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Theory of Inhomogeneous Superconductors near $T = T_c$ *

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The work of Neumann and Tewordt is generalized to obtain the first-order correction (in $1 - T/T_c$) to the Ginzburg-Landau expression for the free energy of an inhomogeneous superconductor. From this expression, the generalized Neumann-Tewordt equations for the first-order corrections to the solutions of the Ginzburg-Landau equations are derived. For two important geometries, the normal-superconducting wall and the mixed state of type-II superconductors, we show that the free energy can be rewritten so that it involves only the solutions of the Ginzburg-Landau equations. We apply this formalism to the calculation of the N-S wall energy, where we calculate σ_{NS} as a function of T and ξ_0/l for $\kappa = 1/\sqrt{2}$, and to the mixed state of type-II superconductors, where we calculate H_{c1} as a function of T, κ, and ξ_0/l for singly and doubly quantized isolated vortices.

I. INTRODUCTION

Neumann and Tewordt[1] have obtained from earlier work by Tewordt[2] a general expression for the first-order correction in $1 - T/T_c$ to the Ginzburg-Landau[3] (GL) expression for the free energy of an inhomogeneous superconductor near $T = T_c$. They used this result to derive an expression for the free energy of an isolated vortex explicitly in terms of the order parameter, the magnetic field, and the superfluid velocity; the differential equations for the corrections to the GL equations were derived and solved for a mesh of values of κ and the mean free path. The solutions of the differential equations were then used to evaluate the free energy of a singly quantized isolated vortex and hence the lower critical field H_{c1}.

In Sec. II, we review the Neumann-Tewordt (NT) theory and derive an expression for the first-order correction to the GL free energy; our expression is explicitly in terms of the order parameter, the magnetic field, and the superfluid velocity, and is valid for all geometries. From this expression we derive the generalized NT equations for the corrections to the solutions of the GL equations; these equations are believed valid for all geometries. We then expand the free-energy expression in powers of $1 - T/T_c$ and show that the term involving the solutions of the generalized NT equations can be written as the integral of a divergence; for two important geometries, the mixed state of type-II superconductors and the normal-superconducting wall problem, we show that this integral can be written solely in terms of the solutions of the GL equations—it is not necessary to solve the generalized NT equations in order to evaluate the free energy.

In Sec. III, we apply the formalism developed in Sec. II to the calculation of the normal-superconducting wall energy for $\kappa = 1/\sqrt{2}$ and general values of the mean free path.

In Sec. IV, we apply the formalism to calculate the lower critical field H_{c1} as a function of κ and the mean free path for both singly and doubly quantized isolated vortices. Good agreement with the results of NT for the single-quantum case is found.

The results of this paper are used in the succeeding paper[4] to calculate κ_c, the critical value of κ for type-II superconductivity, near $T = T_c$.

II. FREE-ENERGY AND DIFFERENTIAL EQUATIONS

The NT expression[1] for the Gibbs free energy of an inhomogeneous superconductor in an applied field $\vec{H}_a$, relative to the Meissner state, is

$$4\pi \Delta G / H_c^2 \lambda^3 = \int d^3r \,[h^2 - 2\vec{h}\cdot\vec{h}_a + \tfrac{1}{2}(1 - f^2)^2 + \kappa_3^{-2}(\vec{\nabla} f)^2 + f^2 v^2] + (1 - t)\int d^3r\, P(\vec{r}) \ , \quad (2.1)$$

where

$$P(\vec{r}) = Q(\vec{r}) + [\eta_{4d}\delta_{ij}\delta_{kl} + \eta_{4c}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})] \times (O_i O_j \psi)(O_k O_l \psi)^* \quad (2.2)$$

and

$$Q(\vec{r}) = \eta_c f^2 (1 - f^2)^2 - \eta_k (1 - f^2)[\kappa_3^{-2}(\vec{\nabla} f)^2 + f^2 v^2] + \eta_w \kappa_3^{-2} f^2 (\vec{\nabla} f)^2 \ . \quad (2.3)$$

The notation in Eqs. (2.1)–(2.3) is essentially that of NT[1]: The reduced temperature is $t = T/T_c$; the microscopic magnetic field in reduced units is $\vec{h} = \vec{H}(\vec{r})/\sqrt{2}H_c = \vec{\nabla}\times\vec{a}$ and the applied field in reduced units is $\vec{h}_a = \vec{H}_a/\sqrt{2}H_c$; $\vec{a}$ is given by $\vec{a} = \vec{A}(\vec{r})/\sqrt{2}H_c\lambda$ and lengths are measured in units of the penetration depth λ; ψ is given by $\psi = \Psi/\Psi_\infty$, where Ψ is the GL order parameter and Ψ_∞ is its value in wholly superconducting material; f is a real quantity defined by $\psi = f e^{i\Phi}$ and the gauge-invariant superfluid velocity $\vec{v}$ is defined by

$$\vec{v} = \kappa_3^{-1}\vec{\nabla}\Phi - \vec{a} \ , \quad (2.4)$$

where

$$\kappa_3 = \sqrt{2}(2e/\hbar c)H_c\lambda^2 ; \tag{2.5}$$

$\vec{O}$ is the operator $\vec{O}=\kappa_3^{-1}\vec{\nabla}-i\vec{a}$; and the repeated index summation convention has been used in Eq. (2.2) for $P(\vec{r})$. The various η's and the quantity ϕ {defined by the expansion

$$\kappa_3 = \kappa[1+(1-t)\phi]+O(1-t)^2 \tag{2.6}$$

of κ_3 in powers of $1-t$} are, except for η_c, functions of the mean free path l through their dependence on $\alpha=\pi\xi_0/2\gamma l = 0.882\xi_0/l$; the expressions are

$$\eta_c = -31\zeta(5)/98\zeta^2(3), \quad \eta_k = -4(2S_{41}+S_{32})/7\zeta(3)S_{21},$$
$$\eta_w = -8S_{41}/7\zeta(3)S_{21}, \quad \eta_{4d} = -\alpha S_{33}/2S_{21}^2, \tag{2.7}$$
$$\eta_{4c} = -3S_{23}/10S_{21}^2, \quad \phi = 2\eta_c - \eta_k - S_{12}/S_{21},$$

where $\zeta(x)$ is the Riemann ζ function and

$$S_{ij}(\alpha) = \sum_{n=0}^{\infty}(2n+1)^{-i}(2n+1+\alpha)^{-j}. \tag{2.8}$$

The expression for $P(\vec{r})-Q(\vec{r})$ given in Eq. (2.2) is not satisfactory for many calculations and it is necessary to work out the terms to obtain the explicit dependence on f, $\vec{v}$, and $\vec{h}$. The coefficients of η_{4d} and η_{4c} have been obtained by NT for the case of an isolated vortex [where $f=f(r)$, $\vec{v}=v(r)\hat{\theta}$, and $\vec{h}=h(r)\hat{z}$]. The calculation of these coefficients is considerably more difficult for the general case of an inhomogeneous superconductor; the details are without interest and we give only the final result

$$P(\vec{r})-Q(\vec{r}) = (\eta_{4d}+3\eta_{4c})\{[\kappa_3^{-2}\nabla^2 f - fv^2]^2 + \kappa_3^{-2}f^{-2}[\vec{\nabla}\cdot(f^2\vec{v})]^2\} + \eta_{4c}\kappa_3^{-2}[f^2h^2 - 4f\vec{h}\cdot(\vec{v}\times\vec{\nabla}f)]$$
$$+\eta_{4c}2\kappa_3^{-2}\vec{\nabla}\cdot[\tfrac{1}{2}\kappa_3^{-2}\vec{\nabla}(\vec{\nabla}f)^2 - \kappa_3^{-2}(\vec{\nabla}f)(\nabla^2 f) + v^2\vec{\nabla}f^2 + f^2(\vec{v}\cdot\vec{\nabla})\vec{v} - \vec{v}\,\vec{\nabla}\cdot(f^2\vec{v})]. \tag{2.9}$$

In Appendix A we show that this result can be simplified to

$$P(\vec{r})-Q(\vec{r}) = (\eta_{4d}+3\eta_{4c})(\kappa_3^{-2}\nabla^2 f - fv^2)^2 + \eta_{4c}\kappa_3^{-2}[f^2h^2 - 4f\vec{h}\cdot(\vec{v}\times\vec{\nabla}f)]; \tag{2.10}$$

the omitted terms are either identically zero to the order in $1-t$ to which we work or vanish on integration for the geometries of interest. Our expression (2.10) contains the result of NT for a singly quantized isolated vortex as a special case; the coefficients of all terms but that for η_{4c} are identical as they stand, while those for η_{4c} can be shown to be equal by an integration by parts.

The differential equations for $\vec{v}$ and f can be obtained by the usual method[1]; on defining

$$f = f_0 + (1-t)f_1 + \cdots, \tag{2.11}$$
$$\vec{v} = \vec{v}_0 + (1-t)\vec{v}_1 + \cdots, \tag{2.12}$$
$$\vec{h} = \vec{h}_0 + (1-t)\vec{h}_1 + \cdots \tag{2.13}$$

and expanding κ_3 according to Eq. (2.6), we obtain the GL equations[3]

$$\kappa^{-2}\nabla^2 f_0 - f_0 v_0^2 - f_0(f_0^2-1) = 0, \tag{2.14}$$
$$\vec{\nabla}\times\vec{h}_0 = f_0^2\vec{v}_0 \quad\text{or}\quad \vec{\nabla}\times(\vec{\nabla}\times\vec{v}_0) + f_0^2\vec{v}_0 = 0 \tag{2.15}$$

and the generalized NT equations

$$\kappa^{-2}\nabla^2 f_1 - (3f_0^2 - 1 + v_0^2)f_1 - 2f_0\vec{v}_0\cdot\vec{v}_1$$
$$= (\eta_c+\eta_{4d}+3\eta_{4c})f_0(1-f_0^2)(1-3f_0^2) + 2\phi f_0(f_0^2-1+v_0^2) + \kappa^{-2}(6\eta_{4d}+18\eta_{4c}-\eta_k-\eta_w)f_0(\vec{\nabla}f_0)^2$$
$$+(2\eta_{4d}+6\eta_{4c}+\eta_k-\eta_w)f_0^3v_0^2 + [(\eta_k+\eta_w)f_0^2-\eta_k]f_0(1-f_0^2) + \eta_{4c}\kappa^{-2}(3f_0h_0^2 + 2f_0^3v_0^2), \tag{2.16}$$

$$-\vec{\nabla}\times(\vec{\nabla}\times\vec{v}_1) - f_0^2\vec{v}_1 - 2f_0f_1\vec{v}_0 = (2\eta_{4d}+6\eta_{4c}-\eta_k)f_0^2(1-f_0^2)\vec{v}_0 + \eta_{4c}\kappa^{-2}\{2\vec{h}_0\times\vec{\nabla}(f_0^2) - f_0^4\vec{v}_0 + \vec{\nabla}\times[\vec{v}_0\times\vec{\nabla}(f_0^2)]\}. \tag{2.17}$$

These last two equations reduce to the differential equations of NT for f_1 and $\vec{v}_1$ in the case of an isolated vortex.

In general, the evaluation of ΔG from Eq. (2.1) requires the solution of Eqs. (2.16) and (2.17) for f_1, $\vec{v}_1$, and $\vec{h}_1$; even for simple geometries where the partial differential equations go over into ordinary differential equations, a large amount of numerical work is required. For some geometries of interest, however, the unpleasant task of solving Eqs. (2.16) and (2.17) can be avoided if we do not require f_1 and $\vec{v}_1$ as functions of $\vec{r}$ but need only calculate the free-energy difference ΔG. To show this, we go back to expression (2.1) for ΔG and expand in powers of $1-t$; the leading terms are

$$4\pi\Delta G/H_c^2\lambda^3 = -2\int d^3r\,\vec{h}_a\cdot\vec{h}(\vec{r}) + \epsilon_0 + (1-t)\epsilon_1, \tag{2.18}$$

where

$$\epsilon_0 = \int d^3r\,[h_0^{\,2} + \tfrac{1}{2}(1-f_0^{\,2})^2 + \kappa^{-2}(\vec{\nabla} f_0)^2 + f_0^{\,2} v_0^{\,2}] \tag{2.19}$$

and

$$\epsilon_1 = \int d^3r\,[P_0(\vec{r}) - 2\kappa^{-2}\phi(\vec{\nabla} f_0)^2] + 2\int d^3r\,[\vec{h}_0\cdot\vec{h}_1 - f_0 f_1(1-f_0^{\,2}) + \kappa^{-2}\vec{\nabla} f_0\cdot\vec{\nabla} f_1 + f_0 f_1 v_0^{\,2} + f_0^{\,2}\vec{v}_0\cdot\vec{v}_1]\,; \tag{2.20}$$

the subscript on $P_0(\vec{r})$ indicates that the functions f, $\vec{v}$, and $\vec{h}$ in Eqs. (2.3) and (2.10) are to be replaced by f_0, $\vec{v}_0$, and $\vec{h}_0$, respectively.

On using Eqs. (2.14) and (2.15) and some simple vector relations, we find that ϵ_1 can be rewritten as

$$\epsilon_1 = \int d^3r\,[P_0(\vec{r}) - 2\kappa^{-2}\phi(\vec{\nabla} f_0)^2] + 2\int d^3r\,\vec{\nabla}\cdot[\vec{h}_0\times\vec{v}_1 + \kappa^{-2} f_1\vec{\nabla} f_0]\,. \tag{2.21}$$

The point of this procedure is that for some geometries of interest it is possible to eliminate f_1, $\vec{v}_1$, and $\vec{h}_1$ from the expression for ΔG; one need then solve only the GL equations (2.14) and (2.15) and not (2.16) and (2.17). Since the former equations contain only one parameter (κ) while the latter contain two parameters (κ and α), the calculation of ΔG is made an order of magnitude less difficult and involves only slightly more work than the calculation of the GL free energy.

We have investigated two physical situations—a normal-superconducting wall and the mixed state of type-II superconductors; in both cases it is possible to eliminate f_1, $\vec{v}_1$, and $\vec{h}_1$ completely from the expression for ΔG. The calculation of the normal-superconducting wall energy is described in Sec. III and the mixed state is considered in Sec. IV. One can easily show that to calculate the nth-order correction to the GL free energy for these geometries, one need solve only the differential equations for the $(n-1)$th-order corrections to the solutions of the GL equations.

III. *N-S* WALL ENERGY

The energy of formation of a normal-superconducting interface was considered by GL in their famous paper[3] on the phenomenological theory of superconductivity. This energy is calculated from the one-dimensional solutions of Eqs. (2.14) and (2.15): $f = f(x)$, $\vec{h} = \vec{h}(x)\hat{z}$, and $\vec{v} = v(x)\hat{y}$. Loosely speaking, the sample is divided into two parts, the region $x > 0$, which is "superconducting" in that

$$f \to 1,\; v \to 0,\; h \to 0 \qquad \text{as } x \to +\infty\,, \tag{3.1}$$

and the region $x < 0$, which is "normal" in that

$$f \to 0,\; v \to \text{const} - h_a x,\; h \to h_a \qquad \text{as } x \to -\infty\,. \tag{3.2}$$

The separation into normal and superconducting regions is sharp only in the limits $\kappa \ll 1$ and $\kappa \gg 1$, where h and f, respectively, vary rapidly at the interface. Since the free-energy densities at $x = +\infty$ and $x = -\infty$ must be equal, H_a is required to be equal to H_c (i.e., $h_a = 1/\sqrt{2}$).

In the GL regime (T very close to T_c), the N-S wall energy σ_{NS} is given by

$$4\pi\sigma_{NS}/H_c^{\,2}\lambda = g_0(\kappa)\,, \tag{3.3}$$

where

$$g_0 = \int_{-\infty}^{\infty} dx\,[h_0^{\,2} - \sqrt{2}h_0 + \tfrac{1}{2}(1-f_0^{\,2})^2 + f_0^{\,2}v_0^{\,2} + \kappa^{-2}(f_0')^2] \tag{3.4}$$

to lowest order in $1-t$; f_0, v_0, and h_0 in Eq. (3.4) are the solutions of the one-dimensional versions of Eqs. (2.14) and (2.15). GL[3] obtained the approximate expression

$$4\pi\sigma_{NS}/H_c^{\,2}\lambda = 2\sqrt{2}/3\kappa \tag{3.5}$$

for $\kappa \ll 1$, which shows that $\sigma_{NS} > 0$ for $\kappa \ll 1$. They also showed analytically that $\sigma_{NS} < 0$ for $\kappa \gg 1$ and found from numerical solutions of Eqs. (2.14) and (2.15) that the dividing line occurs at $\kappa = 1/\sqrt{2}$, where $\sigma_{NS} = 0$. An expression for σ_{NS}, valid for $\kappa \gg 1$, is[5]

$$4\pi\sigma_{NS}/H_c^{\,2}\lambda = -\tfrac{4}{3}(\sqrt{2}-1)\,. \tag{3.6}$$

The extension of Eq. (3.3) for σ_{NS} to lower temperatures is, from the results of Sec. II,

$$4\pi\sigma_{NS}/H_c^{\,2}\lambda = g_0(\kappa) + (1-t)g_1(\kappa,\,\alpha)\,, \tag{3.7}$$

where

$$g_1 = 2\int_{-\infty}^{\infty} dx\,\frac{d}{dx}\left[\left(\frac{1}{\sqrt{2}} - h_0\right)v_1 + \kappa^{-2} f_1 f_0'\right] + \int_{-\infty}^{\infty} dx\,\{\eta_c f_0^{\,2}(1-f_0^{\,2})^2 - \eta_k(1-f_0^{\,2})[\kappa^{-2}(f_0')^2 + f_0^{\,2}v_0^{\,2}] + \eta_w\kappa^{-2}(f_0 f_0')^2 + (\eta_{4d} + 3\eta_{4c})[\kappa^{-2}f_0'' - f_0 v_0^{\,2}]^2 - 2\phi\kappa^{-2}(f_0')^2 + \eta_{4c}\kappa^{-2}[f_0^{\,2}h_0^{\,2} + 4 f_0 f_0' v_0 h_0]\}\,. \tag{3.8}$$

By virtue of the boundary conditions [Eqs. (3.1) and (3.2)], the first integral in Eq. (3.8) is identically zero and we are left with an expression for g_1 which involves only f_0, h_0, and v_0. Thus it is necessary to solve only the GL equations to obtain the correction to the GL results for σ_{NS}.

We have solved the one-dimensional versions of Eqs. (2.14) and (2.15) numerically for $\kappa = 1/\sqrt{2}$ and give in Table I values of $g_1(1/\sqrt{2},\,\alpha)$ as a function of α; numerical results for $g_0(\kappa)$ and $g_1(\kappa,\,\alpha)$ for $\kappa \neq 1/\sqrt{2}$ will be published separately.[6] The values given in Table I are used in the following article[4] to calculate the critical value of κ as defined by $\sigma_{NS} = 0$.

IV. MIXED STATE OF TYPE-II SUPERCONDUCTORS

In the mixed state of type-II superconductors, f, $\vec{v}$, and $\vec{h}$ are independent of z, $\vec{h}$ is in the z direction, and $\vec{v}$ has no component in the z direction.

TABLE I. $g_1(\kappa, \alpha)$ [defined by Eq. (3.8)] as a function of α for $\kappa = 1/\sqrt{2}$.

α	$g_1(1/\sqrt{2}, \alpha)$
0	−0.0209
0.1	−0.0275
0.2	−0.0328
0.5	−0.0441
1	−0.0548
2	−0.0650
4	−0.0735
10	−0.0823
20	−0.0873
50	−0.0920
100	−0.0943
∞	−0.0979

Our expression (2.1) for ΔG, per unit length in the z direction, becomes

$$4\pi\Delta G/H_c^2\lambda^2 = -2Nh_a 2\pi p/\kappa_3 + \epsilon_0 + (1-t)\epsilon_1 , \quad (4.1)$$

where N is the number of flux unit cells in the sample. ϵ_0 and ϵ_1 are given by

$$\epsilon_0 = \int d^2r \left[h_0^2 + \tfrac{1}{2}(1-f_0^2)^2 + \kappa^{-2}(\vec{\nabla} f_0)^2 + f_0^2 v_0^2\right] \quad (4.2)$$

and

$$\epsilon_1 = \epsilon_{10} + \epsilon_{11} ; \quad (4.3)$$

$$\epsilon_{10} = \int d^2r \left[P_0(\vec{r}) - 2\phi\kappa^{-2}(\vec{\nabla} f_0)^2\right] \quad (4.4)$$

involves only the solutions of the GL equations (2.14) and (2.15) and

$$\epsilon_{11} = 2\int d^2r\, \vec{\nabla}\cdot\left[\vec{h}_0 \times \vec{v}_1 + \kappa^{-2} f_1 \vec{\nabla} f_0\right] \quad (4.5)$$

involves the solutions of both the GL equations and the generalized NT equations (2.16) and (2.17). In deriving Eq. (4.1) we have used the relation

$$\int d^2r\, \vec{h}(\vec{r})\cdot\hat{z} = N \lim 2\pi r \vec{v}(\vec{r})\cdot\hat{\theta} \quad \text{as } r \to 0$$
$$= N 2\pi p/\kappa_3 , \quad (4.6)$$

where p is the number of flux quanta per unit cell; a proof of Eq. (4.6) is given in Appendix B. As in the case of the calculation of the N-S wall energy, it is possible to eliminate the dependence of ϵ_{11} on the solutions of the generalized NT equations. The calculation is given in Appendix C and we quote here only the result

$$\epsilon_{11} = N4\pi h_0(0)(-\phi p/\kappa) , \quad (4.7)$$

where $h_0(0)$ is the microscopic magnetic field (in reduced units) at the center of the flux line to lowest order in $1-t$.

The above results can be used for approximate calculations of the properties of type-II superconductors near $T = T_c$ for all values of the applied field H_a between H_{c1} and H_{c2}; work in this direction is in progress. We restrict our attention here to the case of an isolated vortex $[f = f(r),\ \vec{v} = v(r)\hat{\theta},\ \vec{h} = h(r)\hat{z}]$, in which case the partial differential equations (2.14) and (2.15) reduce to ordinary differential equations.

The case $p = 1$ for an isolated vortex was considered by Abrikosov[7] for $\kappa \gg 1$. The generalization of his result for H_{c1}/H_c to $p > 1$ is[5]

$$H_{c1}/H_c = (p/\sqrt{2}\kappa)\ln\kappa . \quad (4.8)$$

The first calculation of H_{c1}/H_c for $\kappa \approx 1$ was made by Harden and Arp,[8] who obtained numerical solutions of Eqs. (2.14) and (2.15) for the isolated vortex geometry. These calculations are of course based on the GL equations.

NT[1] calculated the first-order corrections to GL results; they solved the isolated vortex forms of Eqs. (2.14)–(2.17), evaluated the integrals in Eq. (2.18), and obtained H_{c1}/H_c by setting ΔG equal to zero. We have repeated their calculations using the formalism described above, which requires only the solutions of the GL equations. We follow the notation of NT[1] in describing the correction to the GL result by

$$H_{c1}/H_c = (H_{c1}/H_c)\big|_{\rm GL}[1 + (1-t)\delta_1(\kappa, \alpha)] ; \quad (4.9)$$

the values of H_{c1}/H_c at $T = T_c$ are given as a function of κ in Table II and $\delta_1(\kappa, \alpha)$ is plotted in Fig. 1. Good agreement with the previous results of NT[1] is found.

The GL theory of a doubly quantized isolated vortex was considered by Matricon[9]; some of his results are given in the review article by Fetter and Hohenberg.[10] We describe our results for H_{c1}/H_c for $p = 2$ by

$$H_{c1}/H_c = (H_{c1}/H_c)\big|_{\rm GL}[1 + (1-t)\delta_2(\kappa, \alpha)] ; \quad (4.10)$$

we give in Table II the values of H_{c1}/H_c at $T = T_c$ as a function of κ and plot $\delta_2(\kappa, \alpha)$ in Fig. 2.

The above results are used in the following article[4] to calculate the critical value of κ for $T < T_c$ from the definition $H_{c1} = H_c$.

APPENDIX A

On going from Eq. (2.9) to (2.10), we have dis-

TABLE II. H_{c1}/H_c at $T = T_c$ as a function of κ for singly quantized ($p = 1$) and doubly quantized ($p = 2$) isolated vortices.

	H_{c1}/H_c at $T = T_c$	
κ	$p = 1$	$p = 2$
$1/\sqrt{2}$	1.0	1.0
1	0.8180	0.8553
2	0.5490	0.6288
5	0.3169	0.4080
10	0.2019	0.2809
20	0.1242	0.1842
50	0.0624	0.0985

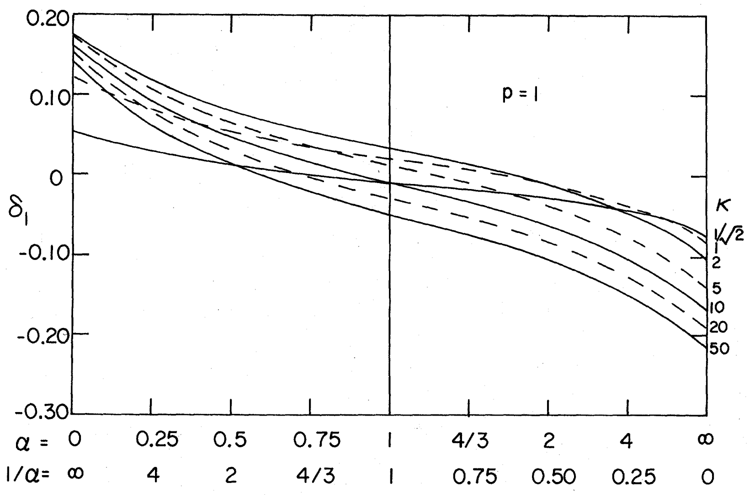

FIG. 1. $\delta_1(\kappa, \alpha)$ as a function of α with κ as a parameter. The values of κ are given at the right-hand side of the graph; the abscissa is linear in α for $\alpha < 1$ and linear in $1/\alpha$ for $\alpha > 1$.

carded two terms; here this step is justified.

The first term which has been neglected is

$$(1-t)(\eta_{4d}+3\eta_{4c})\kappa_3^{-2}\int d^3r\, f^{-2}[\vec{\nabla}\cdot(f^2\vec{v})]^2 .$$

On taking the divergence of the second GL equation (2.15) and using the vector identity $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{V})=0$, where $\vec{V}$ is any vector, we find that

$$\vec{\nabla}\cdot(f_0^2\vec{v}_0)=0 . \qquad \text{(A1)}$$

Thus the above term can be neglected for the purpose of calculating the first-order correction to the GL theory; it does, however, contribute to higher-order corrections.

The second term which has been neglected is

$$\int d^3r\,\vec{\nabla}\cdot\vec{Z} ,$$

where

$$\vec{Z}=(1-t)2\eta_{4c}\kappa_3^{-2}[\tfrac{1}{2}\kappa_3^{-2}\vec{\nabla}(\vec{\nabla}f)^2-\kappa_3^{-2}(\vec{\nabla}f)(\nabla^2 f)+v^2\vec{\nabla}(f^2)+f^2(\vec{v}\cdot\vec{\nabla})\vec{v}-\vec{v}\vec{\nabla}\cdot(f^2\vec{v})] . \qquad \text{(A2)}$$

The neglect of this term must be justified separately for each geometry.

For the N-S wall problem, where $f=f(x)$, $\vec{v}=v(x)\hat{y}$, and $\vec{h}=h(x)\hat{z}$, the above expression reduces to

$$\int_{-\infty}^{\infty} dx\,\vec{\nabla}\cdot\vec{Z}=(1-t)2\eta_{4c}\kappa_3^{-2}\,2v^2 f\,\frac{df}{dx}\bigg|_{x=-\infty}^{\infty} ,$$

which vanishes by virtue of the boundary conditions (3.1) and (3.2).

For the mixed state of type-II superconductors, we show that the above expression vanishes when the integral is taken over a unit cell of the vortex lattice and thus that it vanishes when the integral is taken over the entire volume of the superconductor. The divergence theorem can be used to write

$$\int_{\text{cell}} d^2r\,\vec{\nabla}\cdot\vec{Z}=\oint_{C_1} dl\,\hat{n}\cdot\vec{Z} , \qquad \text{(A3)}$$

where C_1 is the path along the boundary of the unit

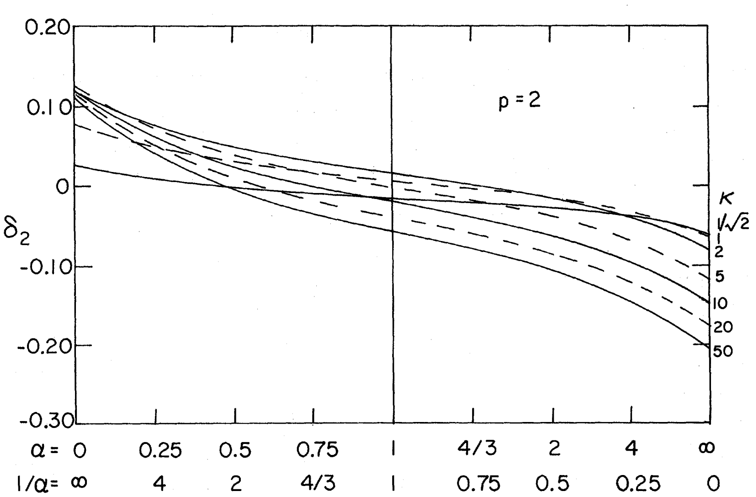

FIG. 2. $\delta_2(\kappa, \alpha)$ as a function of α with κ as a parameter. The values of κ are given at the right-hand side of the graph; the abscissa is linear in α for $\alpha < 1$ and linear in $1/\alpha$ for $\alpha > 1$.

cell and $\hat{n}$ is the unit outward normal to C_1. Since $\vec{v}$ and $\vec{\nabla} f$ at the boundary are normal and parallel to the boundary, respectively, the integral along the path C_1 vanishes and we obtain the desired result

$$\int_{\text{cell}} d^2r\, \vec{\nabla} \cdot \vec{Z} = 0 \ . \tag{A4}$$

The justification for step (A3) is the following: If f and $\vec{v}$ are expanded as

$$f = g_1 r^p + g_2 r^{p+2} + g_3 r^{p+4} + \cdots \ , \tag{A5}$$

$$\vec{v} = p\hat{\theta}/\kappa_3 r + \vec{w}_1 r + \vec{w}_2 r^3 + \cdots \ , \tag{A6}$$

where g_1, $\vec{w}_1$, etc., are independent of r, one finds, for $p > 1$, that $\vec{Z} \to 0$ as $r \to 0$; for $p = 1$, it is necessary to make the reasonable assumption that the coefficient g_1 is independent of θ to obtain this result.

APPENDIX B

To prove Eq. (4.6), we must evaluate the integral

$$I = \int_{\text{cell}} d^2r\, \vec{h}(\vec{r}) \cdot \hat{z} \ . \tag{B1}$$

Since I is just the flux through the unit cell, we have

$$I = p\, \frac{hc}{2e}\, \frac{1}{\sqrt{2}H_c \lambda^2} \ ; \tag{B2}$$

$hc/2e$ is the flux quantum. On using Eq. (2.5), we obtain

$$I = 2\pi p/\kappa_3 \tag{B3}$$

or

$$\int d^2r\, \vec{h}(\vec{r}) \cdot \hat{z} = N 2\pi p/\kappa_3 \ . \tag{B4}$$

Equation (B3) can be used to determine the behavior of $\vec{v}$ near the center of a vortex; since $\vec{h} = -\vec{\nabla} \times \vec{v}$, we find, with the help of Stokes's theorem, that

$$I = 2\pi p/\kappa_3 = -\oint_{C_1} \vec{v} \cdot d\vec{l} - \oint_{C_2} \vec{v} \cdot d\vec{l} + \int_{A_2} d^2r\, h(r) \ , \tag{B5}$$

where C_1 is along the boundary of the unit cell, C_2 is along the circumference of a circle of radius $R \to 0$ about the axis of the vortex, and A_2 is the area inside this circle.

On assuming that $\vec{v}$ at the boundary of the unit cell is perpendicular to the boundary and that $\vec{v}(\vec{r}) \to v(r)\hat{\theta}$ as $r \to 0$, we find, from Eq. (B5), that

$$\vec{v}(\vec{r}) \to p\hat{\theta}/\kappa_3 r \quad \text{as } r \to 0 \ . \tag{B6}$$

APPENDIX C

To evaluate the expression (4.5) for ϵ_{11}, we use the divergence theorem to write

$$\epsilon_{11}/N = \oint_{C_1} dl\, \hat{n} \cdot \vec{Y} + \oint_{C_2} dl\, \hat{n} \cdot \vec{Y} + \int_{A_2} d^2r\, \vec{\nabla} \cdot \vec{Y} \ , \tag{C1}$$

where

$$\vec{Y} = 2(\vec{h}_0 \times \vec{v}_1 + \kappa^{-2} f_1 \vec{\nabla} f_0) \ . \tag{C2}$$

The integral along C_1 is zero since $\vec{v}$ and $\vec{\nabla} f$ at the boundary of the unit cell are, respectively, perpendicular and parallel to the boundary. The integral over the area A_2 is easily seen to approach zero as $R \to 0$ and we are left with

$$\epsilon_{11}/N = \lim 2\pi R \{ -\hat{r} \cdot [h_0(0)\hat{z} \times p\hat{\theta}\, (\kappa_3^{-1} - \kappa^{-1})/R(1-t) + \hat{r}O(R^{2p-1})] \} \ , \tag{C3}$$

$$= -4\pi p h_0(0)\phi/\kappa \ . \tag{C4}$$

Equation (C4) is just Eq. (4.7).

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