Random-phase approximation in the fractional-statistics gas

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The random-phase approximation for a gas of particles obeying $\frac{1}{2}$ fractional statistics, in the context of Feynman perturbation theory performed in the fermion representation, is shown to yield a gauge-invariant Meissner effect with full screening in the ground state, a coherence length comparable with the in interparticle spacing, and a linearly dispersing undamped collective mode.

It was recently proposed by one of us[1] that the charge carriers in high-temperature superconductors might obey $\nu=\frac{1}{2}$ fractional statistics,[2] and that this might be the cause of the charge-2 superfluidity. In this paper, we strengthen this point of view by explicitly calculating the linear response of such a system to an applied external electromagnetic potential. The key step in this calculation is the use of random-phase approximation (RPA) to account for the long-range gauge potentials associated with the fractional statistics. The resulting response function exhibits a Meissner effect and also closes the gap in the unperturbed collective-mode spectrum, yielding a linear spectrum in the long-wavelength limit. This latter effect is the inverse of the "plasmonization" of low-lying collective modes in an electron gas. These results imply that the quantum-mechanical ground state implicit in the random-phase approximation is a true superfluid, and in particular exhibits broken symmetry.

In a first-quantized fermion representation, the many-body Hamiltonian takes the form

$$\mathcal{H}=\sum_j \frac{1}{2m}\,|\mathbf{p}_j+\mathbf{A}_j(\mathbf{r}_j)|^2\,, \tag{1}$$

where $\mathbf{r}$ denotes a two-dimensional vector in the x-y plane and where

$$\mathbf{A}_j(\mathbf{r}_j)=\hbar(1-\nu)\sum_{k\neq j}\frac{\hat{\mathbf{z}}\times\mathbf{r}_{jk}}{|\mathbf{r}_{jk}|^2}\,. \tag{2}$$

Here ν characterizes the specific form of the fractional statistics: $\nu=0$ corresponds to a fermion representation of noninteracting bosons and $\nu=\frac{1}{2}$ is the case of current interest. The system may be thought of physically as spinless fermions interacting through long-range magnetic vector potentials, including three-body contributions associated with the terms proportional to $|\mathbf{A}_j|^2$.

We first consider the mean field $\bar{\mathbf{A}}$ generated by the average density ρ of the particles. Replacing the sum in Eq. (2) by an integral, we find

$$\bar{\mathbf{A}}(\mathbf{r})=\rho\pi\hbar(1-\nu)(\hat{\mathbf{z}}\times\mathbf{r})=\tfrac{1}{2}\mathbf{B}\times\mathbf{r}\,. \tag{3}$$

Here $\mathbf{B}=2\pi\rho\hbar(1-\nu)\hat{\mathbf{z}}$ is an equivalent uniform mean magnetic field that defines the corresponding magnetic length $a=(\hbar/B)^{1/2}$ and cyclotron frequency $\omega_c=B/m$. We use this mean field to define an unperturbed one-body Hamiltonian

$$\mathcal{H}_0=\sum_j \frac{1}{2m}\,|\mathbf{p}_j+\bar{\mathbf{A}}(\mathbf{r}_j)|^2\,, \tag{4}$$

the eigenfunctions $\phi_{jn}(\mathbf{r})$ and eigenvalues $\varepsilon_{jn}=(n+\frac{1}{2})\hbar\omega_c$ of which are those associated with the Landau levels in the field B.[3] With this definition of $\mathcal{H}_0$, the analysis becomes an expansion in the perturbation Hamiltonian

$$\mathcal{H}_1=\mathcal{H}-\mathcal{H}_0$$
$$=\sum_j\frac{1}{m}[(\mathbf{p}_j+\bar{\mathbf{A}}_j)\cdot(\mathbf{A}_j-\bar{\mathbf{A}}_j)+\tfrac{1}{2}\,|\mathbf{A}_j-\bar{\mathbf{A}}_j|^2]\,. \tag{5}$$

Note that the interactions implicit in $\mathcal{H}_1$ couple to the particles through the mean-field density and current-density operators, defined by

$$j_0(\mathbf{r})=\sum_j\delta(\mathbf{r}-\mathbf{r}_j)\,, \tag{6}$$

and

$$\mathbf{j}(\mathbf{r})=\sum_j\tfrac{1}{2}\{\mathbf{p}_j+\bar{\mathbf{A}}(\mathbf{r}_j),\delta(\mathbf{r}-\mathbf{r}_j)\}\,. \tag{7}$$

The *physical* density operator J_0 is the same as j_0, but the physical current density

$$\mathbf{J}(\mathbf{r})=\sum_j\tfrac{1}{2}\{\mathbf{p}_j+\mathbf{A}_j(\mathbf{r}_j),\delta(\mathbf{r}-\mathbf{r}_j)\}\,, \tag{8}$$

differs from $\mathbf{j}(\mathbf{r})$ by an internal diamagnetic contribution.

The problem of interest is the linear response to an external electromagnetic field, described by a potential $A_\mu^{\rm ext}(\mathbf{r},t)$, where μ runs over 0, x, and y for the time and space components. The perturbation Hamiltonian associated with this field is

$$\Delta\mathcal{H}(t)=-\int[\mathbf{A}^{\rm ext}(\mathbf{r},t)\cdot\mathbf{J}(\mathbf{r},t)-A_0^{\rm ext}(\mathbf{r},t)J_0(\mathbf{r},t)]d\mathbf{r}\,. \tag{9}$$

The linear response has two contributions,[4,5] a diamagnetic part proportional to the density and a paramagnetic part proportional to the retarded correlation function of J_μ and J_ν:

$$\Delta_{\mu\nu}(1,2) = -i\langle[\hat{\mathbf{J}}_\mu(1),\hat{\mathbf{J}}_\nu(2)]\rangle\theta(t_1-t_2) . \tag{10}$$

In this expression, the angular brackets denote an average in the exact ground state, the caret denotes Heisenberg representation, and 1 denotes a space-time point $\mathbf{r}_1 t_1$. The linear response in Fourier space, defined by

$$4\pi\langle J_\mu(\mathbf{q},\omega)\rangle = -K_{\mu\nu}(\mathbf{q},\omega)A_\nu^{\text{ext}}(\mathbf{q},\omega) \tag{11}$$

is given specifically in terms of Δ by

$$K_{\mu\nu} = \rho\delta_{\mu\nu}(1-\delta_{\mu 0}) + \Delta_{\mu\nu}(\mathbf{q},\omega) . \tag{12}$$

The first step in obtaining an approximate expression for Δ is to introduce the unperturbed correlation function

$$\mathcal{D}^0_{\mu\nu}(1,2) = -i\langle T[\hat{\mathbf{j}}_\mu(1)\hat{\mathbf{j}}_\nu(2)]\rangle_0 , \tag{13}$$

where the subscript 0 denotes an expectation value in the unperturbed ground state. We then perform a perturbation expansion for the *mean-field* correlation function

$$\mathcal{D}_{\mu\nu}(1,2) = -i\langle T[\hat{\mathbf{j}}_\mu(1)\hat{\mathbf{j}}_\nu(2)]\rangle , \tag{14}$$

using the usual Feynman rules of field theory.[6] The interaction Hamiltonian contains long-range potentials similar to those familiar from the electron gas. As in that case, the leading contributions at long wavelengths arise from the repeated "bubble" diagrams (the RPA) in which the same momentum transfer $\mathbf{q}$ appears on each interaction line.[7] One of these [see Fig. 1(a)] arises from the $(\mathbf{p}+\mathbf{A})\cdot\mathbf{A}$ term in $\mathcal{H}_1$. Since this part of the interaction involves all three components of A_μ, it couples the various components of $\mathcal{D}^0_{\lambda\mu}$. For example, the first-order contribution to $\mathcal{D}_{00}$ involves both $\mathcal{D}^0_{0y}$ and $\mathcal{D}^0_{y0}$ (we take $\mathbf{q}$ along $\hat{\mathbf{x}}$). The three-body interactions lead to three RPA-like diagrams [Figs. 1(b)-1(d)], but only the first of these is divergent. The significant RPA-like contributions reduce to an expression for $\mathcal{D}_{\mu\nu}$ of the form

$$\mathcal{D} = \mathcal{D}^0 + \mathcal{D}^0\mathcal{V}\mathcal{D} , \tag{15}$$

where $\mathcal{V}$ is the 3×3 Hermitian potential matrix

$$\mathcal{V} = \frac{(1-\nu)2\pi}{q^2}\begin{pmatrix} 1 & 0 & iq \\ 0 & 0 & 0 \\ -iq & 0 & 0 \end{pmatrix} . \tag{16}$$

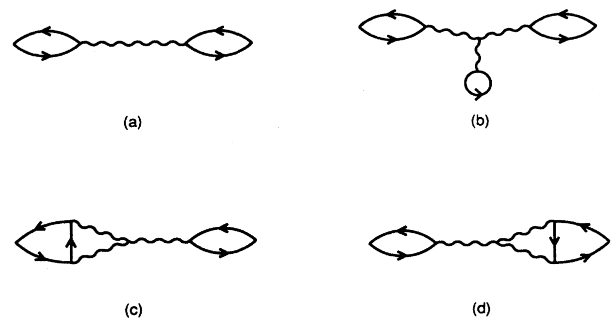

FIG. 1. First-order diagrams relevant for RPA description. (a) Two-body term, (b) three-body term that must be retained, and (c) and (d) three-body terms that are negligible in comparison with that in (b).

The final step in the calculation is to correct the matrix $\mathcal{D}$, which is defined in terms of the mean-field currents $\mathbf{j}$, by adding the "internal" diamagnetic contribution associated with $\delta\mathbf{J} = \mathbf{J} - \mathbf{j}$. In the long-wavelength limit, we obtain

$$\Delta \cong \Delta^{\text{RPA}} = (1+\rho\mathcal{U}^\dagger)\mathcal{D}(1+\rho\mathcal{U}) , \tag{17}$$

where

$$\mathcal{U} = \frac{(1-\nu)2\pi}{q}\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{18}$$

The final linear-response kernel $K_{\mu\nu}^{\text{RPA}}$ follows by combining Eqs. (12) and (17).

Given these expressions, it remains only to determine the unperturbed matrix $\mathcal{D}^0(\mathbf{q},\omega)$. Specializing to the case of $\nu = \frac{1}{2}$, and taking length and energy units for which a and $\hbar\omega_c$ are both unity, we obtain

$$\mathcal{D}^0(\mathbf{q},\omega) = \frac{1}{\pi}\begin{pmatrix} q^2\Sigma_0 & q\omega\Sigma_0 & -iq\Sigma_1 \\ q\omega\Sigma_0 & \omega^2\Sigma_0 - 1 & -i\omega\Sigma_1 \\ iq\Sigma_1 & i\omega\Sigma_1 & \Sigma_2 \end{pmatrix} , \tag{19}$$

where

$$\Sigma_i(\mathbf{q},\omega) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{e^{-x}x^{n-1}}{(n-1)!(\omega^2-n^2)}\{(1-\delta_{n1})(n-x)^i + (n+1)^{-1}(n+1-x)^{2-i}[n(n+1)-(2n+3)x+x^2]^i\} , \tag{20}$$

and $x = q^2/2$. We note that the resulting $K_{\mu\nu}^{\text{RPA}}$ is manifestly gauge invariant[4] because the three-component vector with elements $(-\omega, q, 0)$ is an eigenvector with zero eigenvalue.

The Meissner effect follows from the static limit of the response function $K(q) = K_{yy}(q,\omega=0)$. A direct expansion for $q \rightarrow 0$ yields the relation

$$K^{\text{RPA}}(q) = \rho[1 - \tfrac{3}{8}q^2 + O(q^4)] . \tag{21}$$

Here, the 1 is the diamagnetic contribution, and the remainder arises from the paramagnetic part. As in the usual BCS theory,[4] the paramagnetic contribution vanishes for long wavelengths, leaving a full Meissner effect, with all the particles contributing to the effective superconducting density. Comparing the form of Eq. (21) with the corresponding result for the phenomenological Pippard[8] kernel

$$K^P = \rho[1 - (q\xi_0)^2/5 + O(q^4)] , \tag{22}$$

we obtain a zero-temperature coherence length ξ_0 of $(15/8)^{1/2}a$, which is comparable with the interparticle spacing.

The collective modes associated with density fluctuations occur at the poles of the response function Δ_{00}. In the present RPA, these arise from the zeros of the determinant, since the singularities at $\omega = n$ cancel identically. Expanding for small **q** and ω, we obtain

$$\mathrm{Im}\Delta_{00}^{\mathrm{RPA}} = -[q/2v_s + O(q^3)]\delta(\omega - v_s q) , \tag{23}$$

where the sound speed v_s is $\sqrt{2}$ in units of $\omega_c a$. This value agrees with that calculated from the total energy per particle $E = \hbar\omega_c$ of the unperturbed system with two filled Landau levels, in the manner

$$v_s = \left[\frac{\partial}{\partial\rho} \left(\rho^2 \frac{\partial E}{\partial \rho} \right) \right]^{1/2} . \tag{24}$$

The corresponding Hartree-Fock energy is smaller by a factor 29/32, which implies that the Hartree-Fock sound speed is 5% smaller than this value. Note that the pole in Δ_{00} is *sharp,* with no background continuum of the sort found in a Fermi liquid. Note also that the structure factor $\mathbf{S}(\mathbf{q})$ vanishes linearly for small q, as in the case both in a Fermi liquid and a Bose superfluid, in contrast to the quadratic behavior of the unperturbed structure factor. This difference reflects the presence of superfluid density fluctuations in the ground state implicit in the RPA.

We note finally that the RPA Hall conductance, given at small q and ω by

$$\sigma_{xy}^{\mathrm{RPA}} = \frac{1}{i\omega} K_{xy}^{\mathrm{RPA}} \approx \frac{1}{4\pi} \frac{(v_s q)^2}{\omega^2 - (v_s q)^2} , \tag{25}$$

is almost certainly an artifact of the calculation, attributable to neglect of nonsingular diagrams. A Hall conductance of this form also results for the case of $\nu = 0$, which is a Bose gas.

The present paper has shown how Feynman diagrams for the coupled density and current correlation functions of the fractional-statistics gas can be summed to yield physically sensible results, and that these include the Meissner effect and presence of a sharp Goldstone mode. The same techniques should prove valuable in considering other aspects of the problem, such as the effect interparticle repulsions.

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