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Calculation of the Self-Focusing of Electromagnetic Radiation in Semiconductors

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A theory for the self-focusing of laser light in InSb is presented. The nonlinear dielectric constant is calculated in the effective-mass approximation and a closed expression is obtained. Similarly closed expressions are obtained for the higher-harmonic-generation coefficients. The equations of self-focusing are shown to follow from a variational principle thereby providing us with a generalization of Fermat's principle. Detailed calculations give, for typical laser conditions, a self-focusing length of ~2 mm.

I. INTRODUCTION

As a consequence of the invention of the laser, a host of nonlinear optical effects have been discovered. One of the most interesting effects is the self-focusing of light beams. This topic has received much attention in the literature, both theoretically and experimentally.[1] Several mechanisms have been proposed for the self-focusing phenomenon. Among them are the Kerr effect, electrostriction, thermal perturbation of the medium, nonlinear electronic polarization, and forward stimulated Brillouin scattering. At first the effect was studied in liquids but the investigations quickly expanded to include solids, vapors, and plasmas.[2]

In this paper we propose a new mechanism for the self-focusing of electromagnetic waves in degenerate semiconductors such as InSb. The effect arises because of the velocity-dependent mass of the conduction electrons. As we will see this yields a nonlinear dielectric constant. The connection of this with self-focusing will be spelled out in detail later. The existence of the velocity-dependent mass has been exploited successfully in another nonlinear optical effect—the mixing of light waves.[3] It was demonstrated there that the nonlinear currents can be quite substantial. Thus we expect to find a fairly strong self-focusing effect also.

Whenever one is interested in transmitting intense radiation through a crystal the necessity for estimating the self-focusing effect becomes important. Thus it could play an important role in such experiments as parametric conversion, harmonic generation, optical mixing, self-induced transparency, and laser design studies. In addition self-focusing provides us with a procedure for injecting a very large field into a limited region in the crystal. This could lead to interesting studies of the dynamics of hot conduction electrons or the generation of lattice imperfections. By studying the self-focusing profile it is possible to draw conclusions about the nonlinear dielectric constant of the sample.

In Sec. II we derive an expression for the nonlinear dielectric constant. At first we develop a series expansion for the dielectric function ϵ in the field strength. In the cold plasma approximation we are able to obtain a closed formula for ϵ. In addition we derive closed expressions for the $(2n+1)$th harmonic-generation coefficient.

In Sec. III the self-focusing phenomenon is studied. A variational principle is used to obtain a generalization of Fermat's principle. By making the standard eikonal approximation a simple mechanical analog appears relating self-focusing to central force motion.

In Sec. IV detailed calculations are carried out for InSb. We find that under typical laser conditions we can obtain self-focusing in approximately 2 mm.

Finally Sec. V summarizes the main limitations of the theory.

II. CALCULATION OF NONLINEAR DIELECTRIC CONSTANT

We consider a degenerate semiconductor such as InSb with n electrons per cm^3 in the conduction band. It is assumed that n is sufficiently small that collective effects play a negligible role. Also the crystal anisotropy will be neglected. As shown by Kane,[4] the dynamics of these electrons, owing to their interaction with the lattice, is governed to a high degree of accuracy by the Hamiltonian,

$$H_0=[(E_g/2)^2+E_g p^2/2m^*]^{1/2} . \qquad (1)$$

Here E_g is the gap energy separating the bottom of the conduction band from the top of the valence band, m^* is the effective mass of the electron near the band's bottom, and p is the electronic momentum. As we go to high p, Eq. (1) becomes less accurate but our attention will be confined to fairly low p.

The Hamiltonian in Eq. (1) formally resembles that of a relativistic electron. Indeed if we define an equivalent speed of light c^* by

$$c^*=(E_g/2m^*)^{1/2} , \qquad (2a)$$

we obtain

$$H_0 = [m^{*2}c^{*4} + c^{*2}p^2]^{1/2} . \qquad (2b)$$

As we will be interested only in the long-wavelength response of the semiconductor, we will not resort to a quantum description. We note that in a relativistic situation the velocity and the momentum are not linearly related. Whereas the electronic current is proportional to the velocity, the temporal evolution of the momentum is governed linearly by the electromagnetic field. Thus we have a nonlinear dependence of the current on the field. It is this effect which is responsible for the nonlinear dielectric constant to be derived shortly.

The introduction of the electromagnetic field in Eq. (2b) is accomplished via the minimal substitution $\vec{p} \to \vec{p} + (e/c)\vec{A}$. This leads to

$$\frac{d}{dt} \frac{m^* \vec{v}}{[1-(v/c^*)^2]^{1/2}} = -e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) . \qquad (3)$$

Assume we have a wave $\vec{E} = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$ passing the electron. Then the right-hand term of Eq. (3) may be written as

$$-eE_0 \cos(kz - \omega t)\left[\hat{E}_0 + \frac{\sqrt{\epsilon_L}\, v}{c} \hat{v} x (\hat{k} \times \hat{E}_0)\right] ,$$

where ϵ_L is the lattice dielectric constant. Since $v < c^*$ and c^* is roughly two orders of magnitude smaller than c, the magnetic term is negligible. Hence, unlike the situation in a true relativistic plasma, magnetic effects are negligible. Consequently we have

$$\frac{d}{dt} \frac{m^* \vec{v}}{[1-(v/c^*)^2]^{1/2}} = -e\vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t) . \qquad (4)$$

In general the integration of Eq. (4) is complicated because of the fact that $\vec{r}$ depends on t in the right-hand term. If we examine the phase of the cosine in detail, however, we notice that

$$\frac{d}{dt}(\vec{k} \cdot \vec{r} - \omega t) = -\omega\left(1 - \frac{\sqrt{\epsilon_L}\, v}{c} \hat{k} \cdot \hat{v}\right) . \qquad (5)$$

Hence, consistent with the neglect of the magnetic field, we may neglect the factor $\sqrt{\epsilon_L}\, v/c$ and treat $\vec{r}$ as though it were constant. Equation (4) may be integrated to give

$$\frac{m^* \vec{v}}{[1-(v/c^*)^2]^{1/2}} = \vec{q}_0 + \frac{e\vec{E}_0}{\omega} \sin(\vec{k} \cdot \vec{r} - \omega t) , \qquad (6)$$

where $\vec{q}_0$ is the initial momentum of the electron. In integrating Eq. (4) we have implicitly assumed that the electromagnetic field has been adiabatically switched on.

The current density vector in the sample is given by $\vec{J} = -ne\vec{v}$. Thus, from Eq. (6) we obtain

$$\vec{J} = -nec^* \frac{\vec{q}_0 + (e\vec{E}_0/\omega)\sin\xi}{\{(m^*c^*)^2 + [\vec{q}_0 + (e\vec{E}_0/\omega)\sin\xi]^2\}^{1/2}} , \qquad (7)$$

where we have let $\xi = \vec{k} \cdot \vec{r} - \omega t$. Equation (7) clearly displays the nonlinear dependence of $\vec{J}$ on $\vec{E}_0$. We note that in addition to having a fundamental, all odd harmonics of ω will be present (the even harmonics will disappear upon averaging). In order to obtain the average current of the electrons, we must integrate $\vec{J}$ over the Fermi-Dirac momentum distribution. The procedure of first solving Newton's equation of motion and then performing an average is equivalent to solving the Boltzmann equation in an external field. Thus

$$\langle \vec{J} \rangle = \frac{\int d\vec{q}_0 f_0 \vec{J}}{\int d\vec{q}_0 f_0} , \qquad (8)$$

where

$$f_0 = [1 + \exp\beta(\mathcal{E} - \mu)]^{-1} . \qquad (9)$$

Here $\beta = 1/k_B T$ and

$$\mathcal{E} = [m^{*2}c^{*4} + c^{*2}q^2]^{1/2} . \qquad (10)$$

In Eq. (9) the chemical potential μ is approximately the "rest energy" m^*c^{*2} plus the Fermi energy.

Since in this paper we are interested only in self-focusing and not harmonic generation, the only part of Eq. (7) that is important is the part which varies as $\sin\xi$. It is important to note that there is no $\cos\xi$ dependence. Thus $\vec{J}$ and $\vec{E}_0$ are 90° out of phase with each other and there is no absorption. There is, however, a small amount of coupling to the higher harmonics. This effect will subsequently be omitted. In general we have

$$\langle \vec{J} \rangle = \sum_{n=0}^{\infty} \vec{J}_n \sin(2n+1)\xi . \qquad (11)$$

Hence

$$\vec{J}_n = \frac{1}{\pi} \int_0^{2\pi} d\xi \, \langle \vec{J} \rangle \sin(2n+1)\xi . \qquad (12)$$

In Appendix A we develop a series expansion for $\vec{J}_n$. For $\vec{J}_0$ the leading terms are

$$\vec{J}_0 = -\frac{ne^2}{m^*\omega} \vec{E}\left[\left\langle \frac{1 + \frac{2}{3}\Delta}{(1+\Delta)^{3/2}} \right\rangle - \frac{3}{8}\left(\frac{eE_0}{m^*c^*\omega}\right)^2 \left\langle \frac{1}{(1+\Delta)^{7/2}} \right\rangle + \cdots \right] , \qquad (13)$$

where $\Delta = (q_0/m^*c^*)^2$. If we write the dielectric constant as

$$\epsilon = \epsilon_0 + \epsilon_2 E_0^2 + \epsilon_4 E_0^4 + \cdots , \qquad (14)$$

then

$$\epsilon_0 = \epsilon_L - \frac{4\pi n e^2}{m^*\omega^2} A_0 , \qquad (15a)$$

and

$$\epsilon_2 = \frac{3}{8}\left(\frac{e}{m^*c^*\omega^2}\right)^2 \frac{4\pi n e^2}{m^*} A_2 . \qquad (15b)$$

In Eq. (15) we have included the lattice dielectric constant. Here A_0 and A_2 refer to the first and second averages appearing in Eq. (13).

In the case where the Fermi energy is low and the temperature is also low, A_0 and A_2 may be replaced by unity. The former condition guarantees that the Fermi level lies in the nonrelativistic region. Thus $\langle q_0^2/(m^*c^*)^2\rangle$ can be neglected, as well as higher powers. The latter condition is imposed so that the high-temperature tail of the Fermi distribution does not contribute significantly. Thus we will assume that

$$(\hbar^2/2m^*)(3\pi^2 n)^{2/3} \ll E_g , \tag{16a}$$

and

$$k_B T \ll E_g . \tag{16b}$$

We will call the conditions imposed by Eq. (16) the "cold dilute plasma" assumptions.

Under these assumptions it is possible to go back to Eq. (12) and obtain a closed expression for J_n. Details for obtaining this formula are presented in Appendix B. For the fundamental harmonic this gives

$$J_0 = -nec^*\sqrt{\delta}\, F(\tfrac{1}{2}, \tfrac{1}{2}; 2; \delta) , \tag{17}$$

where $\delta = \gamma/(1+\gamma)$ and $\gamma = (eE_0/m^*c^*\omega)^2$. For small field strengths it reduces to Eq. (13) (except for the thermal averages). In the limit of high field strengths it becomes constant. The dielectric constant corresponding to Eq. (17) is

$$\epsilon = \epsilon_L - \frac{4\pi n e^2}{m^*\omega^2} \frac{F(\frac{1}{2}, \frac{1}{2}; 2; \delta)}{(1+\gamma)^{1/2}} . \tag{18}$$

For high field strengths ($\gamma \to \infty$) we have

$$\epsilon \to \epsilon_L \quad \text{as} \quad E_0 \to \infty . \tag{19}$$

The dielectric constant rises monotonically from its value ϵ_0 of Eq. (15a) to ϵ_L as E_0 is increased from zero to infinity. The coefficient ϵ_4 of Eq. (14) will be negative.

If we were to define the plasma frequency as the root of the equation $\epsilon = 0$ then for small fields we have

$$\omega_p^2 = 4\pi n e^2/m^*\epsilon_L . \tag{20}$$

In general the plasma frequency would be dependent on the field strength. In order to have propagation in the crystal we must have $\omega > \omega_p$, of course.

Having obtained an expression for the nonlinear dielectric constant let us now proceed to study its effect on the self-focusing of electromagnetic radiation.

III. SELF-FOCUSING EQUATIONS

The fact that the dielectric constant displays a nonlinear behavior has a profound influence on the propagation of electromagnetic waves through the medium. For a beam whose intensity profile is not homogeneous we obtain gradients in the dielectric function. These result in a self-refraction of the beam. For the case where $\epsilon_2 > 0$ the beam will refract into the region where the field intensity is greatest. This produces a self-focusing effect. Since all beams in nature possess fluctuations, self-focusing will occur even for "uniform" beams. The requirement for self-focusing is that the refraction effect be sufficient to overcome the diffraction tendency. The latter arises from the confinement of the beam to some finite width.

A comprehensive review of the electrodynamics of self-focusing has been presented by Akhmanov, Suhkorokov, and Khokolov.[5] We present here an alternate derivation of the self-focusing equations which proceeds directly from a variational principle. Our approach yields several benefits not enjoyed by the derivation of the previous authors. Firstly, we have the computational advantage of having a variational principle. In principle this could be used computationally to optimize trial focusing trajectories. Secondly it provides us with a natural extension of Fermat's principle.

We begin with the expression for the Lagrangian density for an electromagnetic wave interacting with a nonlinear dielectric

$$\mathcal{L} = (\epsilon_L E^2 - B^2)/8\pi + \int \vec{P}\cdot d\vec{E} = \mathcal{L}_0 + \mathcal{L}_I , \tag{21}$$

where $\vec{P}$ is the polarization vector. The variation principle is that

$$\delta(1/\tau)\int_0^\tau dt \int d\vec{r}\,\mathcal{L} = 0 , \tag{22}$$

where we assume the end points of time $(0, \tau)$ are held fixed. In the free part of the Lagrangian $\mathcal{L}_0$ we regard the fields as stemming from the vector potential $\vec{A}$ (in Coulomb gauge)

$$\vec{E} = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t} , \tag{23a}$$

and

$$B = \nabla \times \vec{A} . \tag{23b}$$

Upon realizing that, to first order in $\mathcal{L}_I$, the direction of $\vec{A}$'s polarization is unaltered, we look for a solution of the form

$$\vec{E} = E_0\hat{i}\cos(\omega t - kz - kS) = E_0\hat{i}\cos\Delta . \tag{24}$$

We look for stationary monochromatic solutions so E_0 and S are time independent and slowly varying functions of space. Azimuthal symmetry will be assumed for E_0 and S.

The time integration of the action principle can be executed with the effect that $\cos^2\Delta$ and $\sin^2\Delta$ are replaced by their average values $\frac{1}{2}$. In the integration of $\mathcal{L}_I$ we introduce the electric susceptibility

$$\vec{P} = \chi(E_0^2)\vec{E} \ . \tag{25}$$

The main variation of $\vec{P}\cdot d\vec{E}$ is through the $\vec{E}$ part of $\vec{P}$ rather than the χ part, so we replace $\int \vec{P}\cdot d\vec{E}$ by $\chi(\bar{E}_0^2)\int \vec{E}\cdot d\vec{E}$. For $\bar{E}_0^2$ we take $E_i^2/2f^2$ [cf. Eq. (27b)]. Hence we obtain the variation principle

$$-\frac{\epsilon_L}{16\pi}\,\delta\int d\vec{r}\left\{\frac{1}{k^2}\left[\left(\frac{\partial E_0}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial E_0}{\partial R}\right)^2\right]\right.$$
$$\left.+E_0^2\left[2\,\frac{\partial S}{\partial z}+\left(\frac{\partial S}{\partial z}\right)^2+\frac{1}{2}\left(\frac{\partial S}{\partial R}\right)^2\right]-\frac{4\pi\chi(\bar{E}_0^2)}{\epsilon_L}\,E_0^2\right\}=0 \ . \tag{26}$$

From this it is a simple matter to derive the self-focusing equations of Akhmonov, Sukhorokov, and Khokolov.[5] They follow if we neglect the $(\partial E_0/\partial z)^2$ and $(\partial S/\partial z)^2$ terms in the spirit of the eikonal equations. Henceforth we will make this approximation. In analogy with the approach of Akhmanov, Sukhorokov, and Khokolov[5] we pick the following trial functions:

$$S=\varphi(z)+\tfrac{1}{2}R^2\beta(z) \ , \tag{27a}$$

and

$$E_0^2=[E_i/f(z)]^2\,e^{-2R^2/(af)^2} \ . \tag{27b}$$

In the above equation E_i is the initial prevailing field intensity in the crystal, $f(z)$ is the dimensionless radius of the beam, and a is the initial beam radius. The boundary conditions are

$$\varphi(0)=0=\beta(0) \ , \quad f(0)=1 \ , \tag{28a}$$

and, denoting derivatives with respect to z by a prime,

$$\varphi'(0)=0=\beta'(0)=f'(0) \ . \tag{28b}$$

If we perform the R integration and examine Eq. (26) we see that z now plays the role conventionally alloted to the time variable. The quantities φ, β, and f are now dynamical variables in this "time" variable. The R integration is elementary and leads to a Lagrangian

$$L=-\frac{\epsilon_L E_i^2a^2}{32}\left[\varphi'-\frac{4\pi\chi}{\epsilon_L}+\left(\frac{1}{afk}\right)^2+\left(\frac{\beta af}{2}\right)^2+\beta'\left(\frac{af}{2}\right)^2\right] \ . \tag{29}$$

Note that the term φ' is superfluous since it is an exact differential. The corresponding Hamiltonian is

$$H=\frac{\epsilon_L E_i^2a^2}{32}\left[-\frac{4\pi\chi}{\epsilon_L}+\left(\frac{1}{afk}\right)^2+\left(\frac{\beta af}{2}\right)^2\right] \ . \tag{30}$$

The equations of motion for P_β inform us that $\beta=f'/f$, so we obtain finally that

$$H=\frac{\epsilon_L E_i^2a^2}{32}\left[\left(\frac{af'}{2}\right)^2+\left(\frac{1}{afk}\right)^2-\frac{\epsilon-\epsilon_L}{\epsilon_L}\right] \ , \tag{31}$$

where we have used the fact that $\epsilon=\epsilon_L+4\pi\chi$.

Equation (31) can be given a simple mechanical analog. Let

$$m=\epsilon_L E_i^2a^4/64 \ , \tag{32a}$$

$$L=\epsilon_L E_i^2a^2/(32k) \ , \tag{32b}$$

$$V=E_i^2a^2[\epsilon_L-\epsilon(E_i^2/2f^2)]/32 \ , \tag{32c}$$

then Eq. (31) becomes the Hamiltonian for a particle of mass m and angular momentum L moving in a potential V. Thus

$$H=\frac{m}{2}\left(\frac{df}{dz}\right)^2+\frac{L^2}{2mf^2}+V. \tag{33}$$

All our familiar notions regarding potential motion can be transferred directly to the self-focusing problem.

In particular we find the concept of effective potential[6] to be quite important. Thus we let $V_{\text{eff}}=V+L^2/2mf^2$. If V_{eff} has an absolute minimum then we have the possibility of self-focusing. This corresponds to the mechanical analog of having bound noncircular orbits. At the minimum of V_{eff} we would have self-trapping which is analogous to circular orbits.

In a self-focusing situation the focal length is given by the "time" required to go from one root of Eq. (33) to another. Denoting the two f "radii" by 1 and F we have

$$Z_f=\int_F^1\frac{df}{[(2/m)(H-V_{\text{eff}})]^{1/2}} \ . \tag{34}$$

In the particular case that ϵ is quadratic in E_i/f this integral can be carried out. Then only if

$$\left.\frac{L^2}{2m}+f^2V\right|_{f=0}<0 \tag{35}$$

can we obtain self-focusing. This equation represents the competition between the diffraction and refraction effects. It defines a critical power below which no self-focusing can occur. The beam will, in this approximation, shrink to zero radius (i.e., $F=0$) in a distance

$$Z_f^0=[-(2/m)f^2V_{\text{eff}}]^{-1/2} \ . \tag{36}$$

In the more general case we must resort to explicit integration of Eq. (34).

Transcribing Eqs. (35) and (36) back into the optical language we obtain the conventional formulas for the critical power and focal length.[5] The field at the entrance to the crystal is related to the incident power through the relation

$$E_i=\frac{8}{1+\sqrt{\epsilon_L}}\left(\frac{P}{ca^2}\right)^{1/2} \ . \tag{37}$$

Thus the critical power becomes

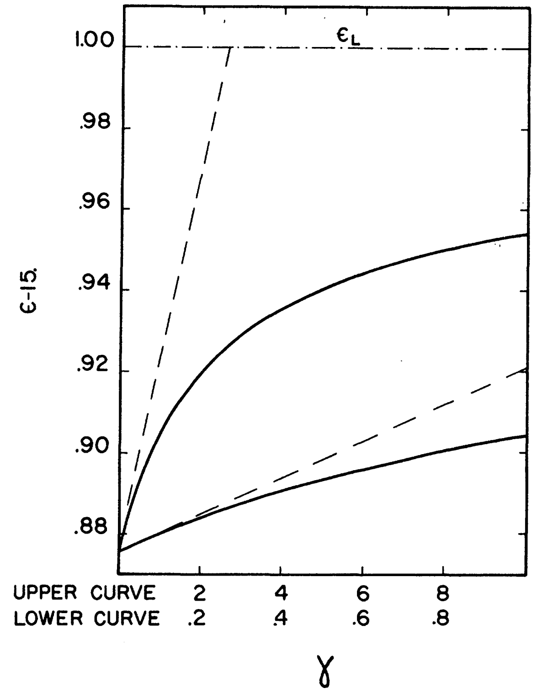

FIG. 1. Nonlinear dielectric constant as a function of the dimensionless parameter $\gamma = (eE_0/m^*c^*\omega)^2$. The dashed line is the quadratic approximation to ϵ.

$$P_{cr} = \left(\frac{1+\sqrt{\epsilon_L}}{8}\right)^2 \frac{2c\epsilon_L}{\epsilon_2 k^2} \quad , \tag{38}$$

and the focal length is

$$Z_f^0 = \tfrac{1}{2} ka^2 (P/P_{cr} - 1)^{-1/2} \quad . \tag{39}$$

From Eq. (39) we see that a real solution for Z_f^0 can only exist for $P > P_{cr}$. If P were very much larger than P_{cr} the k dependence of Z_f^0 would disappear. This indicates that the diffraction is not important in this limit.

IV. RESULTS FOR InSb

Having derived detailed expressions for the nonlinear dielectric constant and having reviewed the theory of self-focusing we now proceed to apply the formalism to a case of practical interest. InSb was chosen for several reasons. It has a low electronic effective mass, $m^* = m_e/60$ (where m_e is the mass of a free electron), and relatively small gap energy, $E_g = 0.234$ eV. We find $c^* = 1.11 \times 10^8$ cm/sec. The ratio of the velocity of light to c^* is 270. This validates the approximations of Eqs. (4) and (6). Here ϵ_L was taken to be 16.

If the experiment is performed at liquid-nitrogen temperature ($T = 77$ °K) the free-carrier absorption is minimal. The radiation frequency is taken to be $\omega = 1.742 \times 10^{14}$ rad/sec corresponding to the 10.81-μ line (in vacuum) of the CO_2 laser. For this frequency two-photon absorption is energetically forbidden and therefore it is easy to transmit the radiation through the crystal. The absorption length was estimated to be $\gtrsim 1$ cm.

In order to keep the free-carrier absorption reasonably small the carrier concentration was taken to be $n = 2\times 10^{16}$ per cm^3. This gives a plasma frequency of 1.54×10^{13} rad/sec. Since $\omega \gg \omega_p$ the neglect of cooperative plasma effects is justified. Then ϵ will never differ very much from ϵ_L, although, of course, it is just this difference which is responsible for self-focusing.

For the above parameters, the cold dilute plasma approximation is well justified. Thus [see Eq. (16)] $\hbar^2(3\pi^2 n)^{2/3}/(2m^* E_g) = 0.07$ and $k_B T/E_g = 0.03$. Then we find $\epsilon_0 = 15.86$ and $\epsilon_2 = 1.26\times 10^{-7}$ esu. In Fig. 1 we have plotted the dielectric constant ϵ as a function of the dimensionless variable γ defined in Eq. (17). The dashed line represents the quadratic approximation to γ. Note that the departure from the quadratic approximation sets in fairly early at $\gamma \simeq 0.2$. For high γ the saturation effect is exhibited.

From Eq. (39) it seems as if we can make Z_f^0 as small as we want by simply increasing P. Unfortunately when the incident intensity exceeds a critical mean intensity, surface ionization will occur. This critical intensity I_{ion} is approximately 3×10^7 W/cm^2. To minimize Z_f we would, in general, like to work close to I_{ion}.

If we take the above parameters then the critical power of Eq. (38) is $P_{cr} = 552$ W. The argument of ϵ involves $E_0^2 = \frac{1}{2} E_i^2$. Thus we take

$$\gamma = (e/m^*c^*\omega_0)^2 \tfrac{1}{2} E_i^2 \ , \tag{40}$$

where E_i is defined in Eq. (27b). For I_{ion} we take $I_{ion} = (c/8\pi) E_0^2$ and obtain

$$\gamma < \left(\frac{e}{m^*c^*\omega_0}\right)^2 \frac{32\pi}{(1+\sqrt{\epsilon_L})^2 c} I_{ion} \equiv \gamma_{ion} \quad . \tag{41}$$

We find $\gamma_{ion} = 0.107$. Our calculations will be made for $\gamma = 0.10$. This corresponds to an input power of 2560 W, where we have taken the radius of the beam to be 0.0054 cm (five vacuum wavelengths).

In the quadratic approximation Eq. (39) gives 1.79 mm. for the focal length. The results of the numerical integration are presented in the next few figures. In Fig. 2 the effective potential is presented. The repulsive part of the potential represents the diffraction effect while the attractive part is due to the refraction. The horizontal dashed line indicates the trajectory followed by the normalized beam radius f. Also shown is the quadratic approximation to the effective potential. The trajectory in this case would go to $f = 0$. In the exact case it goes only to $f = 0.12$. Thus there is a fivefold shrinkage in beam size and a corresponding sixty-five-fold increase in central beam intensity.

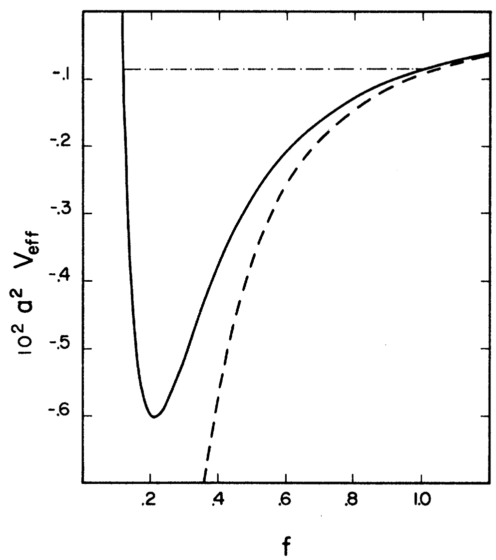

FIG. 2. Effective potential as a function of the dimensionless width of the beam f. The dashed curve is the quadratic approximation. The horizontal dashed line is the trajectory of the beam radius.

The revised focal length is 2.06 mm. After focusing to the minimum the beam size will bounce back to its initial position. This process will continue periodically with period $2Z_f$.

In Fig. 3 the width of the beam a is presented as a function of z. The γ parameter was held fixed at $\gamma = 0.1$ while a/λ was set equal to 5, 10, and 15.

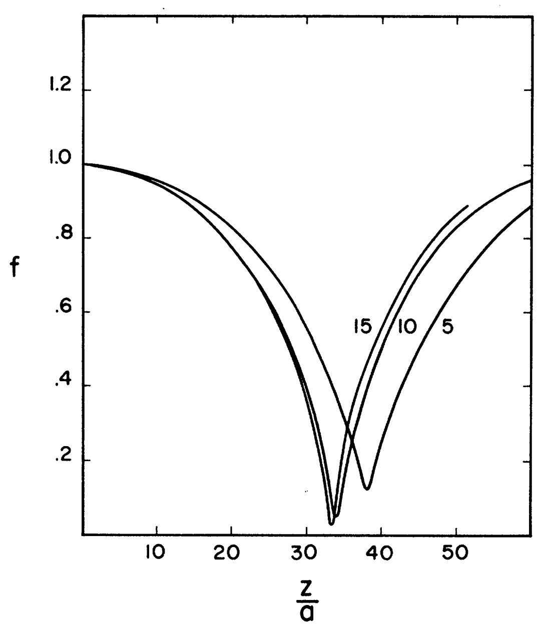

FIG. 3. Dimensionless beam radius f as a function of the dimensionless distance down the beam z/a. Curves are drawn for values of $a/\lambda = 5$, 10, and 15.

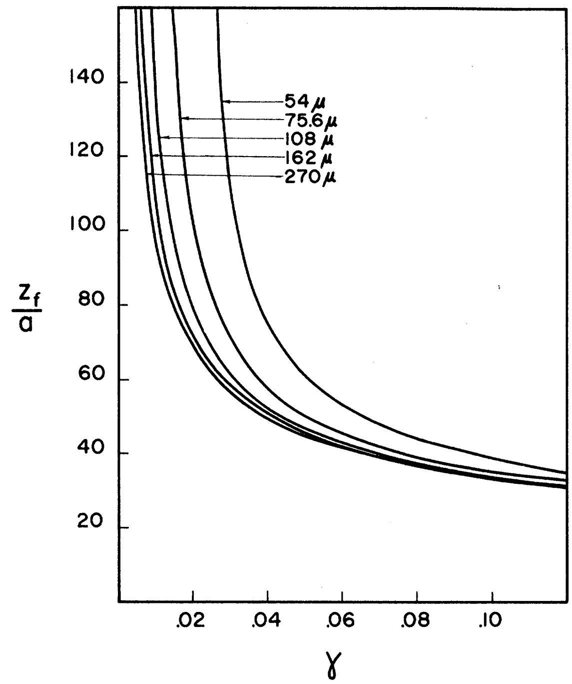

FIG. 4. Ratio of focal length to beam radius as a function of the dimensionless parameter γ for five different initial beam radii a.

The vertical scale is the dimensionless beam size and the horizontal scale is z/a. We have only shown one-half of the "period" but the curve is symmetrical about the turning point. In Fig. 4 the ratio of focal length to beam radius is plotted as a function of γ for several a values. (We remind the reader that $\lambda_{vac} = 10.81\mu$.) Because of our neglect of losses, the lower curves are not to be taken too literally. They are only given to illustrate the trends.

V. CONCLUSION

In conclusion we see that self-focusing in InSb is possible with currently available technology. A summary of some of the constraints obeyed in arriving at this conclusion is presented pictorially in Fig. 5. The curves drawn pertain to the quadratic approximation to ϵ, but similar conclusions follow from the exact ϵ case.

We plot P/P_{cr} vs a/λ in Fig. 5. For $P/P_{cr} < 1$ we are in a region where the diffraction is so strong relative to the self-refraction that no self-focusing can occur. On the other hand, for small a/λ and large P/P_{cr}, surface ionization breakdown will occur. This limitation might, in principle, be lifted by prefocusing the beam or by coating the surface appropriately. Also shown are the lines of constant focal length for $Z_f^0 = 1$, 2, 3, and 4 mm. For large focal length ($\lesssim 1$ cm) we have another limitation—namely, that of absorption by the solid for the

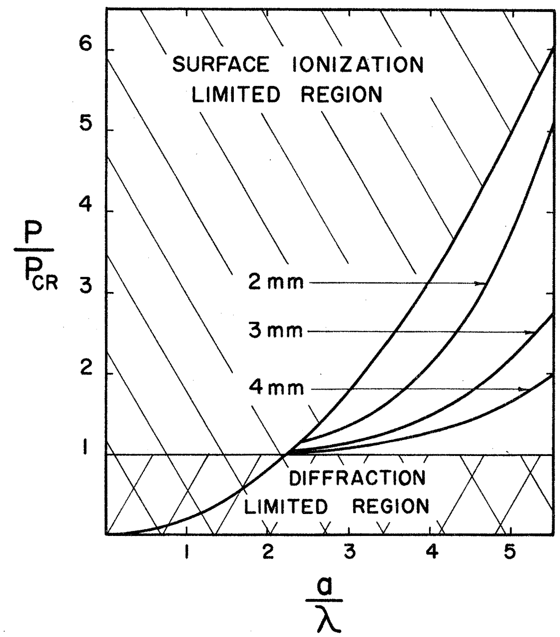

FIG. 5. Region where self-focusing is possible in the P/P_{cr}–a/λ plane. Four lines of constant focal length are indicated.

present choice of electron density.

Our paper shows that it is possible to pick the parameters in such a way that self-focusing can be observed.

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APPENDIX A: POWER-SERIES EXPANSION OF $\vec{J}_n$

In this Appendix we derive an expression for the power-series expansion of $\vec{J}_n$, the nth harmonic of the current. Unfortunately it does not appear possible to compress the expression into a known closed function. Nevertheless the expansion should be useful for evaluating the higher-order nonlinear currents.

Letting $\vec{A}=(e\vec{E}_0/\omega)\sin\xi$, we can write Eq. (7) more concisely as

$$\vec{J}=-nec^*\nabla_A[(m^*c^*)^2+(\vec{q}_0+\vec{A})^2]^{1/2}\ . \quad (A1)$$

Letting

$$\alpha=(c^*/\mathcal{E})^2(2\vec{q}_0\cdot\vec{A}+A^2)\ , \quad (A2)$$

and employing Eq. (10) we obtain

$$\vec{J}=-ne\mathcal{E}\nabla_A(1+\alpha)^{1/2}\ . \quad (A3)$$

The square root is expanded as

$$(1+\alpha)^{1/2}=1+\tfrac{1}{2}\alpha+\sum_{n=2}^{\infty}(-\tfrac{1}{2})_n\frac{(-\alpha)^n}{n!}\ . \quad (A4)$$

Likewise $(-\alpha)^n$ is expanded into the sum

$$(-\alpha)^n=(c^*/\mathcal{E})^{2n}\sum_{m=0}^{n}\binom{n}{m}(2\vec{q}_0\cdot\vec{A})^{n-m}A^{2m}(-)^n\ . \quad (A5)$$

Equation (8) calls for an angular average on the momenta. Thus

$$\langle(\vec{q}_0\cdot\vec{A})^{n-m}\rangle=(q_0A)^{n-m}f_{nm}/(n-m+1)\ , \quad (A6)$$

where we have defined f_{nm} as 1 or 0 depending on whether $n+m$ is even or odd. The gradient operation is trivial and gives

$$\nabla_A A^{m+n}=(m+n)A^{m+n-1}\hat{A}\ . \quad (A7)$$

Equation (12) requires an integration to project out a given harmonic. This is accomplished through the following formula:

$$\frac{1}{\pi}\int_0^{2\pi}d\xi\,\sin(2j+1)\xi\,\sin^{m+n-1}\xi =(-)^j(2)^{m+n-2}\binom{m+n-1}{\frac{1}{2}(m+n)+j}\ . \quad (A8)$$

Combining Eqs. (A1)–(A8) we obtain

$$J_j=-ne\mathcal{E}\hat{E}_0\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-)^n(2c^*)^{2n}\times\left(\frac{eE_0}{4\omega}\right)^{m+n-1}Q_{mn}X_{mn}^{(j)}\ , \quad (A9)$$

where

$$Q_{mn}=\langle q_0^{n-m}/\mathcal{E}^{2n}\rangle\ , \quad (A10)$$

and

$$X_{mn}^{(j)}=\frac{1}{n!}(-\tfrac{1}{2})_n\binom{n}{m}f_{nm}\frac{m+n}{n-m+1}\times(-)^j\binom{m+n-1}{\frac{1}{2}(m+n)+j}\ . \quad (A11)$$

APPENDIX B: $\vec{J}_n$ IN COLD PLASMA APPROXIMATION

In this Appendix we derive an expression for $\vec{J}_n$ in the cold plasma approximation. This is equivalent to putting $\vec{q}_0=0$ in Eq. (7). Upon combining this with Eq. (12) we obtain

$$\vec{J}_n=\frac{-nec^*}{\pi}\int_0^{2\pi}d\xi\,\frac{\vec{A}}{[(m^*c^*)^2+A^2]^{1/2}}\sin(2n+1)\xi\ , \quad (B1)$$

where $\vec{A}=(e\vec{E}_0/\omega)\sin\xi$. Let

$$\gamma=(eE_0/m^*c^*\omega)^2\ , \quad (B2)$$

and

$$\delta=\gamma/1+\gamma\ . \quad (B3)$$

Then we have

$$\vec{J}_n = -nec^*\sqrt{\delta}\,\frac{E_0}{\pi}\int_0^{2\pi} d\xi\,\frac{\sin\xi\,\sin(2n+1)\xi}{[1-\delta\cos^2\xi]^{1/2}}\ , \qquad \text{(B4)}$$

where $0 \leq \delta < 1$. We expand $\sin(2n+1)\xi$ as

$$\sin(2n+1)\xi = \sum_{k=0}^{n} A_k \sin^{2k+1}\xi\ , \qquad \text{(B5)}$$

with

$$A_k = \frac{(-)^k(2n+1)}{(2k+1)!}\prod_{j=1}^{k}\left[(2n+1)^2-(2j-1)^2\right]\ . \qquad \text{(B6)}$$

The following integral is required:

$$\int_0^{2\pi}\frac{\sin^{2k+2}\xi}{[1-\delta\cos^2\xi]^{1/2}}\,d\xi$$

$$= \frac{2\sqrt{\pi}\,\Gamma(k+\frac{3}{2})}{(k+1)!}\,F(\tfrac{1}{2},\tfrac{1}{2};k+2;\delta)\ , \qquad \text{(B7)}$$

where F is the Gauss hypergeometric function. Thus we find

$$\vec{J}_n = -nec^*\sqrt{\delta}\,\frac{2}{\sqrt{\pi}}\,\hat{E}_0\sum_{k=0}^{n}A_k\,\frac{\Gamma(k+\frac{3}{2})}{(k+1)!}\times F(\tfrac{1}{2},\tfrac{1}{2};k+2;\delta)\ . \qquad \text{(B8)}$$

Thus J_n is expressible as a finite sum of closed functions. Equation (B8) is valid for arbitrary field strength.

In the particular case where $n=0$ we find

$$J_0 = -nec^*\sqrt{\delta}\,F(\tfrac{1}{2},\tfrac{1}{2};2;\delta)\ . \qquad \text{(B9)}$$

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