PHYSICAL REVIEW B VOLUME 4, NUMBER 7 1 OCTOBER 1971

Tunneling in the Normal-Metal-Insulator-Superconductor Geometry Using the Bogoliubov Equations of Motion*

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(Received 22 March 1971)

The quasiparticle (or thermal current) transmission probability $W(E)$ for excitations going from a normal to a superconducting region through an oxide layer (N-I-S geometry) is calculated from first principles using the Bogoliubov–de Gennes equations of motion. We also work out the transmission probability $\overline{W}(E)$ for electrical currents. For thick oxide layers ($W_{NN} \ll 1$), we find that both $W(E)$ and $\overline{W}(E)$ are given by $W_{NN}\, E/(E^2-\Delta^2)^{1/2}$ for $E > \Delta$. This is in agreement with the tunneling Hamiltonian approach. In the opposite limit of no oxide layer ($W_{NN}=1$), we find that $W(E)$ goes smoothly over into the expression obtained by Andreev for N-S junctions. On the other hand, $\overline{W}(E)$ reduces to unity, as expected. All our results are for sharp interfaces.

I. INTRODUCTION

Since the pioneering work of Cohen *et al.*,[1] it has been customary to compute the electrical current transmission coefficient $\overline{W}_{NS}(E)$ of the N-I-S system (see Fig. 1) using the tunneling Hamiltonian approach (for a review of this type of calculation, see Duke[2]). In the present paper, we make use of the Bogoliubov–de Gennes equations, treating the N-S junction as an *off-diagonal* step-function potential and the oxide layer as a *diagonal* potential barrier. We compute the transmission coefficients for both electrical currents and quasiparticle currents. The latter is denoted by $W_{NS}(E)$ and is appropriate for thermal transport studies. We believe that this is the first time such calculations have been reported for the N-I-S geometry (the case of the N-S junction alone was first considered by Andreev[3]). The advantage of our approach is that we are not limited to thick junctions, such that the coupling of the normal and superconducting metals is weak. Indeed as the oxide-layer thickness goes to zero, our expression for $W_{NS}(E)$ goes smoothly over to Andreev's results, as expected.

Section II is devoted to calculation of $W_{NS}(E)$. Since the method of solving the Bogoliubov equations and matching the solutions at the N-I and I-S interfaces is straightforward, we will be somewhat brief. For similar calculations on the N-S, N-S-N, and S-N-S geometries, we refer the reader to some previous work by the authors.[4] By taking the derivative with respect to temperature,[3,4] our expression for $W_{NS}(E)$ may be used in calculating the extra resistance ΔR for a heat current flowing normal to the oxide layer. Our results generalize some earlier work by Griffin and Maki[5] on ΔR using the tunneling Hamiltonian approach.

In Sec. III, we compute the transmission coefficient $\overline{W}_{NS}(E)$ for electrical currents. For $E < \Delta$, we take into account the induced supercurrent flow, following the work of Kummel.[6] Our general expression for $\overline{W}_{NS}(E)$ reduces to that obtained[1,2] using first-order time-dependent perturbation theory in conjunction with a tunneling Hamiltonian of the kind

$$T=\sum_{\vec{k},\vec{k}'}\left[T_{\vec{k}\vec{k}'}\, a^*_{\vec{k}R}a_{\vec{k}'L}+\text{H.c.}\right], \tag{1.1}$$

with $T_{\vec{k}\vec{k}'}$ being the tunneling amplitude between two *normal* metals (or, more precisely, taken to be independent of the excitation energy).

Finally, in Sec. IV, we briefly discuss some extensions of our model calculations. The most serious limitation of the results given here is that they do not include over-the-barrier transmission. We also comment on the relation between our work and recent microscopic theories of tunneling.

II. QUASIPARTICLE TRANSMISSION COEFFICIENT

We shall take our quasiparticle to be incident from the normal side (see Fig. 1). The wave-function components in the various regions are as follows:

for $x<0$

$$\bar{u}_N(x)=U_+e^{ik_+x}+U_-e^{-ik_+x}, \qquad \bar{v}_N(x)=V_+e^{ik_-x}; \tag{2.1}$$

for $0<x<L$

$$\bar{u}_I(x)=U^I_+e^{\kappa x}+U^I_-e^{-\kappa x}, \qquad \bar{v}_I(x)=V^I_+e^{\kappa x}+V^I_-e^{-\kappa x}; \tag{2.2}$$

for $x>L$

$$\bar{u}_S(x)=U^+_+e^{ik^+x}+U^-_-e^{-ik^-x}, \qquad \bar{v}_S(x)=BU^+_+e^{ik^+x}+B^{-1}U^-_-e^{-ik^-x}. \tag{2.3}$$

Our notation follows Ref. 4. The Bogoliubov amplitudes are given by

$$u(\vec{r})=\bar{u}(x)e^{i\vec{k}_{\parallel}\cdot\vec{r}_{\parallel}},$$

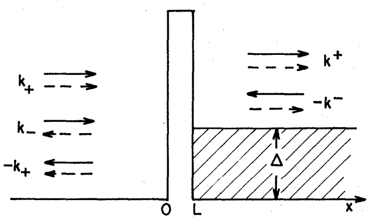

FIG. 1. *N-I-S* geometry with incident excitation (k_+) from the normal side. The solid arrows denote the direction of the propagation vector (phase velocity) while the dashed arrows denote the direction of the group velocity of the various waves.

$$v(\vec{r}) = \bar{v}(x) e^{i\vec{k}_\parallel \cdot \vec{r}_\parallel}\,, \tag{2.4}$$

where $\vec{k}_\parallel$ is the wave vector parallel to the oxide layer, while

$$k_\pm = (k_F^2 - k_\parallel^2 \pm 2mE/\hbar^2)^{1/2}\,,$$
$$k^\pm = (k_F^2 - k_\parallel^2 \pm 2m\Omega/\hbar^2)^{1/2}\,, \tag{2.5}$$

with

$$\Omega = (E^2 - \Delta^2)^{1/2}\,, \quad E > \Delta$$
$$= i(\Delta^2 - E^2)^{1/2}\,, \quad E < \Delta\,. \tag{2.6}$$

For (2.3) to solve the Bogoliubov equations, we must have

$$B^{\pm 1} = \Delta/(E \pm \Omega)\,. \tag{2.7}$$

We might note that B is the ratio of the holelike and particlelike amplitudes of a k^+ wave (and the inverse ratio for a k^- wave):

$$B^2 = (v_{k^+}/u_{k^+})^2 = (u_{k^-}/v_{k^-})^2\,. \tag{2.7$'$}$$

According to (2.7), this *ratio* is the same as for a homogeneous superconductor, in which case

$$v_{k^+}^2 = \tfrac{1}{2}(1 + \Omega/E) = u_{k^-}^2\,,$$
$$u_{k^+}^2 = \tfrac{1}{2}(1 - \Omega/E) = v_{k^-}^2\,. \tag{2.7$''$}$$

The nonoscillatory waves in the potential barrier of height V_0 (measured with respect to the Fermi energy E_F) are described by

$$\kappa_\pm = [k_\parallel^2 + 2m(V_0 \mp E)/\hbar^2]^{1/2} \simeq \kappa \equiv [k_\parallel^2 + 2mV_0/\hbar^2]^{1/2}\,. \tag{2.8}$$

Since these waves are nonoscillatory and $V_0 \gg E$, we have not made any distinction between κ_+ and κ_- in (2.2). The extension of our calculation for $V_0 \lesssim E$ (in which case one has over-the-barrier transmission as well as tunneling) will be deferred to another paper.

In order to determine eight of the nine coefficients in Eqs. (2.1), (2.2), and (2.3), we use the continuity of $\bar{u}(x)$, $\bar{v}(x)$, $\bar{u}'(x)$, and $\bar{v}'(x)$ at $x=0$ and $x=L$. In contrast with the calculations of Ref. 4, we cannot restrict ourselves to scattering processes (incident → reflected) which do not change the sign of the wave vector. The diagonal potential barrier gives rise to processes $k_+ \to -k_+$, in addition to the processes $k_+ \to k_-$ induced by the off-diagonal potential. As a result, we must make use of the continuity of *both* the wave functions and their derivatives. After some algebra, we find

$$|U_+^+|^2 = |B^{-1}|^2 \left[1 + \left(\frac{\kappa^2 + k^2}{2\kappa k}\right)^2 \sinh^2 \kappa L\right] |V_+|^2\,,$$

$$|U_-^-|^2 = |B|^2 \left[\left(\frac{\kappa^2 + k^2}{2\kappa k}\right)^2 \sinh^2 \kappa L\right] |V_+|^2\,, \tag{2.9}$$

$$U_+ = B^{-1}\left[1 + \left(\frac{\kappa^2 + k^2}{2\kappa k}\right)^2 (1 - B^2)\sinh^2 \kappa L\right] V_+\,.$$

We have made use of the approximation

$$k^\pm \simeq k_\pm \simeq k \equiv (k_F^2 - k_\parallel^2)^{1/2} \tag{2.10}$$

in writing down these final results.

The quasiparticle transmission coefficeint $W(E)$ is the ratio of the quasiparticle current density $M(x)$ in the transmitted and incident quasiparticle waves, with

$$M(x) = \frac{\hbar}{m}\,\mathrm{Im}\left(\bar{u}^*(x)\frac{\partial}{\partial x}\bar{u}(x) - \bar{v}^*(x)\frac{\partial}{\partial x}\bar{v}(x)\right)\,. \tag{2.11}$$

By incident waves, we mean those which give rise to a quasiparticle current which is in the same direction as the transmitted current (positive x direction in Fig. 1). We find

$$W_{NS}(E) \simeq (1 - B^2)[\,|U_+^+|^2 + B^{-2}|U_-^-|^2]/|U_+|^2 \quad \text{if } E > \Delta\,, \tag{2.12}$$

where again (2.10) has been used to simplify the final expression. One may prove that $W_{NS}(E<\Delta) = 0$ if it is recalled that $|B|^2 = 1$ for $E < \Delta$. Substituting the results given in (2.9) into (2.12), we have

$$W_{NS}(E) = (1 - B^2)\,\frac{1 + (1 + B^2)\alpha}{[1 + (1 - B^2)\alpha]^2} \quad \text{for } E > \Delta\,, \tag{2.13}$$

where

$$\alpha(k_\parallel) \equiv \alpha \equiv \left(\frac{\kappa^2 + k^2}{2\kappa k}\right)^2 \sinh^2 \kappa L\,. \tag{2.14}$$

This dimensionless parameter α has a very weak dependence on E as result of our use of (2.8). However, it does depend significantly on the value of $k_\parallel$ both through k and κ. While we shall leave this dependence on $k_\parallel$ implicit in most of the following discussion, we would like to emphasize that our computed transition rates [such as (2.13)] depend quite strongly on $k_\parallel$ through their dependence on α. In the limit $\Delta \to 0$, we have $B \to 0$ and hence (2.13) simplifies to

$$W_{NN}(k_\parallel) \equiv W_{NN} = 1/(1+\alpha) , \qquad (2.15)$$

which is the usual transmission coefficient for an ordinary potential barrier. For thick barriers $\kappa L \gg 1$, we may approximate (2.15) by the WKB expression

$$W_{NN} \simeq \alpha^{-1} \simeq 16\left(\frac{\kappa k}{\kappa^2 + k^2}\right)^2 e^{-2\kappa L} \ll 1 . \qquad (2.16)$$

In the limit of $L \to 0$, we find $\alpha \to 0$, and (2.13) then reduces to

$$W_{NS}(E) \simeq 1 - B^2 = \frac{2(E^2 - \Delta^2)^{1/2}}{E + (E^2 - \Delta^2)^{1/2}} . \qquad (2.17)$$

This is the well-known Andreev[3] result for the transmission coefficient across a N-S junction, without any oxide layer.

Another important limiting case of (2.13) is obtained for E which satisfies

$$(1 - B^2) \gg \alpha^{-1} . \qquad (2.18)$$

Since $B(E = \Delta) = 1$, this inequality is only valid for E somewhat larger than Δ. For $\alpha \gg 1$, (2.13) can be well approximated by

$$W_{NS}(E) \simeq W_{NN} \frac{1+B^2}{1-B^2} = W_{NN} \frac{E}{\Omega} . \qquad (2.19)$$

We conclude that the transmission coefficient (defined in terms of the quasiparticle current density) for a particlelike excitation going from the normal side to the superconducting side is given by the ordinary transmission coefficient across the potential barrier times the BCS density of states (normalized to that in a normal metal). This is identical with the lowest-order result obtained by the tunneling Hamiltonian approach in its simplest form,[5] i.e., when the transfer amplitude $T_{\vec{k}\vec{k}'}$ is taken to be independent of the excitation energy so that $|T|^2$ is given by (2.16).

Clearly for E very close to Δ, (2.18) is no longer valid. One may show that for $\alpha \gg 1$, $W_{NS}(E)$ as given by (2.13) reaches a maximum value of $\frac{1}{2}$ at

$$E_0 \simeq \Delta(1 + 1/8\alpha^2 + \cdots) . \qquad (2.20)$$

At the threshold Δ, the transmission coefficient is proportional to $(E-\Delta)^{1/2}\alpha$.

An examination of the structure of (2.13) indicates that $W_{NS}(E)$ has a maximum for any $\alpha > \frac{1}{2}$. As α gets smaller and smaller, the peak shifts to higher energies and is progressively washed out. In Fig. 2, we give a few examples. We emphasize that for typical oxide layers, it is adequate to use the approximation given by (2.19).

Without giving any details, let us simply state that we would obtain the same transmission coefficient as in (2.13) if we had chosen our incident excitation to be holelike with wave vector $-k_-$. In place of (2.1), we would now have

$$\bar{u}_N(x) = U_- e^{-ik_+ x} ,$$
$$\bar{v}_N(x) = V_+ e^{ik_- x} + V_- e^{-ik_- x} . \qquad (2.21)$$

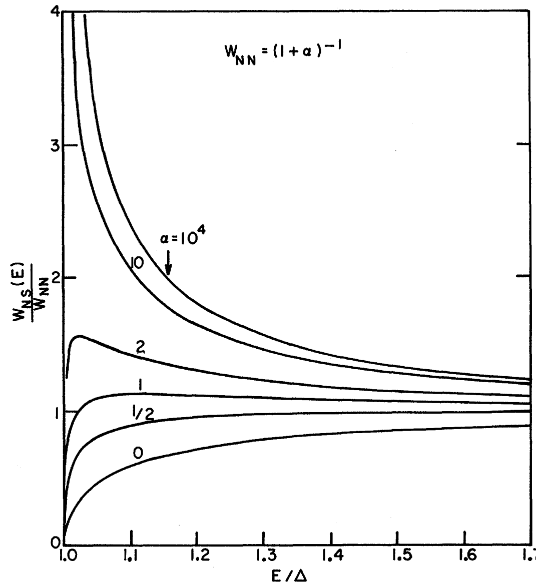

FIG. 2. Quasiparticle current transmission coefficient given by Eq. (2.13) (for the geometry in Fig. 1) as a function of the reduced energy E/Δ. The parameter α is defined in Eq. (2.14). We note that $W_{NS}(E \leq \Delta) = 0$.

As discussed in Ref. 4, the total heat current carried by quasiparticles moving from N to S in the geometry shown in Fig. 1 is given by

$$Q_{NS}^{\perp} = \sum_{\vec{k}_\parallel, k_+} E_k M_k^{\perp}(x<0) f(E_k) + \sum_{\vec{k}_\parallel, -k_-} E_k M_k^{\perp}(x<0) f(E_k) . \qquad (2.22)$$

Here the first term is the contribution due to incident particlelike excitations and the second is due to the incident holelike excitations. The number of such excitations is given by the Fermi-Dirac distribution $f(E_k)$. As in Ref. 4,[7] we may reduce (2.22) to

$$Q_{NS}^{\perp} \simeq \frac{V}{\pi^2 \hbar} \int_0^{k_F} dk_\parallel k_\parallel \int_\Delta^D dE \times E f(E) |U_+|^2 W_{NS}(E, \alpha(k_\parallel)) , \qquad (2.23)$$

where D is some upper cutoff which depends on $k_\parallel$. Because of the Fermi factor, the integral over E in (2.23) depends only very weakly on the value of D and hence we need not specify it any more precisely.

We shall discuss the two limits $\alpha = 0$ and $\alpha \gg 1$.

Normalization gives (for $E < \Delta$)

$$|U_+|^2 = \frac{1}{V}\frac{1}{1+B^2} \qquad \text{if } \alpha = 0 \tag{2.24a}$$

$$\simeq \frac{1}{V}\left[1 + O\left(\frac{1}{\alpha}\right)\right] \quad \text{if } \alpha \gg 1. \tag{2.24b}$$

Making use of (2.13), we obtain

$$Q_{NS}^{\perp} \simeq \frac{1}{\pi^2\hbar}\int_0^{k_F} dk_{\parallel}\, k_{\parallel}\int_{\Delta}^{D} dE\, E\, \frac{\Omega}{E}\, f(E) \tag{2.25}$$

in the absence of any oxide layer ($\alpha = 0$), and

$$Q_{NS}^{\perp} \simeq \frac{1}{\pi^2\hbar}\int_0^{k_F} dk_{\parallel}\, k_{\parallel}\int_{\Delta}^{D} dE\, E\, \frac{E}{\Omega}\, W_{NN}(k_{\parallel}) f(E) \tag{2.26}$$

in the opposite limit ($W_{NN} \simeq \alpha^{-1} \ll 1$). It is interesting to note that (2.26) leads to the same result as obtained by Griffin and Maki[5] using the tunneling Hamiltonian approach.

On the other hand, (2.25) does not appear to agree with the well-known result of Andreev,[3] who obtained

$$Q_{NS}^{\perp} \simeq \frac{1}{\pi^2\hbar}\int_0^{k_F} dk_{\parallel}\, k_{\parallel}\int_{\Delta}^{D} dE\, \Omega\, \frac{2E}{E+\Omega}\, f(E)\ . \tag{2.27}$$

The reason for this difference is that Andreev used

$$|U_+|^2 = 1/V\ . \tag{2.28}$$

However, as discussed in Ref. 4, the normalization constant is energy dependent and while (2.28) is correct for $E < \Delta$, we obtain (2.24a) for $E > \Delta$. Our result, (2.25), for the heat current going from N to S appears to be reasonable if we remember that the heat current is proportional to the group velocity of the excitations going from $N \to S$, which is given by

$$v_S(E) \equiv \frac{1}{\hbar}\frac{\partial E_k}{\partial k} = \frac{\hbar k}{m}\frac{\Omega}{E}\ . \tag{2.29}$$

In this connection, we might note explicitly that a generalized group velocity may be defined using (2.11). This is discussed at some length by Kummel.[6]

III. TRANSMISSION COEFFICIENT FOR ELECTRIC CURRENTS

The electrical current density $J(x)$ carried by a quasiparticle wave (described by the Bogoliubov amplitudes $\bar{u}$ and $\bar{v}$) is given by

$$J(x) = \frac{e\hbar}{m}\,\mathrm{Im}\left(\bar{u}^*(x)\frac{\partial}{\partial x}\bar{u}(x) + \bar{v}^*(x)\frac{\partial}{\partial x}\bar{v}(x)\right)\ . \tag{3.1}$$

This should be compared with the quasiparticle current density $M(x)$ given in (2.11). We shall find that the ratio of the transmitted to the incident electrical current, denoted by $\overline{W}_{NS}(E)$, may be quite different from (2.13).

In calculating $\overline{W}_{NS}(E)$, we shall use an alternative way of dealing with the N-I-S system to that used in Sec. III. That is, we shall work out the wave functions for a N-S system with a zero-range diagonal potential barrier at $x = 0$, denoted by $\lambda\delta(x)$. The wave-function components are given by (2.1) for $x < 0$ and (2.3) for $x > 0$. We may determine four of the five amplitudes by using the continuity of $\bar{u}(x)$ and $\bar{v}(x)$ at $x = 0$ and the fact that the derivatives are discontinuous,

$$\begin{aligned}(\hbar^2/2m)[\bar{u}'(0^+) - \bar{u}'(0^-)] &= \lambda\bar{u}(0)\ ,\\ (\hbar^2/2m)[\bar{v}'(0^+) - \bar{v}'(0^-)] &= \lambda\bar{v}(0)\ .\end{aligned} \tag{3.2}$$

One finds

$$\begin{aligned} U_-^- &= iB\bar{\lambda}V_+\ , & U_- &= -(1-B^2)B^{-1}i\bar{\lambda}(1-i\bar{\lambda})V_+\ ,\\ U_+^+ &= B^{-1}(1-i\bar{\lambda})V_+\ , & U_+ &= B^{-1}[1+(1-B^2)\bar{\lambda}^2]V_+\ ,\end{aligned} \tag{3.3}$$

where $\bar{\lambda} \equiv \lambda m/\hbar^2 k$ and use has been made of (2.10). Substituting these coefficients into (2.12), we find (2.13) once again if we make the identification $\bar{\lambda}^2 = \alpha$. The advantage of treating the oxide layer as a δ function of strength $\lambda = \hbar^2 k\alpha^{1/2}/m$ is that the amplitudes in (3.3) are much simpler than the equivalent ones which the procedure used in Sec. II entails. We wish to emphasize, however, that the results for $\overline{W}_{NS}(E)$ which are obtained in this section are identical to those which would result if the more complicated procedure of Sec. II were used.

Calculating the transmission $\overline{W}_{NS}$ and reflection $\overline{R}_{NS}$ coefficients for electrical currents, we have (for $E > \Delta$)

$$\overline{W}_{NS}(E) = (1+B^2)\,\frac{|U_+^+|^2 - B^{-2}|U_-^-|^2}{|U_+|^2 + |V_+|^2}\ , \tag{3.4}$$

$$\overline{R}_{NS}(E) = \frac{|U_-|^2}{|U_+|^2 + |V_+|^2}\ . \tag{3.5}$$

Making use of the coefficients in (3.3), we obtain ($E > \Delta$)

$$\begin{aligned}\overline{W}_{NS}(E) &= \frac{(1+B^2)[1+(1-B^2)\alpha]}{B^2 + [1+(1-B^2)\alpha]^2}\ ,\\ \overline{R}_{NS}(E) &= \frac{(1-B^2)^2\alpha(1+\alpha)}{B^2+[1+(1-B^2)\alpha]^2}\ .\end{aligned} \tag{3.6}$$

We note that these results satisfy

$$\overline{W}_{NS}(E) + \overline{R}_{NS}(E) = 1\ . \tag{3.7}$$

As with our discussion of (2.13), it is useful to consider two limits of (3.6):

$$\overline{W}_{NS}(E) \simeq 1 - \alpha\frac{(1-B^2)^2}{(1+B^2)} + \cdots \quad \text{for } \alpha \ll 1 \qquad (3.8a)$$

$$\simeq \alpha^{-1}\left(\frac{1+B^2}{1-B^2}\right) - \alpha^{-2}\,\frac{1+B^2}{(1-B^2)^2} + \cdots \quad \text{for } \alpha \gg 1\ . \qquad (3.8b)$$

Thus we find that in the absence of an oxide layer, an electrical current carried by Bogoliubov excitations is unaffected by a *NS* boundary. This agrees with results given in the last section of Ref. 4.[8] In the opposite limit of a thick oxide layer, we see that the electrical current transmission coefficient given by (3.8b) is identical to that for thermal (or quasiparticle) currents, as given by (2.19).

In Fig. 3, we have plotted $\overline{W}_{NS}(E)$ in (3.6) for several intermediate values of α. As with the case of thermal currents discussed in Sec. II, our results are not limited to oxide layers with a small tunneling cross section. Of course, for the realistic case of $\alpha \gg 1$, our result for $\overline{W}_{NS}(E)$ agrees exactly with that obtained using first-order time-dependent perturbation theory in conjunction with a tunneling Hamiltonian.[1,2] The net transmitted electrical current due to the tunneling of single excitations moving from N to S is

$$J^{\perp}_{NS} \simeq \frac{e}{\pi^2\hbar}\int_0^{k_F} dk_{\parallel}\, k_{\parallel}\int_{\Delta}^{D} dE\,\frac{E}{\Omega}\,W_{NN}(k_{\parallel})f(E)\ . \qquad (3.9)$$

In concluding this section, we wish to comment on a few features of the preceding calculations. In using the wave functions given by (2.1) in the thermal current problem, the incident part is given by a pure particlelike wave k_+ while the reflected part is a superposition of the particlelike wave $-k_+$ and the holelike wave k_-. On the other hand, for electrical current flow, the incident part is given by a superposition of the particlelike wave k_+ and the holelike wave k_-, while the reflected part is a pure particlelike wave $-k_+$. Moreover, the transmitted electrical current [as given by the wave functions in (2.3)] in the superconducting side is interesting in that one has holelike waves $-k^-$. One might try to obtain an alternative solution (for $x>0$) given by

$$\overline{u}_s(x) = U_+^+ e^{ik^+x} + U_+^- e^{ik^-x}\ ,$$
$$\overline{v}_s(x) = BU_+^+ e^{ik^+x} + B^{-1}U_+^- e^{ik^-x}\quad . \qquad (3.10)$$

Calculation shows that this solution requires that $\overline{v}_N(x)=0$. However, the wave function in (3.10) is not bounded since

$$e^{ik^{\pm}x} \simeq e^{ikx \mp sx}, \quad s \equiv (m/\hbar^2 k)(\Delta^2 - E^2)^{1/2} \qquad (3.11)$$

for $E<\Delta$.

Finally, we wish to study briefly the transmission coefficient for electrical currents carried by excitations with $E<\Delta$. With (2.1) and (2.3), this is given by

$$\overline{W}_{NS}(E<\Delta) = (1+|B|^2)\,\frac{[\,|U_+^+|^2 - |B^{-1}|^2|U_-^-|^2]\,e^{-2sx}}{|U_+|^2 + |V_+|^2}\quad . \qquad (3.12)$$

As with the case $E>\Delta$ [see (3.4)], one finds that there is no contribution from the interference terms to the transmitted current, i.e., terms involving $e^{i(k^+ + k^{-*})x}$ and $e^{-i(k^{+*}+k^-)x}$. Making use of

$$B^{\pm 1} = [E \mp i(\Delta^2 - E^2)]^{1/2}/\Delta \quad \text{for } E<\Delta\ , \qquad (3.13)$$

we may reduce (3.12) to

$$\overline{W}_{NS}(E<\Delta) = \frac{e^{-2sx}}{1 + 2[(\Delta^2 - E^2)/\Delta^2]\alpha(\alpha+1)}\ . \qquad (3.14)$$

The actual transmitted electrical current ($x>0$) carried by an excitation of energy $E<\Delta$ is given by

$$J^{T}_{NS}(x) = 2e\hbar k/m\,|V_+|^2 e^{-2sx}\ . \qquad (3.15)$$

Making use of the normalization condition and (3.3), we obtain

$$1/V = \tfrac{1}{2}\,|V_+|^2[\,|1+(1-B^2)\alpha|^2 + |1-B^2|^2 + 1]\ , \qquad (3.16)$$

which may be simplified to

$$|V_+|^2 = \frac{1}{V}\left(1 + 4\,\frac{\Delta^2 - E^2}{\Delta^2}\,\alpha(\alpha+1)\right)^{-1}\ . \qquad (3.16')$$

The transmitted electrical current is exponentially damped in a distance $\sim s^{-1}$ because quasiparticles with $E<\Delta$ are not stable in the superconducting side.

The reflection coefficient for electrical currents carried by quasiparticles is found to be

$$\overline{R}_{NS}(E<\Delta) = \frac{2[(\Delta^2 - E^2)/\Delta^2]\alpha(\alpha+1)}{1 + 2[(\Delta^2 - E^2)/\Delta^2]\alpha(\alpha+1)}\ . \qquad (3.17)$$

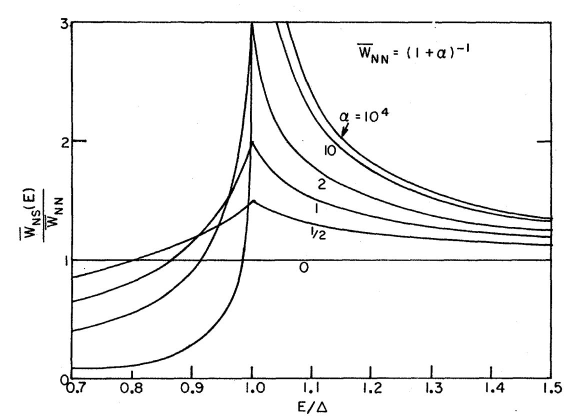

FIG. 3. Electrical current transmission coefficient $\overline{W}_{NS}$ given by (3.6) for $E>\Delta$ and (3.19) for $E<\Delta$, as a function of the reduced energy E/Δ. For large values of α (such as 10^4), $\overline{W}_{NS}(E<\Delta) \simeq 0$.

We note that for $\alpha=0$, $\overline{R}_{NS}(E<\Delta)=0$ and hence $\overline{W}_{NS}(E<\Delta)=1$ if we use (3.7). On the other hand, for $\alpha\gg1$, we find (3.17) gives $\overline{R}_{NS}(E<\Delta)=1$ and hence $\overline{W}_{NS}(E<\Delta)=0$. One may argue that these results for $\overline{W}_{NS}(E<\Delta)$ differ from that obtained by *direct* calculation of the transmitted current because we have not included the induced supercurrent contribution in (3.15) or (3.14). It may be easily verified that if we take (3.1) to be the electrical current density, it is not always conserved, since

$$\nabla\cdot\vec{J}(\vec{r},t)+\frac{\partial}{\partial t}\rho(\vec{r},t)=\frac{4e}{\hbar}\,\mathrm{Im}(\Delta\overline{v}\,\overline{u}^*)\,, \qquad (3.18)$$

where the charge density is defined by

$$\rho(\vec{r},t)\equiv e[|\overline{u}|^2-|\overline{v}|^2]\,.$$

While the right-hand side of (3.18) vanishes trivially in the normal region (since $\Delta=0$), it is finite in the superconducting side if we use (2.3) for the transmitted excitations. As discussed at some length by Kummel,[6] this problem arises because we have neglected to include the supercurrent (due to Cooper pairs) produced by the transmitted quasiparticles. The simplest way of correctly including this supercurrent in calculating the transmission coefficient is to use (3.17) in conjunction with (3.7), which gives

$$\overline{W}_{NS}(E<\Delta)=\frac{1}{1+2[(\Delta^2-E^2)/\Delta^2]\alpha(\alpha+1)}\,. \qquad (3.19)$$

We have plotted (3.19) in Fig. 3, normalizing the result with respect to $\overline{W}_{NN}$. We note that $\overline{W}_{NS}$ for $E<\Delta$ and $\overline{W}_{NS}$ for $E>\Delta$ [given by (3.6)] both equal unity at $E=\Delta$. This is another indication of the fact that the electrical current is not proportional to the group velocity of the excitation (this vanishes at $E=\Delta$). The limiting expression $\overline{W}_{NS}(E)=1$ for $\alpha=0$ is in agreement with measurements on the electrical resistance of superconductors in the intermediate state. On the other hand, there is no electrical current through a typical oxide layer for $E<\Delta$, a result consistent with $\overline{W}_{NS}(E<\Delta)\simeq0$ for $\alpha\gg1$.

IV. CONCLUDING REMARKS

In this paper, we have given a detailed discussion of thermal and electrical currents in the *N-I-S* geometry, with the incident waves coming from the normal side. According to detailed balance, in thermal equilibrium, these currents must be equal in magnitude to those flowing from the superconducting side. Explicit calculations for the *S-I-N* geometry will be given elsewhere.

It is clear that similar calculations can be done using the Bogoliubov equations for *S-I-S* and *S-N-S* geometries. Recently, Kulik[9] has given a detailed analysis of the Josephson supercurrent in the *S-N-S* geometry, assuming that there is phase difference between the two superconductors. Using the wave functions given in Ref. 4, one may easily compute the *quasiparticle* contribution to the thermal and electrical currents flowing from left to right in the *S-N-S* geometry in the case of zero phase difference. We obtain

$$Q^{\perp}_{SS}\propto\int_0^D d\Omega\, Ef(E)(1-B^2)|U^+_+|^2W_{SS}(E)$$

$$=\int_\Delta^D dE\, EW_{SS}(E)f(E) \qquad (4.1)$$

for the thermal current[10] and

$$J^{\perp}_{SS}\propto\int_0^D d\Omega f(E)(1+B^2)|U^+_+|^2W_{SS}(E)$$

$$=\int_\Delta^D dE\,(E/\Omega)W_{SS}(E)f(E)\,, \qquad (4.2)$$

where

$$W_{SS}(E)\equiv W_{SS}(E,k_\parallel)\equiv W_{SS}(E;k^+-k^-)$$

is the quasiparticle transmission probability given by Eq. (2.34) of Ref. 4. It is a straightforward matter to extend such calculations and discuss how the heat current depends on the phase difference in the *S-N-S* and *S-I-S* geometries. The latter was studied some years ago using a tunneling Hamiltonian approach.[11]

In conclusion, let us summarize what has been accomplished in this paper. By using a somewhat idealized model of a normal-metal–insulator–superconductor junction, we have been able to find analytical expressions for the thermal and electrical current transmission coefficients. The simplicity of the model allowed an essentially exact solution of the Bogoluibov equations of motion to be obtained. Nowhere did we have to assume that the two sides of the oxide layer were weakly coupled or resort to perturbation theory. However, in the limit of a small tunneling probability, our general expressions essentially reduced to those found using first-order perturbation theory in a tunneling interaction.[1,5]

In our model calculation, we have assumed that the normal and superconducting regions can be described in terms of well-defined quasiparticles. We have neglected all band-structure and many-body effects.[2] Moreover, the oxide layer has been treated as a static potential barrier. In order to simplify our analysis, the potential height V_0 was taken to be sufficiently large so that over-the-barrier transmission of excitations (such as discussed in Ref. 4) was negligible and approximation (2.8) could be invoked. Certainly it would be more realistic from an experimental point of view to decrease the value of the parameter α defined in (2.14) by lowering V_0 rather than decreasing the oxide-layer thickness L.

It seems clear that if we had not made use of approximation (2.8), the analog of the parameter α in (2.14) would depend explicitly on E as well as on $k_{\parallel}$. The fact that it depends on the true excitation energy E rather than on the free-particle energy is of some inportance since these are not necessarily the same. In the limit of weak coupling, then, our expression for the effective transfer amplitude would disagree with that used in the tunneling Hamiltonian approach in that $T_{\vec{k}\vec{k}'}$ would depend on the excitation energy E as well as on $k_{\parallel}$. A similar conclusion has been obtained in recent microscopic calculations[12] of the electrical tunneling current when the latter is small enough to allow first-order perturbation theory to be used.

As a final remark, we should like to emphasize that in the present paper we have only computed the single-particle tunneling current. This should be contrasted with using perturbation theory to compute the so-called second-order tunneling current, which involves the transfer of two excitations across the oxide layer.[13] This second-order current is proportional to $|T|^4$ but has nothing to do with the second term in (3.8b). While the latter also can be viewed as being of order $|T|^4$, it is a higher-order contribution to the single-particle tunneling current. The analogous term does not appear to have been computed directly using the tunneling interaction (1.1) in conjunction with second-order perturbation theory.

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*Work supported by the National Research Council of Canada.

†Supported by a Province of Ontario Fellowship and the Reginald Blyth Fund.

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[8]In this connection, the phrase preceding Eq. (4.1) of Ref. 4 should read "electrical current" instead of "quasiparticle current," and the phrase preceding Eq. (4.6) should read "The quasiparticle current is given by ...."

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