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Transport in a Magnetic Field. IV. Cyclotron Resonance and Related Effects*

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The surface impedance of an ideal slab is calculated for Azbel'-Kaner geometry using the formulas of I and II in this series. The cyclotron-resonance frequency and the subharmonic frequencies are unshifted by particle interactions. This result is dependent on the details of our formulas of I and II. Additional structure in Z is found including a TE surface-plasmon resonance.

I. INTRODUCTION

Charged particles moving in a magnetic field may be excited by illumination by rf energy at the frequency of their cyclotron precession. For free electrons

$$\omega_c = eH/mc, \tag{1}$$

where m is the free-electron mass. Electrons in solids, however, are subject to a variety of forces which complicate the interpretation of those resonances observed. Our purpose in this discussion is to study the interpretation of observed cyclotron resonance in a simple metal.

Charged particles deform the crystal lattice about them; this deformation tends to follow their motion, and thus having to drag a portion of the lattice about, the electrons may be made effectively more massive and the cyclotron-resonance frequency correspondingly reduced.

The electrons also interact with one another, and this might be expected to change their cyclotron mass, the value of m which must be used to make Eq. (1) match the observed resonances.

Azbel'[1] suggested that the quasiparticle mass is measured in cyclotron resonance. This suggestion is based on the Silin transport equation[2] applied to the anomalous-skin-effect frequency range. Luttinger has further considered this problem in an unpublished study. Platzman and Jacobs[3] have reported Luttinger's agreement with Azbel'.

In previous publications[4] we have obtained equations for electron magnetotransport which differ from Silin's in just the combined effects of interactions and cyclotron precession. We apply these transport equations here to cyclotron resonance. We find that the frequencies are unshifted by the interactions, that the bare particle, rather than the quasiparticle, mass is measured, that the observed mass changes[5] should be attributed to the phonon and lattice interactions.

The effects of these interactions are considerably larger than those to be expected from particle-particle interactions by more than one order of magnitude. The question we raise is therefore unlikely to be resolved in direct measurements of cyclotron resonance.

In III[6] of this series, we discussed the conduction-electron spin-resonance (CESR) experiments of Schultz and Dunnifer[7] and the interpretation of them given by Platzman and Wolff[8] using Silin's[2] formulism. The cyclotron mass is used in interpreting these experiments, and the question of whether or not to incorporate interparticle effects influences the algebraic form of the results. Since cyclotron resonance is itself a result of a transport process, however, it would be inconsistent to use our transport formalism on the CESR measurements but using a cyclotron mass interpreted by Silin's equation. We found in III that our transport equations led to a single parameter fit of the data using a cyclotron mass which was unshifted by the interactions. This paper provides the detailed justification for that unconventional usage.

The theory of cyclotron resonance is complicated by the skin effect. Because the electron gas is so highly conducting, rf energy penetrates only a short distance into a sample. In the frequency range of interest, this distance is much smaller than the cyclotron radius. Many different wavelengths are

therefore required for a description, i.e., Fourier analysis, of the rf internal fields, and surface conditions are of some importance.

Cyclotron resonance is observed as a series of peaks in the derivative dZ/dH, where Z is the surface impedance and H the dc magnetic field.[5] We therefore devote our attention to the surface impedance. We write down Maxwell's equations for the interior of the metal. This gives one linear relation between $\vec{E}$ and $\vec{j}$, the Fourier components of the electric field and the current density. A second relation must be found from the equations of motion of the electron gas. To obtain this second linear relation, we write the Liouville equation for the density matrix in a representation appropriate to specular electron reflection at the metallic surface. We express the current density $\vec{j}_{\vec{q}}$ as an integral of certain matrix elements of the density operator, obtaining these matrix elements from solving the Liouville equation. The matrix elements prove to be linear in $\vec{E}_{\vec{q}}$, and this provides our second relation between $\vec{E}_{\vec{q}}$ and $\vec{j}_{\vec{q}}$, the conductivity.

The surface impedance is found from $\vec{E}$ at the sample surface, and this quantity is found by integrating $\vec{E}_{\vec{q}}$, $\vec{j}_{\vec{q}}$ having been eliminated between the two relations. We identify the cyclotron-resonance peaks and an additional collection of higher-frequency resonances associated with modes localized on the surface. We note as mentioned that the cyclotron resonances are unshifted.

II. WAVE EQUATION

From Maxwell's equations we can write a wave equation[9] on the electric fields and current:

$$\frac{\partial^2\vec{E}}{\partial y^2}+\frac{\omega^2\vec{E}}{c^2}=\frac{4\pi}{c^2}\frac{\partial\vec{j}}{\partial t}=\frac{4\pi i\omega\vec{j}}{c^2}\ , \tag{2}$$

all quantities being assumed to vary in time like $e^{i\omega t}$. We let $y>0$ be the interior of the metal and $y<0$ the exterior. The constant magnetic field lies in the $\hat{z}$ direction. If the space $y<0$ were filled with a similar metal sample, we would have

$$\vec{E}(y)=\vec{E}(-y);\qquad \frac{\partial\vec{E}(y)}{\partial y}=-\frac{\partial\vec{E}(-y)}{\partial y}\ . \tag{3}$$

Now $\vec{E}(y)$ is a continuous function, but $\vec{E}'=\partial\vec{E}/\partial y$ has a step discontinuity at the surface, hence

$$\frac{\partial^2\vec{E}}{\partial y^2}=2\vec{E}'\,\delta(y)+\text{nonsingular part.} \tag{4}$$

Fourier analyzing gives

$$2\vec{E}'-q^2\vec{E}_q+(\omega^2/c^2)\,\vec{E}_q-(4\pi i\omega/c^2)\,\vec{j}_q=0, \tag{5}$$

where

$$\vec{E}_q=\int_{-\infty}^{\infty}e^{-iqy}\,\vec{E}(y)\,dy\ . \tag{6}$$

III. TRANSPORT EQUATION

To obtain a solution of (5), we need to know the conductivity $\sigma(q,\omega)=j_q(\omega)/E_q(\omega)$. We have elsewhere[4] given a transport equation for an electron gas disturbed by an electromagnetic field. The conductivity so obtained proves appropriate for the specularly reflected case but this assertion requires demonstration.

The one-electron basis function must vanish at the surfaces $y=0$ and L. Appropriate functions are

$$u_{\vec{k}}=e^{i(k_x x+k_z z)}\sin k_y y\ , \tag{7}$$

where

$$k_y=n\pi/L,\quad n=1,2,\ldots\ .$$

There are twice as many positive values of k_y with these boundary conditions as with periodic boundary conditions, which would give us the condition

$$k_y=\pm\,2\pi n/L\neq 0\ . \tag{8}$$

The total density of states in energy is unchanged.

Under the action of a magnetic field, the wave functions $u_{\vec{k}}$ tend to evolve into

$$v_{\vec{k}}=e^{ik_x x+ik_z z}\cos k_y y. \tag{9}$$

These functions are not independent of the $u_{\vec{k}}$;

$$v_{\vec{k}}=\sum_{\vec{k}'}a_{\vec{k}'}\,u_{\vec{k}'}\ . \tag{10}$$

When

$$k_y=k_e\equiv 2m\pi/L,$$

all the $a_{k_e'}$ vanish. Similarly, when

$$k_y=k_0\equiv(2m+1)\pi/L\ ,$$

the $a_{k_0'}=0$. Thus if it were not for the boundary condition, we could replace the u_{k_0} by the v_{k_e}.

To determine the relative error introduced by such a substitution, we evaluate the coefficients a_k

$$\cos\frac{my\pi}{L}=\sum_{n,m}\left(\frac{1}{m+n}+\frac{1}{m-n}\right)\left(\sin n\frac{y\pi}{L}\right)\left(\frac{2}{\pi}\right)^{1/2}. \tag{11}$$

But the sum on the right-hand side can be written

$$\sum_{p=-\infty}^{\infty}\frac{\sin(2p+1)y\pi/L}{2p+1}\left(\frac{2}{\pi}\right)^{1/2}\cos\frac{my\pi}{L}\ . \tag{12}$$

We can terminate the sum over n to keep $m-n=p$ small, approximating v_k as a finite sum. The measure of the error will be the difference between unity and

$$\sum_{-p_0}^{p_0}\frac{\sin(2p+1)y\pi/L}{2p+1}\left(\frac{2}{\pi}\right)^{1/2}\approx 1. \tag{13}$$

This approximation becomes valid for

$$2p_0 \approx L/y \ . \tag{14}$$

To use this approximation, we must have that the spread in k values, Δk, of functions used for expressing a single v_k, must be $\ll k_F$, and $y_{\min}$, the smallest y value for which (14) holds, should be $\ll$ rf skin depth. In fact, as we shall see, cyclotron resonance is dominated by the $q \cong 0$ terms in E_q so that the latter condition is stronger than necessary. However, for typical experimental conditions,[5] these restrictions can be simultaneously well satisfied. When $y_{\min}$ is of the order of one lattice constant, $\Delta k \approx k_F$ and the skin depth is many lattice constants. We are led to standing waves $\sin k_y y$ and $\cos k_y y$ as basis functions for the formulation of the transport problem. We make the same error by combining $\sin k_y y$ and $\cos k_y y$ into exponential running wave functions appropriate to the bulk problem.

The quantities we calculate are certain integrals over the electron distribution. The important quantities are determined by the angular variation of the distribution over the Fermi surface, and depend only weakly on the precise variation of distribution with energy. If our concern were with quantities depending on more precise information about the energy dependence of the distribution, say the specific heat or Seebeck coefficient, we should need to take better account of the boundary conditions. $\Delta k/k_F$ is not small compared to $\kappa T/E_F$ or to q/k_F for reasonable values of $y_{\min}$. Our expressions contain things like $\mathfrak{N}_{\vec{k}} = n_{\vec{k}+\vec{q}} - n_{\vec{k}}$, which we write

$$\mathfrak{N}_{\vec{k}} = (\vec{q}\cdot\vec{k}/k_F)\delta(k - k_F). \tag{15}$$

However, that this may be spread out over a range Δk, rather than confined strictly to the Fermi surface, proves insignificant as long as $\Delta k \ll k_F$.

From Ref. 4 we have

$$i\hbar\frac{\partial \mathfrak{D}_{\vec{k}}}{\partial t} + \left(\mathcal{E}_{\vec{k}} - \mathcal{E}_{\vec{k}+\vec{q}} - i\hbar\frac{e}{mc}\vec{B}\times\vec{k}\cdot\vec{\nabla}\right)\mathfrak{D}_{\vec{k}} = \mathfrak{N}_{\vec{k}}\left(\frac{ie\hbar}{m\omega}\vec{E}_{\vec{q}}\cdot\vec{k} + \sum_{\vec{k}'} V(\vec{k}\vec{k}')\mathfrak{D}_{\vec{k}'} - 2V(\vec{q})\sum_{\vec{k}'}\mathfrak{D}_{\vec{k}'}\right), \tag{16}$$

where

$$\vec{j}_q = -\frac{e\hbar}{m}\sum_{\vec{k}'}\vec{k}'\,\mathfrak{D}_{\vec{k}'}\,, \tag{17}$$

$$\mathcal{E}_{\vec{k}} = \frac{\hbar^2 k^2}{2m} - \sum_{\vec{k}'} V(\vec{k}\vec{k}')\, n_{\vec{k}'}\,. \tag{18}$$

Consider the right-hand side of (16). $\mathfrak{N}_{\vec{k}}$ is the only rapidly varying function in the neighborhood of $k = k_F$. $\mathfrak{D}_{\vec{k}}$ must thus have the same rapid variation. On the left-hand side, the term in $\vec{B}$ instructs us to take a directional derivative perpendicular to $\vec{B}$ and $\vec{k}$, thus tangential to the Fermi surface. We integrate this equation over $|\vec{k}|$. In this process, the details of the precise behavior of the functions near k_F are washed out, as we mentioned, and we are left with an integrodifferential equation on $\mathfrak{D}$ as a function of angles θ and φ. We measure θ from the magnetic field direction, and obtain

$$j_q = -\frac{e\hbar}{m}\frac{k_F^3}{4\pi^3}\int d\Omega\,(\hat{z}\cos\theta + \hat{y}\sin\theta\cos\varphi + \hat{x}\sin\theta\sin\varphi)\mathfrak{D}(\theta,\varphi) \tag{19}$$

and

$$i\hbar\frac{\partial}{\partial t}\,\mathfrak{D}(\theta,\varphi) + \left(\frac{-\hbar^2}{m^*}\vec{k}_F\cdot\vec{q} - i\hbar\frac{eB}{mc}\frac{\partial}{\partial\varphi}\right)\mathfrak{D}(\theta,\varphi) = -q\cos\varphi\sin\theta\left(\frac{ie\hbar}{m\omega}\vec{E}_q\cdot\vec{k}_F + \sum_{\vec{k}'} V(\vec{k}\vec{k}')\mathfrak{D}_{\vec{k}'} - 2V(\vec{q})\sum_{\vec{k}'}\mathfrak{D}_{\vec{k}'}\right). \tag{20}$$

To solve this equation, we assume steady-state conditions, whence

$$i\hbar\frac{\partial}{\partial t}\,\mathfrak{D}(\theta,\varphi) = -\hbar\omega\,\mathfrak{D}(\theta,\varphi), \tag{21}$$

and treat the integral portions on the right-hand side all as an inhomogeneous field

$$\left(g(\theta)\cos\varphi + i\frac{\partial}{\partial\varphi} + \lambda\right)\mathfrak{D}(\theta,\varphi) = h(\theta,\varphi), \tag{22}$$

$$g = \frac{\hbar k_F q}{m^*\omega_c}\sin\theta, \tag{23}$$

$$\lambda = \omega/\omega_c\,, \tag{24}$$

$$h(\theta,\varphi) = q\cos\varphi\sin\theta\left(\frac{eiE_q k_F}{m\omega\omega_c}\cos\theta + \frac{1}{\hbar\omega_c}\sum_{\vec{k}'}[V(\vec{k}\vec{k}') - 2V(\vec{q})]\mathfrak{D}_{\vec{k}'}\right). \tag{25}$$

Equation (22) has the solution

$$\mathfrak{D}(\theta,\varphi) = -ie^{i(g\sin\varphi + \lambda\varphi)}\times\int^{\varphi} e^{-i(g\sin\varphi' + \lambda\varphi')}h(\theta,\varphi')\,d\varphi'. \tag{26}$$

To proceed with the solution, we need an expression for $h(\theta,\varphi)$. It is important to note, however, before specifying $V(\vec{k},\vec{k}')$ that while $g(\theta)$ and $h(\theta,\varphi)$ contain reference to the electron interaction potential function, $\lambda = \omega/\omega_c = \omega mc/eB$ does not. The factor m is the bare mass, not the quasiparticle mass, as we have elsewhere stressed.[4] In Silin's formulation,[2] m^* will appear here and λ will be interaction dependent.

We write

$$V(\vec{k}\vec{k}') = V_0 + V_1\cos\psi + V_2 P_2(\cos\psi) + \cdots , \quad (27)$$

where ψ is the angle between $\vec{k}$ and $\vec{k}'$. Since $\mathfrak{D}_{\vec{k}}$ is localized to the Fermi surface we can drop the dependence of the interactions on k^2 and k'^2. The functions $P_l(\cos\psi)$ can be reexpressed in terms of the separate angles θ, θ', φ, φ' by use of the addition theorem on spherical harmonics:

$$\cos\psi = \cos\theta\cos\theta' + \tfrac{1}{2}\sin\theta\sin\theta'(e^{i\varphi}e^{-i\varphi'} + e^{-i\varphi}e^{i\varphi'}). \quad (28)$$

For convenience sake we truncate after the term in V_1; including higher terms would be straightforward, though tedious. We must also expand $\mathfrak{D}(\theta', \varphi')$,

$$\mathfrak{D}(\theta', \varphi') = \sum_{l=-\infty}^{\infty} \mathfrak{D}_l(\theta')e^{il\varphi'} . \quad (29)$$

Now we can write $h(\theta, \varphi)$ as a sum of terms in $e^{il\varphi}$:

$$h(\theta, \varphi) = A_0 + A_1 e^{i\varphi} + A_{-1}e^{-i\varphi} + A_2 e^{2i\varphi} + A_{-2}e^{-2i\varphi} . \quad (30)$$

The series terminates at one power of $e^{\pm i\varphi}$ higher than the expansion for $V(\vec{k}, \vec{k}')$:

$$A_0 = \frac{q\sin^2\theta}{\hbar\omega_c}\frac{V_1 k_F^2}{16\pi^2}\int_0^{\pi}\left[\mathfrak{D}_1(\theta') + \mathfrak{D}_{-1}(\theta')\right]\sin^2\theta' d\theta', \quad (31)$$

$$A_2 = A_0 = A_{-2} , \quad (32)$$

$$A_1 = \frac{q\sin\theta}{2}\left[\frac{ieE_q k_F\cos\theta}{m\omega\omega_c} + \frac{k_F^2}{4\pi^2\hbar\omega_c}\left(\int \mathfrak{D}_0(\theta')\sin\theta' d\theta'\,(-V_0 - 2V_1) + \cos\theta\int \mathfrak{D}_0(\theta')\sin\theta'\cos\theta' d\theta' V_1\right)\right] = A_{-1} . \quad (33)$$

To evaluate the integral in Eq. (26), we expand

$$e^{-ig\sin\varphi'} = \sum_n^{\infty}\sum_{m=0}^{n}\frac{g^n}{2^n}\frac{e^{-i(n-2m)\varphi'}(-)^m}{(n-m)!\,m!} . \quad (34)$$

Integrating gives

$$\mathfrak{D}(\theta, \varphi) = e^{i(g\sin\varphi)}\sum_{n,m,j}(-)^m\frac{A_j(\frac{1}{2}g)^n}{(n-m)!\,m!} \times \frac{e^{-i(n-2m-j)\varphi}}{(n-2m-j+\lambda)} ,$$

$$n = 0, \infty; \quad m = -n, n; \quad j = -2, 2. \quad (35)$$

To give a complete solution for $\mathfrak{D}(\theta, \varphi)$ we should need to insert (35) into (31) and (33). This would give a 4×4 algebraic system in the integrals of $\mathfrak{D}_l$ over θ. The solutions of this system would then give the A_j and thus $\mathfrak{D}(\theta, \varphi)$. However, to find j_q requires only finding an integral of $\mathfrak{D}(\theta, \varphi)$ as given by Eq. (19). Furthermore, we observe that

$$A_2 = A_{-2} = A_0 = 0 \quad (36)$$

is a solution of the system consistent with

$$A_1 = A_{-1} \neq 0, \quad A_q \neq 0, \quad (37)$$

that is, the rf fields are transverse to q inside as well as outside the metal. Hence we need only calculate

$$(j_q)_x = -\frac{e\hbar}{m}\frac{k_F^3}{4\pi^3}\int_0^{2\pi}\int_0^{\pi}\sin\theta\cos\theta\,\mathfrak{D}(\theta, \varphi)d\theta\,d\varphi. \quad (38)$$

Now $g(\theta)$ is a symmetric function of θ about the midpoint of the range 0, π; $\sin\theta\cos\theta$, however, is antisymmetric. Therefore only the similarly antisymmetric parts of $\mathfrak{D}(\theta, \varphi)$ can contribute to the integral, (38). But by Eqs. (31)–(33), only A_1 and A_{-1} need be found, and indeed only the antisymmetric parts of those; to find the current, we need only find

$$D_0 = \int \mathfrak{D}_0(\theta)\cos\theta\sin\theta d\theta \quad (39)$$

$$= \frac{1}{2\pi}\int d\varphi\cos\theta\sin\theta\,\mathfrak{D}(\theta, \varphi)d\theta$$

$$= \frac{1}{2\pi}\int d\theta\,d\varphi\cos\theta\sin\theta\,e^{i(g\sin\varphi)}\sum_{n,m,j}\frac{A_j e^{i(2m-n+j)\varphi}(\frac{1}{2}g)^n(-)^{m+1}}{(n-m)!\,m!\,(2m-n+j-\lambda)}$$

$$= \frac{1}{2\pi}\int d\theta\,d\varphi\cos\theta\sin\theta\sum_{l=-\infty}^{\infty}e^{il\varphi}J_l(g)\sum_{n,m,j}A_j(-1)^m\left(\frac{g}{2}\right)^n\frac{e^{-i(n-2m-j)\varphi}}{(n-m)!\,m!\,(n-2m-j+\lambda)}$$

$$= \int d\theta\cos\theta\sin\theta\sum_{n,m,j}\frac{A_j(-1)^m}{m!\,(n-m)!}\left(\frac{g}{2}\right)^n\frac{J_{(n-2m-j)}(g)}{n-2m-j+\lambda}$$

$$= \int d\theta \cos\theta \sin\theta \sum_{j,l=-\infty}^{\infty} A_j \frac{J_l(g)\, J_{l+j}(g)}{l+\lambda}, \tag{40}$$

where the J_ν are Bessel functions. We use Eq. (33), and obtain

$$D_0 = \frac{q}{2} \int \cos^2\theta \sin^2\theta \left(\frac{ieE_q k_F}{m\omega\omega_c} + \frac{k_F^2 V_1}{4\pi^2\hbar\omega_c} \int \mathfrak{D}_0(\theta') \sin\theta' \cos\theta' d\theta' \right) \left(\sum_{l=-\infty}^{\infty} \frac{J_l(g)\, J_{l-1}(g)}{l+\lambda} + \frac{J_l(g)\, J_{l+1}(g)}{l+\lambda} \right) d\theta$$

$$= \frac{q}{2} \int \cos^2 \sin^2\theta \left(\frac{ieE_q k_F}{m\omega_c\omega} + \frac{k_F^2 V_1}{4\pi^2\hbar\omega_c} D_0 \right) \left(\sum_{-\infty}^{\infty} \frac{2l\, J_l^2(g)}{g(l+\lambda)} \right) d\theta. \tag{41}$$

If we call the second quantity in large parentheses $G(g, \lambda)$ we can write

$$D_0 = i\frac{q}{2} \int \cos^2\theta \sin^2\theta\, G(g,\lambda)\, d\theta\, \frac{eE_q k_F}{m\omega\omega_c} \Big/ \left(1 - \frac{q k_F^2 V_1}{8\pi^2\hbar\omega_c} \int \cos^2\theta \sin^2\theta\, G(g,\lambda)\, d\theta \right). \tag{42}$$

The conductivity is obtained by multiplying:

$$\sigma(q,\omega) = -\frac{e\hbar}{m} \frac{k_F^3}{2\pi^2} \frac{D_0}{E_q} = \frac{j_q}{E_q}. \tag{43}$$

IV. SURFACE IMPEDANCE AND CYCLOTRON RESONANCE

The surface impedance is obtained from the ratio of $E(0)$ to $\partial E(0)/\partial y = E'$.

$$Z = -(4\pi i\omega/c^2)\, [E(0)/E']. \tag{44}$$

Using Eq. (6),

$$Z = -(2i\omega/c^2) \int_{-\infty}^{\infty} E_q\, dq/E'.$$

We combine Eq. (43) with (5) to yield E_q in terms of E':

$$\frac{E_q}{E'} = -2\left(\frac{\omega^2}{c^2} - q^2 - \frac{4\pi i\omega}{c^2} \frac{j_q}{E_q} \right)^{-1},$$

$$Z = \frac{4i\omega}{c^2} \int_{-\infty}^{\infty} \left[\frac{\omega^2}{c^2} - q^2 - \frac{e^2 k_F^4 \hbar q}{\pi m^2 c^2 \omega_c} \int_0^\pi \cos^2\theta \sin^2\theta\, G(g,\lambda)\, d\theta \left(1 - \frac{k_F^2 V_1 q}{8\pi^2\hbar\omega_c} \int_0^\pi \cos^2\theta \sin^2\theta\, G(g,\lambda)\, d\theta \right)^{-1} \right]^{-1} dq. \tag{45}$$

This expression, integrated over q, is the surface impedance. The problem of finding Z is thus completely reduced to quadratures. Clearly the important features of this expression are governed by the function $G(g, \lambda)$. As a function of λ, G consists of a series of simple poles at all integer real values of λ except 0. The large responses D_0 at these poles constitute cyclotron resonance and the subharmonics characteristic of Azbel'-Kaner geometry.

Our primary interest is in locating and interpreting well-separated resonances. As λ goes through a resonance, the term in the series for that single resonance will dominate the remainder. We temporarily study $G(g, \lambda)$ by taking the terms singly and throwing away the remainder. For example, for $l = -1$, we have

$$Z = \frac{4i\omega}{c^2} \int_{-\infty}^{\infty} dq \left(\frac{\omega^2}{c^2} - q^2 + \frac{\alpha q^2}{\lambda - 1 - \beta q^2} \right)^{-1}, \tag{46}$$

where

$$\alpha \equiv \frac{e^2 k_F^4 \hbar}{2\pi m^2 c^2 \omega_c q} \int_0^\pi \cos^2\theta \sin^2\theta \left(\frac{2}{g} J_{-1}^2(g) \right) d\theta,$$

$$\beta \equiv \frac{k_F^2 V_1}{8\pi^2\hbar\omega_c q} \int_0^\pi \cos^2\theta \sin^2\theta \left(\frac{2}{g} J_{-1}^2(g) \right) d\theta.$$

As we shall see, cyclotron resonance is dominated by small q. We thus approximate the integrals in α and β by the leading terms in q, whence

$$\alpha = \tfrac{1}{5} (\omega_p/\omega_c)^2 (v_F^2/c^2), \tag{47}$$

$$\beta = \tfrac{1}{20} (R_c)^2\, 2m^* k_F V_1/3\hbar^2\pi^2; \quad R_c \equiv \hbar k_F/m^*\omega_c. \tag{48}$$

With these approximations, the integration in (46) can be performed. Experimentally, it is convenient to study

$$\frac{\partial Z}{\partial H} = \frac{mc\omega H^2}{e} \frac{\partial Z}{\partial \lambda}.$$

Integrating and differentiating in (46) yields

$$\frac{\partial Z}{\partial \lambda} = \frac{i\pi\alpha c}{4\omega} \frac{1}{(\lambda-1)^{1/2}} \times \frac{1}{[\lambda - 1 - \alpha + \beta\omega^2/c^2 + 2\omega\, \beta^{1/2}(1-\lambda)^{1/2}/c]^{3/2}}. \tag{49}$$

We observe a resonance at $\mathrm{Re}\lambda = \mathrm{Re}\omega/\omega_c = 1$ with a square root of a Lorentzian line shape; this is the

cyclotron-resonance line and is unshifted by the interactions. It is important to bear in mind that this equation is valid only near $\lambda = 1$.

Successive poles of $G(g, \lambda)$ occur at higher fields, that is, smaller λ. Referring to Eq. (41), we notice that each successive resonance is associated with a higher-order Bessel function. Upon expansion the leading terms of these Bessel functions are successively higher in powers of q. Thus in place of Eq. (46), we need to consider

$$Z = \frac{4i\omega}{c^2}\int_{-\infty}^{\infty} dq \left(\frac{\omega^2}{c^2} - q^2 + \frac{\alpha' q^{2n}}{\lambda - n - \beta' q^{2n}}\right)^{-1} . \quad (50)$$

Expressions of this form become successively more difficult to integrate; $n = 1$ we have considered; $n \geq 4$ cannot be treated analytically. However, by reducing the fraction in the integrand, this expression can be written as the ratio of two polynomials in q^2:

$$Z = \alpha_0 \int_{-\infty}^{\infty} dq \frac{(\alpha_1 - q^2)\cdots(\alpha_n - q^2)}{(\beta_0 - q^2)\cdots(\beta_n - q^2)} . \quad (51)$$

By writing the factors in the numerator as $(\alpha_i - \beta_i) + (\beta_i - q^2)$ we note that the fraction can always be expanded as a sum of integrals like

$$z = \gamma_0 \int_{-\infty}^{\infty} \frac{1}{(\gamma_1 - q^2)(\gamma_2 - q^2)\cdots(\gamma_m - q^2)} dq , \quad (52)$$

where the γ_i are a subset of the β_i.

This kind of expression can be integrated by the method of residues:

$$z = \gamma_0 \prod_{i,j\neq i} \frac{1}{\gamma_i - \gamma_j} \frac{i\pi}{\gamma_i^{1/2}} . \quad (53)$$

Singularities, i.e., resonances, occur at $\gamma_i = \gamma_j$ and $\gamma_i = 0$. But $\gamma_i = 0$ implies that the denominator of (52) and hence of (51) can be factored. Since the constant term in the denominator is $(\lambda - n)\omega^2/c^2$, we see that at least one resonance occurs at $\lambda = n$. If we approximate $G(g, \lambda)$ less drastically by keeping more than just leading terms, we shall have more complicated q dependences than in (50). However, the constant term in the denominator of the integrand for Z will always be of the form

$$(\lambda - n_1)(\lambda - n_2)\cdots(\lambda - n_p)\,\omega^2/c^2,$$

where the n_i are integers, their number determined by the quality of the approximation. Cyclotron resonance and all its subharmonic resonances occur at

$$\omega = n\,eH/mc . \quad (54)$$

This conclusion is unaffected also by the termination of the expansions of $V(k, k')$. Carrying $V(kk')$ to higher order in $e^{i(\varphi-\varphi')}$ will introduce more terms A_j in the resulting expansion of $h(\theta, \varphi)$. The crucial point, however, is that after the integrand in (45) has been expanded as a series in q^2, the coefficients become singular at integer values of λ only, and these singularities give the cyclotron resonances.

We note finally that our conclusions are in agreement with Kohn[10] who found for the case of large skin depth that the cyclotron-resonance fundamental is rigorously unaffected by particle interactions.

V. ADDITIONAL RESONANCES

Equation (49) for $\partial Z/\partial\lambda$ shows an additional "resonance" at

$$\lambda = \alpha + 1 - 2\omega(\alpha\beta/c)^{1/2} + \beta\omega^2/c^2. \quad (55)$$

However, (49) was obtained by assuming $|\lambda - 1| \ll 1$. This "resonance" occurs at a frequency $\lambda \gg 1$ for typical metallic parameters, hence at the very least, we would have to include the $l = +1$ terms in approximating $G(\lambda, \theta)$. We should thereby have constructed a long-wavelength approximation to G (since $J_l^2 \sim g^{2l} \sim q^{2l}$). In this case in place of $\alpha q^2/(\lambda - 1)$ in Eq. (46) we have

$$\alpha q^2\left(\frac{1}{\lambda - 1} - \frac{1}{\lambda + 1}\right) = \frac{2\alpha q^2}{\lambda^2 - 1} . \quad (56)$$

The interaction terms have been dropped for simplicity. This correction gives resonances in Z at

$$\omega = \pm\omega_c(\alpha + 1)^{1/2}$$
$$= \pm\omega_p v_F/c\sqrt{5} + O(\omega_c/\omega_p).$$

The electronic excitation giving rise to this resonance is a TE surface plasmon. The small-q approximation is not valid, however, so the frequency location here is meaningless.

"Small q" means $g < 1$ which from Eq. (23) leads to

$$qR_c < 1$$

or wavelengths long compared to a cyclotron radius. (A different approach yields a criterion $qv_F/\omega < 1$ in the limit $\omega_c \to 0$. This is slightly less restrictive but not sufficiently so.)

The integral for Z, Eq. (46), can be singular in two characteristically different ways. First the term in α can become infinite, leading to a $1/q^2$ singularity in the integrand; this is the cyclotron-resonance fundamental and is a form of "infrared catastrophe." Also, the complete term in q^2 can vanish leading to an integral of a constant over an infinite range. This "ultraviolet catastrophe" is the TE surface-plasmon effect. But an "ultraviolet catastrophe" means the integral is dominated by large q or terms of high powers of q, which terms we have discarded in obtaining this expression for Z. Hence, taking q small is more a mutilation of Z than an approximation to it.

The location of the resonance in frequency is influenced by our long-wavelength conductivity-mutilation procedure, but its existence is not. In fact,

not only the surface-plasmon resonance but also other types of resonant structure in Z and its derivatives must be expected, all caused by the presence of the surface.

Equation (46) and the resulting magneto-optical resonance it seems to predict were obtained by severely truncating the expansions of $G(g,\lambda)$. The existence of the resonance may be suspect on this account. If we make the truncation less severe, the integration becomes rapidly more difficult and analytic expressions become unavailable. However, we can determine enough in general to be sure that improving the approximation increases rather than decreases the resonant structure in Z and its derivatives.

Ignoring the interaction terms, the surface impedance is of the form

$$\int_0^\infty \frac{1}{a_0+a_1q^2+a_2q^4+\cdots}\,dq\ . \tag{57}$$

The coefficients a_2 and higher are all of the form of sums of terms in $\theta_p(\lambda-p)^{-1}$; a_1 contains a constant as well as a pole, $(\lambda-1)^{-1}$. Truncating the expansion after a_1, $a_j=0$ for $j\geq 2$ gives an obvious singularity in the integral when $a_1(\lambda)$ goes to zero. Inclusion of the a_2 term, however, although it makes the integral convergent at $a_1=0$, does not destroy the singularity. Instead, the original singularity is slightly shifted and an additional weaker singularity is introduced at $a_2(\lambda)=0$.

Equation (57) can be written in the form of (52) and integrated, (53). Singularities occur in the integrand at $\gamma_i^{1/2}=0$ and $\gamma_i=\gamma_j$.

The first singularities lead to cyclotron resonance. The latter occur at $\gamma_i^{1/2}=-\gamma_j^{1/2}$. Consider γ_1 and γ_2 as the confluent roots. The surface impedance is proportional to

$$\frac{1}{\gamma_1^{1/2}}\left(\prod_{j\neq 1,2}\frac{1}{\gamma_1-\gamma_j}\right)\frac{1}{\gamma_1-\gamma_2}+\frac{1}{\gamma_2^{1/2}}\prod_{j\neq 1,2}\frac{1}{\gamma_2-\gamma_j}\,\frac{1}{\gamma_2-\gamma_1}$$

and as $\gamma_1\to\gamma_2$, this becomes

$$\frac{1}{\gamma_1^{1/2}\gamma_2^{1/2}}\,\frac{\gamma_1^{1/2}-\gamma_2^{1/2}}{\gamma_1-\gamma_2}\prod_{j\neq 1,2}\frac{1}{\gamma-\gamma_j}\ ,$$

which is singular only for $\gamma_1^{1/2}=-\gamma_2^{1/2}$. It is further important to note that every term in the sum for Z contains an odd half-integer number of powers of the γ_i. Thus, as a function of the γ_i, the singularities of Z are all branch points and equal in number to n^2, where n is the order of polynomial approximation to $G(g,\lambda)$.

As a function of the a_i, Eq. (57), the singularities of Z may still be seen to be branch points. We have

$$a_0+a_1x^2+\cdots+a_nx^{2n}$$
$$=\gamma_0(x^2-\gamma_1)\,(x^2-\gamma_2)\cdots(x^2-\gamma_n).$$

Equating like powers of x, we obtain

$$a_n=\gamma_0,$$
$$a_{n-1}=\gamma_0\sum_{i=1}^{n}\gamma_i,$$
$$a_{n-2}=\gamma_0\frac{1}{2}\sum_{i\neq j}\gamma_i\gamma_j,$$
$$\vdots$$
$$a_0=\gamma_0\frac{1}{n}\sum_{i\neq j\neq k\cdots}\gamma_i\gamma_j\gamma_k\cdots=\prod_{i=0}^{n}\gamma_i,$$

The γ_i are solutions of the equation

$$a_0+a_1\gamma+a_0\gamma^2+\cdots a_n\gamma^n=0.$$

Hence

$$\frac{\partial\gamma}{\partial a_j}=-\gamma^j\Big/\sum_{i=1}^{n}ia_i\gamma^{i-1}\ .$$

The singularities of $\partial\gamma_i/\partial a_j$, expressed as functions of γ_i, are thus poles. If one now constructs

$$\frac{\partial Z}{\partial a_j}=\sum\frac{\partial Z}{\partial\gamma_i}\,\frac{\partial\gamma_i}{\partial\gamma_j},$$

we have that the singularities of this expression must be branch points. We construct finally

$$\frac{\partial Z}{\partial\lambda}=\sum_i\sum_j\frac{\partial Z}{\partial\gamma_i}\,\frac{\partial\gamma_i}{\partial a_j}\,\frac{\partial a_j}{\partial\lambda}\ .$$

The a_j are all polynomials in λ, in general containing linear terms as well as even powers of λ. At least some of the terms in $\partial a_j/\partial\lambda$ must be constants and therefore $\partial Z/\partial\lambda$ must have branch-point singularities. We have seen one such singularity appear in the approximations which permit expressing Z in simple form.

VI. CONCLUSIONS

The formalism of Ref. 4 leads to cyclotron-resonance frequencies which are unshifted by many-body effects in an interacting system. Additional resonances at other frequencies may also exist in the surface impedance or its derivatives with frequency. We should finally note that the formalisms of Refs. 2 and 4 differ only in the combined effects of orbital cyclotron precession and particle interactions. Hence, this discussion of extra resonances, which is essentially independent of the particle interactions, is likewise independent of the choice of transport equations.

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[1]M. Ya. Azbel', Zh. Eksperim. i Teor. Fiz. 34, 766 (1958) [Sov. Phys. JETP 1, 527 (1958)].
[2]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 33, 495 (1957) [Sov. Phys. JETP 6, 387 (1958)].
[3]P. M. Platzman and K. C. Jabobs, Phys. Rev. 134, A974 (1963).
[4]L. L. Van Zandt, Phys. Rev. B 1, 3217 (1970); 1, 3223 (1970).
[5]C. C. Grimes and A. F. Kip, Phys. Rev. 132, 1991 (1963).
[6]L. L. Van Zandt, Phys. Rev. B 1, 3235 (1970).
[7]S. Schultz and G. Dunnifer, Phys. Rev. Letters 18, 283 (1967).
[8]P. M. Platzman and P. A. Wolff, Phys. Rev. Letters 18, 280 (1967).
[9]The material of this section is well known [for example, S. Rodriguez, Phys. Rev. 112, 1616 (1958)], and is included here to establish notation and to provide equations for reference in later sections.
[10]W. Kohn, Phys. Rev. 123, 1242 (1961).

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Studies of Vibrational Surface Modes. I. General Formulation*

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A general formulation is given for studies of the vibrational properties of systems which have two-dimensional periodicity and one or two surfaces. Although layered structures and other systems with interfaces fall within the scope of this formulation, the principal motivation is to provide a framework for calculating and interpreting vibrational surface properties. No assumption is made concerning crystal structure, surface orientation, the interaction between particles, or the number of particles per unit cell. Also, the treatment is applicable to reconstructed surfaces, surfaces with adsorbed impurity particles, etc., as well as unreconstructed clean surfaces, provided that the two-dimensional periodicity is preserved. A discussion is given of the properties of the vibrational modes: In general, the displacement ellipse for a given mode can have any orientation. For surfaces with "axial-inversion symmetry," however, one axis of the ellipse is always normal to the surface. If the surface has "complete reflection symmetry" with respect to a given plane, then for any two-dimensional wave vector parallel to the plane the modes will separate into two classes: one-third of the modes will be pure shear-horizontal (SH) modes, and the other two-thirds will be polarized strictly in the sagittal plane. It is possible for surface modes of one class to lie within the bulk subbands of the other class. If the crystal has either axial-inversion symmetry or a three-dimensional center of inversion, then the complex dynamical matrix can be reduced to a real, symmetric matrix of the same size. If both symmetries are present, as is the case for many surfaces of interest, then a further reduction is possible. Finally, notations are suggested for distinguishing two-dimensional vectors and for labeling symmetry points in the two-dimensional Brillouin zone associated with a surface.

I. INTRODUCTION

The study of elastic surface waves goes back almost a century, to the investigation by Lord Rayleigh in 1885 of surface waves in an isotropic elastic continuum.[1,2] In 1911, Love treated another type of surface wave which can exist when a macroscopic layer of one material is supported by a substrate of another material,[3] and in 1924 Stoneley considered a type of wave which can propagate along the interface of two materials.[4] Rayleigh waves, Love waves, Stoneley waves, and other waves in more general layered media[5] are of importance in seismology.

In the past 15 years there have been many studies of surface waves in anisotropic media.[6–13] For a surface wave in an anisotropic medium, the displacement $\vec{u}$ at the point with position vector $\vec{r} = (x, y, z)$ is given by[6]

$$\vec{u}(\vec{r}) = \sum_{n=1}^{3} c_n \vec{\eta}^n \exp[i(\vec{q}^n \cdot \vec{r} - \omega t)] \tag{1.1}$$

$$= \sum_{n=1}^{3} c_n \vec{\eta}^n \exp[i(q_z^n z + q_x x + q_y y - \omega t)], \tag{1.2}$$

where (q_x, q_y) is the propagation vector, which is two-dimensional and parallel to the surface plane, and q_z^n is a complex number which determines the attenuation of the wave with distance from the surface. (We take the z axis to be perpendicular to the surface.) For the special case of a true Rayleigh wave in an isotropic medium, there are only two q_z^n, both of which are purely imaginary, and vibrations are limited to the sagittal plane.[14] The