}\left(\frac{\omega'}{2T}\right) \times \int \frac{d\Omega}{4\pi} A_{p,p+q} B_{p,p+q} I(\omega',\theta)\right], \tag{16}$$

$$I(\omega',\theta) = \frac{iA(z_0)}{k_c v_F \sin\theta}\,\frac{1 \pm 2\Delta^2\tau\sqrt{\pi}\, W(z_0)}{z_0 - 2\omega' - i\sqrt{\pi}\,\Delta^2 W_1(z_0 - i/\tau)}. \tag{17}$$

At arbitrary temperature, it appears that it is only possible to evaluate Eq. (16) numerically. However, in the low-temperature limit ($T/k_c v_F \ll 1$) further simplification is possible. In particular, as experiments on clean niobium exhibit little dependence on temperature below 4 °K, we will limit our considerations to the case of $T = 0$ °K,

At $T = 0$ °K, $(1/T)\cosh^{-2}(\omega'/2T)$ is replaced by a δ function. [We note that in the calculation of the thermal conductivity we replace $(\omega'^2/2T)\cosh^{-2}(\omega'/2T)$ by $\langle\omega'^2\rangle \times$ (δ function), where $\langle\omega'^2\rangle = \frac{1}{3}(\pi T)^2$.] Therefore, all we require is $I(\omega', 0)$ at $\omega' = 0$, which is given by

$$I(0,\theta) = \frac{A(ix_0)}{k_c v_F \sin\theta}\,\frac{1 \pm 2\sqrt{\pi}\,\Delta^2\tau W(ix_0)}{x_0 - \sqrt{\pi}\Delta^2 W_1(i(x_0 - 1/\tau))}, \tag{18}$$

where x_0 is determined from

$$x_0 = 1/\tau + \sqrt{\pi}\,\Delta^2 W(ix_0). \tag{19}$$

Making use of the approximate expression for $W(ix_0)$,[6]

$$\sqrt{\pi}\, W(ix_0) = \frac{2}{x_0 + [x_0^2 + \alpha(x_0)]^{1/2}}, \tag{20}$$

where

$$\tfrac{1}{4}\pi \le \alpha(x_0) \le 2, \tag{21}$$

it is easy to see that $I(0,\theta)$ is pure real. Further, using Eq. (20), Eq. (19) may be solved with the result

$$x_0 = \frac{1}{\tau} + \frac{2\Delta^2}{4\Delta^2 + \alpha}\left[\left(4\Delta^2 + \frac{1}{\tau^2} + \alpha\right)^{1/2} - \frac{1}{\tau}\right]. \tag{22}$$

The different terms of Eq. (18) have a simple physical interpretation. $A(ix_0)$, the density of states at the Fermi surface, can be approximated by

$$A(ix_0) = 1 - 2\Delta^2\alpha/[\alpha + 2\Delta^2(2+\alpha)]. \tag{23}$$

Since we have both $1/\tau$, $\Delta^2 \ll 1$, except when $\sin\theta = 0$, the second term in Eq. (23) may be neglected. The denominator of Eq. (18),

$$\begin{aligned} k_c v_F \sin\theta\,[x_0 - \sqrt{\pi}\,\Delta^2 W_1(i(x_0 - 1/\tau))] &= k_c v_F \sin\theta\,\{1/\tau + \sqrt{\pi}\,\Delta^2[W(ix_0) - W_1(i(x_0 - 1/\tau))]\} \\ &= k_c v_F \sin\theta\,\{1/\tau + 2\sqrt{\pi}\,\Delta^2 e^{(x_0 - 1/\tau)^2} + \sqrt{\pi}\,\Delta^2[W(ix_0) - W(i(x_0 - 1/\tau))]\} \\ &\cong k_c v_F \sin\theta\,(1/\tau + 2\sqrt{\pi}\,\Delta^2), \end{aligned} \tag{24}$$

is the inverse of the electron lifetime. The first term is the usual impurity scattering lifetime; the second is due to electron-hole scattering induced by the spatially varying order parameter. A dependence of the electron lifetime on the order parameter was first suggested by Vinen *et al.*[2] as a possible explanation of their experimental measurements of thermal conductivity and ultrasonic attenuation. In the present calculation, the physical origin and form of this dependence are deduced for the first time. Finally, the coherence factor

$$1 \pm 2\sqrt{\pi}\,\Delta^2\tau W(ix_0) = 1 \pm \frac{4\Delta^2\tau}{4\Delta^2 + \alpha}\left[\left(\frac{1}{\tau^2} + 4\Delta^2 + \alpha\right)^{1/2} - \frac{1}{\tau}\right] = 1 \pm 4\Delta^2\tau/\alpha^{1/2}. \tag{25}$$

Combining Eqs. (18) and (23)–(25), we obtain in conventional units

$$I^{\pm}(0,\theta) = \tau\,\frac{1 \pm \mu/(1-z^2)^{1/2}}{1 + \mu/(1-z^2)^{1/2}}, \tag{26}$$

where $z = \cos\theta$ and $\mu = 2\sqrt{\pi}\,(\Delta/k_c v_F)^2 k_c l$.

III. TRANSPORT COEFFICIENTS

It is now a straightforward matter to determine the low-frequency transport coefficients of a clean type-II superconductor near H_{c2}. We simply have to carry out the angular integration over $I(z)$ weighted by the simple angle-dependent functions contained in the various current operators. In this paper we will determine the thermal conductivity κ when (i) the temperature gradient is parallel to the magnetic field and (ii) the temperature gradient is perpendicular to the field. We also calculate the attenuation of longitudinal and transverse sound α in the low-frequency limit for both the perpen-

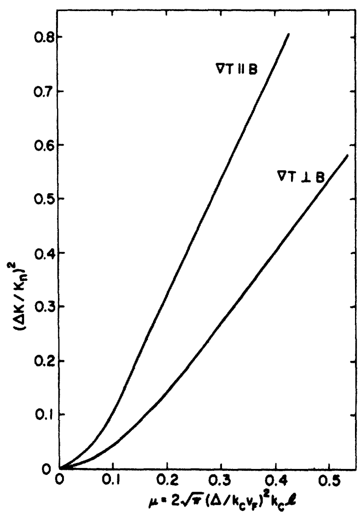

FIG. 1. Thermal conductivity in clean type-II superconductors near H_{c2}. $(\Delta\kappa/\kappa_n)^2$ as a function of $\mu=2\sqrt{\pi}\,(\Delta/k_c v_F)^2 k_c l$ for $\vec{\nabla}T \parallel \vec{B}$ and $\vec{\nabla}T \perp \vec{B}$.

dicular and parallel geometries. Expressions giving the relative change in these coefficients with respect to the normal state are listed below. If we define

$$J(z)=2\mu/[\mu+(1-z^2)^{1/2}]\,, \tag{27}$$

then we find

$$\Delta\kappa_{\parallel}/\kappa_n=-3\int_0^1 dz\, z^2 J(z)$$
$$=-6\mu[(1-\mu^2)J_1+(\tfrac{1}{4}\pi-\mu)]\,, \tag{28}$$

$$\Delta\kappa_{\perp}/\kappa_n=-\tfrac{3}{2}\int_0^1 dz\,(1-z^2)J(z)$$
$$=-3\mu[\mu^2 J_1+(\tfrac{1}{4}\pi-\mu)]\,, \tag{29}$$

$$(\Delta\alpha^T/\alpha_n)(\vec{q}\parallel\vec{B}\text{ or }\vec{e}\parallel\vec{B})$$
$$=-\tfrac{15}{2}\int_0^1 dz\,(1-z^2)z^2 J(z)$$
$$=-15\mu[\mu^2(1-\mu^2)J_1+\tfrac{1}{16}\pi-\tfrac{1}{3}\mu-\tfrac{1}{4}\pi\mu^2+\mu^3]\,, \tag{30}$$

$$(\Delta\alpha^T/\alpha_n)(\vec{q}\perp\vec{B},\ \vec{e}\perp\vec{B})$$
$$=-\tfrac{15}{8}\int_0^1 dz\,(1-z^2)^2 J(z)$$
$$=-\tfrac{15}{4}\mu(\mu^4 J_1+\tfrac{3}{16}\pi-\tfrac{2}{3}\mu+\tfrac{1}{4}\pi\mu^2-\mu^3)\,, \tag{31}$$

$$\Delta\alpha_L^{\parallel}=-\tfrac{5}{4}\int_0^1 dz\,(3z^2-1)^2 J_1(z)$$
$$=-\tfrac{5}{2}\mu[(4-12\mu^2+9\mu^4)J_1-\tfrac{21}{16}\pi+6\mu+\tfrac{9}{4}\pi\mu^2-9\mu^3]\,, \tag{32}$$

$$\Delta\alpha_L^{\perp}/\alpha_n=-\tfrac{5}{4}\int_0^1 dz\,[1-3(1-z^2)+\tfrac{27}{8}(1-z^2)^2]J_1(z)$$
$$=-\tfrac{5}{16}\mu[(8-24\mu^2+27\mu^4)J_1-\tfrac{5}{16}\pi+6\mu+\tfrac{27}{4}\pi\mu^2-27\mu^3]\,, \tag{33}$$

where

$$J_1=\int_0^{\pi/2}\frac{\cos\theta\, d\theta}{\cos\theta+\mu}=\frac{\pi}{2}-\frac{2\mu}{(\mu^2-1)^{1/2}}\tan^{-1}\left(\frac{\mu-1}{\mu+1}\right)^{1/2}. \tag{34}$$

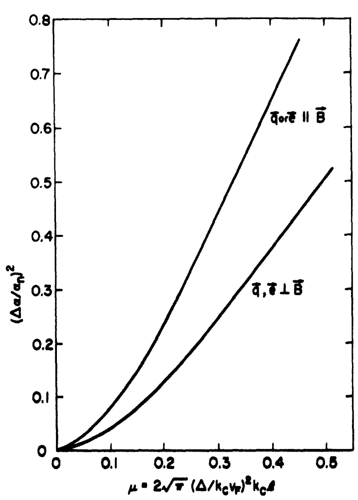

FIG. 2. Attenuation of low-frequency, $ql \ll 1$, transverse sound in clean type-II superconductors near H_{c2}. $(\Delta\alpha/\alpha_n)^2$ as a function of $\mu=2\sqrt{\pi}\,(\Delta/k_c v_F)^2 k_c l$ for $\vec{q}$, $\vec{e}\perp\vec{B}$ and $\vec{q}$ or $\vec{e}\parallel\vec{B}$.

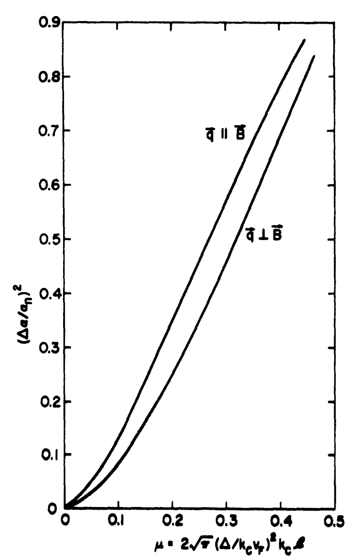

FIG. 3. Attenuation of low-frequency, $ql \ll 1$, longitudinal sound in clean type-II superconductors near H_{c2}. $(\Delta\alpha/\alpha_n)^2$ as a function of $\mu=2\sqrt{\pi}\,(\Delta/k_c v_F)^2 k_c l$ for $\vec{q}\perp\vec{B}$ and $\vec{q}\parallel\vec{B}$.

Equations (27)–(34) show that the changes in thermal conductivity and ultrasonic attenuation are simple functions of the parameter $\mu = 2\sqrt{\pi}(\Delta/k_c v_F)^2 k_c l$ and are, therefore, strongly mean-free-path dependent. In Figs. 1–3, we plot $(\Delta\kappa/\kappa_n)^2$, $(\Delta\alpha^T/\alpha_n)^2$, and $(\Delta\alpha^L/\alpha_n)^2$, respectively, as functions of μ. As is easily seen from the figures, the relative change in the square of the transport coefficients for fixed l can be considered as proportional to Δ^2, that is, $H_{c2} - B$, over a narrow field range close to H_{c2}. Both of these effects are consistent with experiment. Further, we note that the theory correctly predicts the experimentally observed anisotropy in the thermal conductivity near H_{c2}.

Finally, we should point out that we only expect the theory to be valid for $\mu < 1$: For values of $\mu > 1$, that is, for fields $H \ll H_{c2}$ and/or purer samples, it will be necessary to determine the single-particle propagator used in this theory to a higher degree of accuracy.

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[1]K. Maki, in *Superconductivity*, edited by R. Parks (Marcel Dekker, New York, 1969), Chap. 18.

[2]W. F. Vinen, E. M. Forgan, C. E. Gough, and M. J. Hood, Physica (to be published).

[3]U. Brandt, W. Pesch, and L. Tewordt, Z. Physik 201, 209 (1967).

[4]H. A. Cerdeira, W. N. Cottingham, and A. Houghton, Solid State Commun. 7, 1409 (1969); J. Phys. Chem. Solids 31, 1618 (1970); H. A. Cerdeira and A. Houghton, Phys. Rev. (to be published).

[5]W. D. Hibler, III, and M. Cyrot, Phys. Rev. B 1, 3007 (1970).

[6]W. Gautschi, in *Handbook of Math Functions*, edited by M. Abramowitz and I. A. Stegun (Natl. Bur. Std., U. S. GPO, Washington, D. C., 1964), Chap. 7.

PHYSICAL REVIEW B VOLUME 4, NUMBER 3 1 AUGUST 1971

Transport Properties of Clean Type-II Superconductors in the Flux-Flow Regime

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We recalculate the flux-flow resistivity and the Ettingshausen coefficient of clean type-II superconductors in the high-field region. Use is made of a Green's function due to Brandt, Pesch, and Tewordt, which enables accurate calculation of the transport properties in the vicinity of the upper critical field. Both the flux-flow resistivity and the Ettingshausen effect can be compared with the previous calculation in the limit of the small order parameter, although the present expression for the flux-flow resistivity results in a slope at $H = H_{c2}$ larger by a factor of 2. We also find a strong mean-free-path dependence of these coefficients in lower fields.

I. INTRODUCTION

In recent years there has been a great deal of work, both theoretical and experimental, on the transport properties of type-II superconductors[1,2] in the flux-flow regime. The dynamical properties (e.g., flux-flow resistivity and Ettingshausen effect) in the flux-flow regime are of particular interest, since they provide invaluable information on the dynamical behavior of the superconducting order parameter (i.e., the way the order parameter in the vortex state moves in response to an electric field or to a temperature gradient).

This phenomenon can be treated from a microscopic point of view, if we limit ourselves to the vicinity of the upper critical field H_{c2}, where the order parameter $\Delta(\vec{r}, t)$ is small. In fact, making use of a perturbation expansion, where we take the order parameter as a small parameter, vari-