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Some Critical Properties of the Eight-Vertex Model*

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(Received 16 July 1971)

The eight-vertex model solved by Baxter is shown to be equivalent to two Ising models with nearest-neighbor coupling interacting with one another via a four-spin coupling term. The critical properties of the model in the weak-coupling limit are in agreement with the scaling hypothesis. In this limit where $\alpha \rightarrow 0$, the critical indices obey $\gamma/\gamma_0 = \beta/\beta_0 = \nu/\nu_0 = 1 - \frac{1}{2}\alpha$, $\delta/\delta_0 = \eta/\eta_0 = 1$, with the subscripts zero denoting the index values for the ordinary two-dimensional Ising model.

In a recent publication, Baxter¹ has found the free energy for the eight-vertex problem and shown that α is a continuous function of the interaction constants. This continuous variation of a critical index contradicts the hypothesis of smoothness² or universality³ often postulated for near-critical problems.

One way of seeing the source of this behavior is to rephrase the eight-vertex problem as an Ising model.⁴ Imagine a spin placed at the interstitial points of the lattice as in Fig. 1. An arrow to the right (or upward) corresponds to the case in which the adjacent spins are parallel; a leftward or downward arrow makes the adjacent spins antiparallel. Then, the four combinatorial factors a , b , c , and d corresponding to the vertices shown can all be represented by a factor in the partition function

$$Ae^{K^-\sigma_1\sigma_4 + K^+\sigma_2\sigma_3 + \lambda\sigma_1\sigma_2\sigma_3\sigma_4},$$

and we obtain the complete partition function

$$\sum_{\{\sigma_r = \pm 1\}} \prod_{j,k} A \exp(K^+\sigma_{j,k}\sigma_{j+1,k+1} + K^-\sigma_{j+1,k}\sigma_{j,k+1} + \lambda\sigma_{j,k}\sigma_{j+1,k+1}\sigma_{j+1,k}\sigma_{j,k+1}), \quad (1)$$

in which next-nearest-neighbor spins are coupled by interaction constants K^\pm depending upon the direction of the diagonal. The factor λ couples all four spins. The precise connection is that

$$a = Ae^{K^+ + K^- + \lambda}, \\ b = Ae^{-(K^+ + K^-) + \lambda},$$

$$c = Ae^{K^+ - K^- - \lambda}, \\ d = Ae^{-(K^+ - K^-) - \lambda}. \quad (2)$$

The constant A does not, of course, enter into the critical properties.

The Baxter solution shows that this Ising-type problem has a very new kind of singularity at the critical point, namely, one in which the singularity in the specific heat as $\epsilon \sim (b + c + d - a)/a$ goes to zero is of the form $\epsilon^{-\alpha}$ with α being a function of the parameters, namely,⁵

$$\sin \frac{\pi\alpha}{4(1 - \frac{1}{2}\alpha)} = \tanh 2\lambda. \quad (3)$$

This result seems at first to contradict the smoothness hypothesis² which suggests that critical indices should not change their value unless there is a symmetry change. However, this eight-vertex model certainly has a different set of symmetries than the usual two-dimensional Ising model. Notice that at $\lambda = 0$, the lattice with $j + k =$ (even integer) does not interact with the lattice with $j + k =$ (odd integer). Even at $\lambda \neq 0$ for $T > T_c$, i. e., $\epsilon > 0$, the spins on these two sublattices are uncorrelated. Therefore, the Ising form of the eight-vertex model can be viewed as having two lattices with "independent" ferromagnetic transitions which occur at exactly the same temperature. The coupling between these two lattices is of the form

$$\lambda \sum_r u_r; \quad u_r = \mathcal{G}_r^{(1)} \mathcal{G}_r^{(2)}, \quad (4)$$

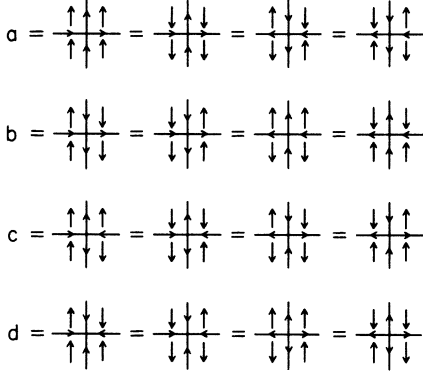


FIG. 1. The correspondence between the Ising-spin configurations, the eight-vertex configurations, and the Boltzmann factors a , b , c , d .

where $\mathcal{G}_r^{(1)}$ and $\mathcal{G}_r^{(2)}$ are the energy densities on the two sublattices. This kind of coupling leaves the spontaneous magnetizations on the two lattices free to point in either the same or in opposite directions. Since this two-sublattice symmetry is very different from that of the usual Ising model, it is not surprising that the critical indices of the Baxter solution are, in general, different from those of the Onsager solution.

A second unexpected feature of the solution is that α varies continuously with λ . The scaling hypothesis usually rules out this idea as is shown in the discussion of Ref. 3. However, there is one special case in which the scaling idea does permit the continuous variation of critical indices—if there is a term in the Hamiltonian of the form of $\lambda \sum_r \tilde{u}_r$ and \tilde{u}_r scales as $1/r^d$ (d denotes the dimension of the lattice).

To see why this particular scaling is so significant, recall the definition of scaling: In the critical region, the phase transition is supposed to be described by fluctuating local quantities, e. g., the magnetization density and the energy density, which we can write as $O_\alpha(r)$. The α distinguishes among different quantities. Let O be a product of n different quantities of this type,

$$O = \prod_{i=1}^n O_{\alpha_i}(r_i), \quad (5a)$$

and take each pair of operators in the product to be separated by a distance $|\vec{r}_i - \vec{r}_j|$ of the order of magnitude of R , with R much greater than a lattice constant and much smaller than a coherence length. Then the statement $O_\alpha(r)$ scales as $1/r^{x_\alpha}$ means precisely⁶ that

$$\langle O \rangle_{K,\lambda} \sim \frac{1}{R^x}, \quad x = \sum_{j=1}^n x_{\alpha_j} \quad (5b)$$

for $n \geq 2$.

Here the x_α 's are critical indices which describe the behavior of the fluctuating variables. For the

ordinary two-dimensional Ising model, the magnetization has an index $x_\sigma = \frac{1}{8}$, and the energy density $x_\epsilon = 1$.

If these indices vary with λ , then the derivative of $\langle O \rangle_{K,\lambda}$ contains a term like $R^{x_\alpha} \ln R$, in particular,

$$\frac{\partial \langle O \rangle_{K,\lambda}}{\partial \lambda} = - \langle O \rangle_{K,\lambda} \sum_{j=1}^n \left(\frac{\partial x_{\alpha_j}}{\partial \lambda} \right) \ln R + \dots, \quad (6)$$

where the \dots represents terms which are not logarithmic in R . Therefore, these logarithmic terms are signals of continuously varying critical indices.

To see how this logarithm can arise, notice that if λ is conjugate to u_r , which contains a term \tilde{u}_r , then

$$\frac{\partial}{\partial \lambda} \langle O \rangle_{K,\lambda} = \sum_r \langle O \tilde{u}_r \rangle_{K,\lambda} + \dots \quad (7)$$

According to the operator algebra concept, when a product of two operators which are relatively close to one another appears inside a correlation function, their product may be approximately replaced according to

$$O_\alpha(r_1) O_\beta(r_2) = \sum_r A_{\alpha\beta,\gamma}(\vec{r}_1 - \vec{r}_2) O_\gamma(\frac{1}{2}(r_1 + r_2)), \quad (8)$$

where, according to scaling,

$$A_{\alpha\beta,\gamma} = \frac{a_{\alpha\beta,\gamma} [(\vec{r}_1 - \vec{r}_2) / (|\vec{r}_1 - \vec{r}_2|)]}{|\vec{r}_1 - \vec{r}_2|^{x_\alpha + x_\beta - x_\gamma}} \quad (9)$$

for separations $|\vec{r}_1 - \vec{r}_2|$ large in comparison to the lattice constant. In the particular case in which O_β is \tilde{u} , which scales as $1/r^d$, then the product in (8) contains a term of the form

$$O_\alpha(r_1) u(r) = \frac{a_\alpha O_\alpha(\frac{1}{2}(r_1 + r))}{|\vec{r}_1 - r|^d} + \dots, \quad (10)$$

when O_α and \tilde{u} are scalars under rotation. Here a_α is, of course, the particular coefficient which appears in the reduction formula (9) when $\alpha = \gamma$ and $O_\beta = \tilde{u}$.

As a result, the sum in (7) contains a succession of terms

$$\frac{\partial}{\partial \lambda} \langle O \rangle_{K,\lambda} = \sum_{j=1}^n \sum_{|\vec{r}-\vec{r}_j| \ll R} \frac{a_j}{|\vec{r}-\vec{r}_j|^d} \langle O \rangle_{K,\lambda} + \dots \quad (11)$$

which corresponds to $|\vec{r} - \vec{r}_j|$ being much smaller than the average separation distance $|\vec{r}_i - \vec{r}_j| \sim R$. Here the \dots include all terms in which all separations are at least of order R . The logarithms then appear in \sum_r . In two dimensions one obtains

$$\sum_{r < R} 1/r^2 \approx 2\pi \ln(R/a_0). \quad (12)$$

When Eqs. (11) and (12) are combined, a set of logarithms appears in the derivative. A comparison with Eq. (6) then shows that

$$\frac{\partial x_\alpha}{\partial \lambda} = -2\pi a_\alpha. \quad (13)$$

We apply this result to the model solved by Baxter

for the particular case $\lambda = 0$. At $\lambda = 0$, the operator

$$\bar{u}_r = \delta \mathcal{G}_r^{(1)} \delta \mathcal{G}_r^{(2)} \quad (14)$$

does indeed scale as $1/r^2$ if $\delta \mathcal{G}_r^{(1)}$ and $\delta \mathcal{G}_r^{(2)}$ are the deviations of the energy on the two sublattices from their critical values. To see this, calculate $\langle \bar{u}_r \bar{u}_0 \rangle$ at $\lambda = 0$ and at the critical point. At $\lambda = 0$, the two sublattices are independent so that

$$\langle \bar{u}_r \bar{u}_0 \rangle = \langle \delta \mathcal{G}_r^{(1)} \delta \mathcal{G}_0^{(1)} \rangle^2. \quad (15)$$

However, the statement that $x_\mathcal{G} = 1$ at $\lambda = 0$ implies that at criticality for large r

$$\langle \delta \mathcal{G}_r^{(1)} \delta \mathcal{G}_0^{(1)} \rangle = q/2\pi r^2. \quad (16)$$

From Ref. 7 we obtain $q = 4/\pi$. (Note that nearest neighbors in the sublattices are separated by $\sqrt{2}$.) The correlation function on the left-hand side of (15) is then $(q/2\pi r^2)^2$ and, consequently, \bar{u}_r scales as $1/r^2$.

Because \bar{u}_r has this special value of the scaling index, the critical phenomena theory indicates that the critical indices can vary continuously in λ . Conversely, if there is no operator with index d , then there can be no continuous variation of the critical indices.

To find the first variation in the critical index for the energy, calculate

$$\delta \mathcal{G}_{r_1}^{(1)} \bar{u}(r_2) \approx \delta \mathcal{G}_{r_1}^{(1)} \delta \mathcal{G}_{r_2}^{(1)} \delta \mathcal{G}_{r_2}^{(2)}. \quad (17)$$

According to Eq. (17), as r_1 approaches r_2 at $\lambda = 0$, the product of the energy fluctuations on lattice (1) can be replaced by a constant divided by $|r_1 - r_2|^2$. In particular,

$$\delta \mathcal{G}_{r_1}^{(1)} \bar{u}(r_2) \approx \frac{q}{2\pi |r_1 - r_2|^2} \delta \mathcal{G}_{r_2}^{(2)}. \quad (18)$$

Note, however, that this result is not of the form (10), needed to reach Eq. (13). To achieve the form (10), we consider the combinations

$$\delta \mathcal{G}_r^\pm = \delta \mathcal{G}_r^{(1)} \pm \delta \mathcal{G}_r^{(2)}, \quad (19)$$

which have a simple symmetry under the interchange of the two lattices. Equations (18) and (19) give

$$\delta \mathcal{G}_{r_1}^\pm \bar{u}(r_2) = \pm \frac{q}{2\pi |r_1 - r_2|^2} \delta \mathcal{G}_{r_2}^\pm$$

when r_1 and r_2 are relatively close together compared to all other distances but $|r_1 - r_2|$ is large compared to a lattice constant. It now follows that \mathcal{G}_r^\pm scales as $1/r^{x_\pm}$ with Eq. (10) giving

$$\frac{dx_\pm}{d\lambda} = \mp q.$$

Since the scaling index is 1 at $\lambda = 0$, we find that for small λ

$$x_\pm = 1 \mp \lambda q. \quad (20a)$$

A similar argument applied to $\sigma_r^{(1)}$ indicates that

at $\lambda = 0$, $dx_\sigma/d\lambda = 0$ so that to first order in λ

$$x_\sigma = \frac{1}{8}. \quad (20b)$$

To derive this result, notice that for r_1 close to r_2 ,

$$\sigma_{r_1}^{(1)} u_{r_2} = \sigma_{r_1}^{(1)} \delta \mathcal{G}_{r_2}^{(1)} \delta \mathcal{G}_{r_2}^{(2)}$$

contains no term which is like $\sigma_r^{(1)}$ since this expression contains a reference to fluctuations on lattice 2. Hence the coefficient a in Eq. (13) vanishes. It follows that $\eta = 2x_\sigma$ does not change to first order in λ .

From these results and scaling theory, we can predict all critical indices to first order in λ . For example, the deviation of energy from criticality contains a singular term of the form

$$\delta \mathcal{G} \sim \xi^{-x_*},$$

where ξ is the correlation length, since x_* is the index which goes with the energy. Also the free energy contains a singular term like

$$\delta F \sim \xi^{-2}.$$

Since $\xi \sim \epsilon^{-\nu}$, $\delta F \sim \epsilon^{2-\alpha}$, and $\delta \mathcal{G} \sim \epsilon^{1-\alpha}$, we find

$$(2 - x_*)\nu = 1$$

or

$$\nu = 1 - q\lambda, \quad (21a)$$

$$\alpha = 2\lambda q. \quad (21b)$$

Equation (21b) is in agreement with Baxter's result, Eq. (3). Similarly, scaling theory implies that

$$\langle \sigma \rangle \sim \xi^{-x_\sigma} = (-\epsilon)^{\nu x_\sigma}$$

on the coexistence curve. Thus, $\beta = \nu x_\sigma$ yields

$$\beta = \frac{1}{8} (1 - q\lambda) \quad (21c)$$

to first order in λ . Thus, we find all the critical indices by using the assumption that scaling holds.

To check this assumption we use first-order perturbation theory. There is a term $\lambda \sum_r \mathcal{G}_r^{(1)} \mathcal{G}_r^{(2)}$ in $-\beta H$. This term may be written as

$$\lambda (\langle \mathcal{G}_r^{(1)} \rangle \langle \mathcal{G}_r^{(2)} \rangle + \Delta \mathcal{G}_r^{(1)} \langle \mathcal{G}_r^{(2)} \rangle + \Delta \mathcal{G}_r^{(2)} \langle \mathcal{G}_r^{(1)} \rangle + \Delta \mathcal{G}_r^{(1)} \Delta \mathcal{G}_r^{(2)}). \quad (22)$$

For simplicity, set $K^+ = K^- = K$. Now consider any expectation value $\langle O \rangle_{K,\lambda}$ where O is a product of terms $O^{(1)} O^{(2)}$ with $O^{(1)}$ containing spins on the first sublattice and $O^{(2)}$ containing spins on the second sublattice. To zeroth order, we may write

$$\langle O \rangle_{K,\lambda} = \langle O^{(1)} \rangle (\epsilon^0) \langle O^{(2)} \rangle (\epsilon^0) + \text{order } \lambda. \quad (23)$$

Here $\epsilon^0 = K_c^0 - K$, with K_c^0 being the critical value of K at $\lambda = 0$.

In first-order perturbation theory,

$$\frac{d}{d\lambda} \langle O \rangle_{K,\lambda} = \langle \mathcal{G}^{(2)} \rangle \langle O^{(2)} \rangle (\epsilon^0) \sum_r \langle \Delta \mathcal{G}_r^{(1)} O^{(1)} \rangle_{K,\lambda=0}$$

$$\begin{aligned}
 & + \langle \mathcal{G}^{(1)} \rangle \langle O^{(1)} \rangle (\epsilon^0 \sum_r \langle \Delta \mathcal{G}_r^{(2)} O^{(2)} \rangle_{K, \lambda=0} \\
 & + \sum_r \langle O^{(1)} \Delta \mathcal{G}_r^{(1)} \rangle_{K, \lambda=0} \langle O^{(2)} \Delta \mathcal{G}_r^{(2)} \rangle_{K, \lambda=0} . \quad (24)
 \end{aligned}$$

Since \mathcal{G} is conjugate to K , the first two terms in (24) generate derivatives with respect to K of $\langle O^{(1)} \rangle_{K, \lambda=0}$ and $\langle O^{(2)} \rangle_{K, \lambda=0}$. To first order in λ , we may replace Eq. (24) by

$$\begin{aligned}
 \langle O^{(1)} O^{(2)} \rangle_{K, \lambda} & \approx \langle O^{(1)} \rangle (\epsilon^*) \langle O^{(2)} \rangle (\epsilon^*) \\
 & + \lambda \sum_r \langle O^{(1)} \Delta \mathcal{G}_r^{(1)} \rangle \langle O^{(2)} \Delta \mathcal{G}_r^{(2)} \rangle , \quad (25)
 \end{aligned}$$

with

$$\epsilon^* = \epsilon^0 - \lambda \langle \mathcal{G} \rangle_{\lambda=0} . \quad (26)$$

Equation (26) gives a renormalized $T - T_c$. Near T_c , $\langle \mathcal{G} \rangle$ is given by

$$\langle \mathcal{G} \rangle = \frac{1}{2} \sqrt{2} - p \epsilon^0 + q \epsilon^0 \ln |\epsilon^0| , \quad (27)$$

with p being a new constant and q being the same as the constant defined by Eq. (16). [Eq. (27) can be obtained, e. g., from Eq. (97) of Ref. 8. Note that $2K_c(0) = \ln \text{ctg}(\frac{1}{3}\pi)$.] To first order in λ , we can write

$$\begin{aligned}
 \epsilon^* & = (\epsilon^0 - \lambda \frac{1}{2} \sqrt{2} + p \lambda \epsilon^0) (1 - \lambda q \ln |\epsilon^0|) \\
 & = \epsilon (1 - \lambda q \ln |\epsilon|) = \epsilon e^{-\lambda q \ln |\epsilon|} = \epsilon |\epsilon|^{-\lambda q} , \quad (28)
 \end{aligned}$$

with

$$\epsilon = \epsilon^0 - \lambda \frac{1}{2} \sqrt{2} + p \lambda \epsilon^0 \quad (29)$$

being essentially $K_c(\lambda) - K$. The shift in K_c given by Eq. (29) checks directly against the value given by Baxter's solution.

Equations (25) and (28) may now be used to evaluate critical indices directly. When $O = \sigma_r^{(1)}$ and $T < T_c$, we find, to first order in λ ,

$$\langle \sigma_r^{(1)} \rangle_{K, \lambda} = \pm (-\epsilon^*)^{\beta_0} = \pm (-\epsilon)^{\beta_0(1-q\lambda)} ,$$

where β_0 is the magnetization index for the Onsager solution. This direct solution then recovers the scaling result (21c). Similarly, the two-spin correlation functions which have the form

$$\langle \sigma_0^{(1)} \sigma_r^{(1)} \rangle_{K, \lambda=0} = \frac{1}{r^{1/4}} f(r/\xi_0)$$

when $\xi_0 \sim |\epsilon^0|^{-1}$ become, to first order,

$$\langle \sigma_0^{(1)} \sigma_r^{(1)} \rangle_{K, \lambda} = \frac{1}{r^{1/4}} f(r/\xi) + \text{order } \lambda^2 \quad (30)$$

when $\xi \sim |\epsilon|^{-1+q\lambda}$. It follows that the two-spin correlation function has a scaling form to first order in λ , and that the coherence length index is correctly given by Eq. (21a). Also, an integration of Eq. (30) over all r gives

$$\gamma = \frac{7}{4} (1 - q\lambda) , \quad (31)$$

as would be predicted by scaling.

In fact, we can see that scaling holds quite generally to first order in λ . Consider the behavior of a product of n spins on the first sublattice at positions $r_1^{(1)}, \dots, r_n^{(1)}$ and m spins on the second at $r_1^{(2)}, \dots, r_m^{(2)}$. If they are far enough separated so that scaling holds at $\lambda = 0$, then

$$\langle \sigma_{r_1} \cdots \sigma_{r_n} \rangle_{K, \lambda=0} = |\epsilon^0|^{n\beta} f_n(\{r_i/\xi_0\}) \quad (32a)$$

while the correlation with an energy density at R takes the form

$$\langle \sigma_{r_1} \cdots \sigma_{r_n} \delta \mathcal{G}_R \rangle_{K, \lambda=0} = \frac{|\epsilon^0|^{n\beta}}{\xi_0} f_{n,1}\left(\left\{\frac{r_i}{\xi_0}\right\}, \frac{R}{\xi_0}\right) . \quad (32b)$$

Then, Eq. (25) implies that to first order in λ ,

$$\begin{aligned}
 & \langle \sigma_{r_1}^{(1)} \cdots \sigma_{r_n}^{(1)} \sigma_{r_1}^{(2)} \cdots \sigma_{r_m}^{(2)} \rangle \\
 & = |\epsilon|^{(n+m)\beta} \left[f_n\left(\left\{\frac{r_i^{(1)}}{\xi}\right\}\right) f_m\left(\left\{\frac{r_j^{(2)}}{\xi}\right\}\right) \right. \\
 & \quad \left. + \lambda \int \frac{d^2 R}{\xi^2} f_{n,1}\left(\left\{\frac{r_i^{(1)}}{\xi}\right\}, \frac{R}{\xi}\right) f_{m,1}\left(\left\{\frac{r_j^{(2)}}{\xi}\right\}, \frac{R}{\xi}\right) \right] , \quad (33)
 \end{aligned}$$

which indicates that for large separations among all the spins their correlation functions obey all the scaling laws, at least to first order in λ .

It now follows that thermodynamic derivatives with respect to magnetic fields inserted in $-\beta H$ as $h_1 \sum_r \sigma_r^{(1)} + h_2 \sum_r \sigma_r^{(2)}$ must obey scaling in the form

$$\frac{\partial^n}{\partial h_1^n} \frac{\partial^m}{\partial h_2^m} \ln Z \Big|_{h_1=h_2=0} \sim |\epsilon|^{(n+m)\beta - [(n+m)-1]2\nu} . \quad (34)$$

In this way, we see that the free energy in the presence of magnetic fields contains a scaling term of the form

$$\epsilon^{2\nu} g\left(\frac{h_1}{\epsilon^\Delta}, \frac{h_2}{\epsilon^\Delta}\right) , \quad (35)$$

with $\Delta = 2\nu - \beta$, and that this scaling term dominates all derivatives with respect to magnetic fields. Therefore the critical exponent $\delta = \Delta/\beta$ does not change to first order in λ .

To check the critical index α in first-order perturbation theory, we consider the free-energy density

$$-\beta f_\lambda = -\beta f_0 + \lambda \langle \mathcal{G}^{(1)} \rangle \langle \mathcal{G}^{(2)} \rangle . \quad (36)$$

From Eqs. (27) and (29) we find that the nonanalytic contributions are

$$-\beta f_{\text{sing}} = -q \epsilon^2 \ln |\epsilon| + \lambda q^2 \epsilon^2 (\ln |\epsilon|)^2 , \quad (37)$$

which in first order in λ can be written

$$(|\epsilon|^{2-2q\lambda} - \epsilon^2)/2\lambda ,$$

which again checks against Eq. (21b).

From Eq. (20a) we obtain the ratio $(2 - x_*)/(2 - x_-)$ = $1 + 2\lambda q$ to first order in λ . To check this ratio we consider the model with an interaction constant

K_1 for the two-spin interactions in the first sublattice and an interaction constant K_2 in the second sublattice. In first order in λ we obtain similarly to Eq. (25)

$$\langle O^{(1)} O^{(2)} \rangle_{K_1 K_2 \lambda} \approx \langle O^{(1)} \rangle (\epsilon_1^*) \langle O^{(2)} \rangle (\epsilon_2^*) + \lambda \sum_r \langle O^{(1)} \Delta \delta_r^{(1)} \rangle \langle O^{(2)} \Delta \delta_r^{(2)} \rangle, \quad (38)$$

with

$$\epsilon_1^* = \epsilon_1^0 - \lambda \langle \mathcal{G}_2 \rangle_{\lambda=0}. \quad (39)$$

In first order in λ we can write

$$\epsilon_1^* = \epsilon_1 - \lambda q \epsilon_2 \ln |\epsilon_2|, \quad (40)$$

$$\epsilon_1 = \epsilon_1^0 - \lambda \frac{1}{2} \sqrt{2} + p \lambda \epsilon_2^0, \quad (41)$$

and similar equations hold for ϵ_2 and ϵ_2^* . Therefore the critical line $\epsilon_1 = 0$ for $\lambda = 0$ moves to $\epsilon_1^* = 0$ at λ , that is,

$$\epsilon_1 = \lambda q \epsilon_2 \ln |\epsilon_2| \quad (42a)$$

in first-order perturbation theory. This equation as well as

$$\epsilon_2 = \lambda q \epsilon_1 \ln |\epsilon_1| \quad (42b)$$

can be written, in first order in λ ,

$$|\epsilon_1 + \epsilon_2| = |\epsilon_1 - \epsilon_2|^{1+2\lambda q}. \quad (43)$$

Since $\epsilon_1 + \epsilon_2$ and $\epsilon_1 - \epsilon_2$ are conjugate to the energy

densities \mathcal{G}_\pm , the exponent in Eq. (43) is expected to be $(2 - x_*)/(2 - x_*)$, in agreement with Eq. (20a).

APPENDIX

When shown the results of this paper, Wilson drew our attention to a similar problem in field theory which was studied by Wilson,⁹ Callen,¹⁰ and Symanzik.¹¹ In field theory the operator Φ^4 corresponds to the operator \tilde{u}_r . In the free-field limit Φ^4 has the critical index (dimension) d , but in first-order perturbation theory its critical index changes. This leads to a breakdown of scaling.

According to Baxter's solution, the critical index α changes continuously with λ , Eq. (3). Therefore we expect \tilde{u}_r to scale like $1/r^2$ for any λ . Wilson and Fisher urged us to show this in first-order perturbation theory.

To see this, note that for $r_1 \neq r$ the operator

$$O_\alpha(r_1) \tilde{u}(r) = \tilde{u}(r_1) \tilde{u}(r)$$

is even under the Kramers-Wannier (KW) transformation of one sublattice only. (Under the KW transformation of sublattice 1 $\delta \mathcal{G}_r^{(1)}$ is odd and $\delta \mathcal{G}_r^{(2)}$ is even.) Since $\tilde{u}(r)$ is odd under this transformation, a_α vanishes. Thus, according to Eq. (13), the critical exponent x_u does not change in first order in λ .

We are indebted to Dr. K. Wilson for his comments.

*Work supported in part by the National Science Foundation.

†On leave from the Institut für Festkörperforschung der Kernforschungsanlage Jülich, Germany.

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