Correlated percolation with long-range interactions

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We study the critical behavior of the q-state Potts model with long-range interactions decaying asymptotically as $\sim R^{-(d+\sigma)}$ in the presence of random-T_c impurities correlated over large distances such that the correlations fall off as $\sim R^{-(d-a)}$, $a<d$. We find that the renormalization-group scaling equations have a new fixed point in the appropriate double ϵ, x expansion, where $\epsilon=3\sigma-d$ and $x=a-\sigma$. This fixed point, however, is never both stable and physical.

I. INTRODUCTION

The relevance of long-range interactions for the critical behavior of various physical systems which can be described by the ϕ^3-field models (e.g., the percolation, the Yang-Lee edge singularity problem, and the Ising spin glass) has been discussed recently in the literature.[1–5] It is now understood that the long-range interactions decaying at large distances as $\sim R^{-(d+\sigma)}$ are relevant (i.e., leading to a new kind of critical behavior) for $\sigma<2$ when the exponent of the correlation function in the presence of short-range (SR) interactions $\eta_{\rm SR}$ is negative, and for $\sigma<2-\eta_{\rm SR}$ if $\eta_{\rm SR}$ is positive. A well-known physical example is the bond-dilute Ising ferromagnet, which exhibits the percolation transition at low temperatures $T\rightarrow 0$ near the critical concentration of ferromagnetic bonds $p=p_c$. Stephen and Grest[6] have shown using the replica formalism that in the case of short-range interactions this percolation transition can be mapped onto the q-state Potts model with $q=2^n$, where the replica number n has to be eventually taken to zero, corresponding to the $q-1$ Potts limit. Stephen and Aharony[7] have applied a similar approach to study the crossover from thermal to percolation transition in the presence of various types of long-range interactions, i.e., dipolar, power-law decaying, exponentially decaying, and z-spin interactions. In the case of power-law decaying interactions of the type $\sim R^{-(d+\sigma)}$, the quadratic term of the effective Hamiltonian which describes the transition at low temperatures has been found proportional to $r+C^\sigma k^\sigma$, where $r=0$ determines the mean-field transition, and C corresponds to the effective range of fluctuations. A more careful analysis by Aharony and Stauffer (see Ref. 8), indicates that a third-order term, which leads to a shift in the upper critical dimensionality, plays an important role in the limit $T\rightarrow 0$, similar to the case of short-range interactions.[6] Mean-field arguments and qualitative fluctuation corrections seem to indicate that the size of the critical region shrinks gradually with growing range of interaction and eventually vanishes when the number of interacting spins z becomes infinite.[8] In that case, the whole system represents a single cluster so that the percolation transition disappears. Although the precise relation between the size of the critical region and the details of interaction is not available, one cannot rule out the existence of a finite critical region in the case of interactions decaying as $\sim R^{-(d+\sigma)}$, since in the limit $\sigma\rightarrow 2$ (short-range interactions) the critical region is certainly finite. It has been shown[1,2] that the percolation transition in the presence of this type of long-range interactions belongs to a new universality class.

New features are expected when the bond occupations are independently determined by occupation probabilities which vary from site to site and are correlated over large distances (correlated percolation). The usual (i.e., uncorrelated) percolation corresponds to the case of short-range or δ correlations between the occupation probabilities. The physical origin of the long-range correlations with a power-law decay at large distances has been discussed by Weinrib.[9] The same author has shown that the bond-correlated percolation can be described by the limit $q\rightarrow 1$ of the q-state Potts model with random interactions $J(R)$ correlated over a finite spatial range such that the correlations decay algebraically, i.e., $[J(R)J(0)]\sim R^{-(d-a)}$, $a<d$. This type of disorder is relevant for the critical behavior when the range of correlations is such that the generalized Harris criterion[9] $2-\nu(d-a)>0$ is satisfied. However, when the random short-range interactions $J(R)$ are infinitely correlated in a subspace of ϵ_d dimensions ($\epsilon_d=2+\delta$), corresponding to the so-called extended defect problem, the absence of a stable accessible fixed point in a double ϵ,δ expansion has been demonstrated by Stolan *et al.*[10]

In the present work we consider the critical behavior of the q-state Potts model with long-range interactions decaying as $\sim R^{-(d+\sigma)}$ and with random-T_c impurities isotropically correlated such that the correlations fall off as $\sim R^{-(d-a)}$ at large distances R. In the limit $q\rightarrow 1$, this model is expected to describe percolation critical behavior of the random Ising ferromagnet with long-range interactions and correlated bond-occupation probabilities. It will be demonstrated that the renormalization-group scaling equations have a new fixed point in the appropriate double ϵ,x-expansion ($\epsilon=3\sigma-d, x=a-\sigma$). It turns out, however, that this fixed point is not physical in the entire domain of attraction. This conclusion applies equally to the percolation limit as well as to the other physical realizations[11] of the q-state Potts model. Thus, the situation is similar to the case when the impurities form an extended defect. In fact, we will show below that in the presence of long-range interactions the results are independent of the type of impurity correlations.

In the following section the renormalization-group analysis of the model is presented. In the final section we give a short summary and a discussion of the results.

II. RENORMALIZATION-GROUP ANALYSIS AND RESULTS

We start with the continuous version of the q-state Potts model in k space[1,2,9]

$$\mathcal{H}=\int \frac{d^dk}{(2\pi)^d}\sum_i \left[\tfrac{1}{4}(r+ck^2+bk^\sigma)Q_{ii}(k)Q_{ii}(-k) + \int \frac{d^dp}{(2\pi)^d}[\phi(p)Q_{ii}(k)Q_{ii}(-k-p)+wQ_{ii}(p)Q_{ii}(k)Q_{ii}(-k-p)] \right] , \quad (1)$$

where $i=1,2,\ldots,q$, Q_{ii} are elements of a traceless tensor $\sum_i Q_{ii}=0$, and the random potential $\phi(k)$ is assumed to be Gaussian distributed with zero mean and variance

$$[\phi(k)\phi(-k)]=uk^{-a} . \quad (2)$$

The quadratic terms proportional to ck^2 and bk^σ $(\sigma<2)$ represent the cases of short- and long-range interactions, respectively. When the long-range interactions are dominant, i.e., $c=0$ in (1), the physically interesting case is obtained for $a\approx\sigma$. Thus we will set $a=\sigma+x$ in order to perform a double expansion in $\epsilon=3\sigma-d$ and $x=a-\sigma$.

By analogy with pure systems[1-3] we expect that the long-range interactions are relevant to the critical behavior for $\sigma<2$ when the exponent $\eta_{\rm SR}$ of the correlation function at the short-range fixed point is negative, and for $\sigma<2-\eta_{\rm SR}$ if it is positive. In the case of short-range interactions the exponent $\eta_{\rm SR}$ of the pure Potts model is given by[1]

$$\eta_{\rm SR}=\frac{1}{4}\frac{(q-2)\hat{\epsilon}}{10-3q} . \quad (3)$$

Similarly, it can be shown using the recursion relations from Ref. 9 that the value of the exponent $\eta_{\rm SR}$ at the random short-range fixed point for general q is

$$\eta_{\rm SR}=\frac{2(q-1)\hat{\epsilon}+q\hat{x}}{30-19q} . \quad (4)$$

Here $\hat{\epsilon}=6-d$ and $\hat{x}=a-2$. In case of percolation, expression (4) becomes negative for $\hat{x}$ negative or equivalently when $a<2$.

In order to perform the long-range (LR) expansion, it suffices to drop the term proportional to ck^2 and fix b to one. Thus, having defined the propagator as $G(p)=(r+p^\sigma)^{-1}$, we find that the critical exponent $\eta_{\rm LR}$ from the diagrams in Fig. 1(a) has the value $[K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)]$:

$$\eta_{\rm LR}=2-\sigma-\left[\frac{\sigma}{2}\right]^2\frac{1}{\sigma+1}\left[2^43^2\left[1-\frac{2}{q}\right]K_dw^2\frac{\epsilon}{\sigma-2} + 2^5K_du\frac{\epsilon+x}{\sigma-2}\right] . \quad (5)$$

Assuming further that w^2 and u are of the order $O(\epsilon,x)$ at the long-range fixed point, we conclude that the correction terms in the brackets in Eq. (5) are already of a higher order in ϵ and x as long as $\sigma<2$. Thus we find that in the presence of long-range interactions the exponent $\eta_{\rm LR}$ retains its classical value

$$\eta_{\rm LR}=2-\sigma , \quad (6)$$

which is correct at least to $O(\epsilon^2,\epsilon x,x^2)$. In the limit $\sigma\to2$ and assuming that[12] $\epsilon/(\sigma-2)\to-1$ as well as $(\epsilon+x)/(\sigma-2)\to-1$, expression (5) reduces to the one obtained in the case of short-range interactions as given by Eq. (4), provided that w^2 and u are at the corresponding fixed-point values.

Having determined the value of the exponent $\eta_{\rm LR}$ as given by Eq. (6), we can obtain a set of recursion relations for the relevant parameters in the case of long-range interactions using the diagrams in Fig. 1 (see also Ref. 9),

$$\frac{dr}{dl}=\sigma r-2^43^2\left[1-\frac{2}{q}\right]w^2(1-2r)-2^5u(1-r) , \quad (7)$$

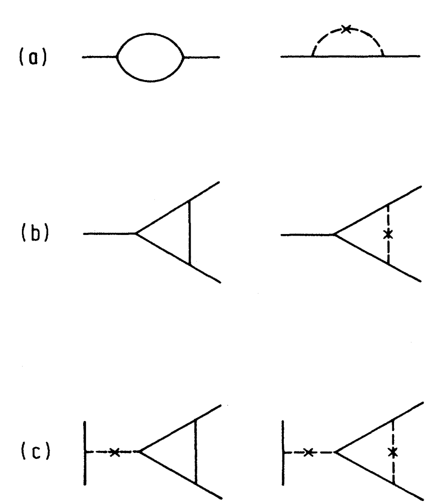

FIG. 1. One-loop diagrams contributing to the scaling equations for (a) r, (b) w, and (c) u. Dashed line with a cross carrying the momentum k represents uk^{-a}.

$$\frac{dw}{dl}=\tfrac{1}{2}\epsilon w+2^5 3^2\left[1-\frac{3}{q}\right]w^3+2^5 3uw\ , \tag{8}$$

$$\frac{du}{dl}=(\epsilon+x)u+2^6u^2+2^6 3^2\left[1-\frac{2}{q}\right]uw^2\ , \tag{9}$$

where $\epsilon=3\sigma-d$ and $x=a-\sigma$, as already stated. The fixed points of these equations are listed in Table I along with the corresponding scaling exponents λ_r, λ_1, and λ_2 for three relevant operators.

The stability of a fixed point under the renormalization-group transformation requires that all scaling exponents associated with the physically relevant perturbations, e.g., $\lambda_{1,2}$ in our case, must be negative. Therefore, according to Table I, the Gaussian and the pure fixed point are stable with respect to the presence of disorder when $\lambda_2^G=x+\epsilon$ and $\lambda_2^P=x+\epsilon/(3-q)$ are both negative. On the other hand, when λ_2^G,λ_2^P are positive, the corresponding fixed points are not stable and disorder is relevant. To leading order in ϵ and x we have

$$\lambda_2\sigma=2-\nu(d-a)\ . \tag{10}$$

When the right-hand side of Eq. (10) is positive, we expect according to the generalized Harris criterion[9] a crossover to another fixed point which is controlled by the disorder (see Fig. 2 for the case $q=1$). It turns out that to the right of the solid lines in Fig. 2 only the unphysical fixed point is stable. However, the corresponding fixed point value u_U in this entire region is negative, in contrast to the physical meaning of the parameter u, which allows only positive values. In a similar way, we find that at the random fixed point both scaling exponents $\lambda_{1,2}$ listed in Table I are negative within the shaded area of Fig. 2 where, in turn, u_R is negative, thus rendering this fixed point unphysical. Therefore, neither the random nor the unphysical fixed point is accessible in the physical systems described by model (1) when the pure and the Gaussian fixed points are unstable due to the presence of disorder. The absence of a stable accessible fixed point could be interpreted as a smearing of the percolation transition in the presence of disorder. Below we will show that the same conclusion applies to the phase transition at a general value of q.

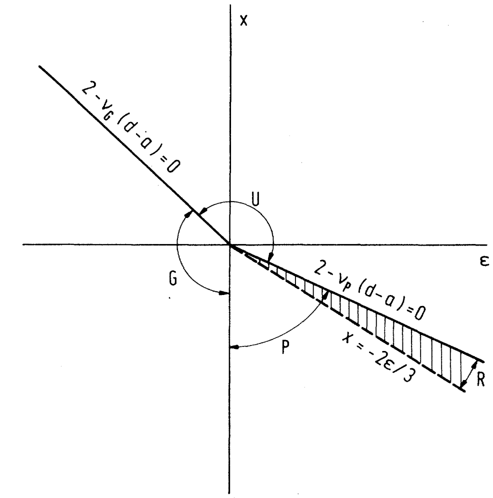

FIG. 2. Regions of stability of the fixed points: Gaussian (G), pure (P), unphysical (U), and random (R), drawn schematically for $q=1$.

The scaling exponents of two relevant operators at the random fixed point (see Table I) are

$$\lambda_{1,2}^R=-\tfrac{1}{2}\epsilon\left[1\pm\frac{\sqrt{D(\xi)}}{3-2q}\right]\ , \tag{11}$$

where $\xi=x/\epsilon$ and

$$D(\xi)=36(3-q)\xi^2+12(9-2q)\xi+4q^2-12q+33\ . \tag{12}$$

Both exponents $\lambda_{1,2}^R$ in Eq. (11) are real for $q<2$, while they can be either real or complex for $q>2$. In the latter case the equation $D(\xi)=0$ has two real roots at

$$\xi_{1,2}^0=\frac{1}{6(3-q)}[2q-9\pm(2q-3)\sqrt{(q-2)}]\ , \tag{13}$$

so that $D(\xi)<0$ for $\xi_2^0<\xi<\xi_1^0$. Thus, for $q<2$ both scaling exponents given by Eq. (11) are real and negative as long as $D(\xi)<(3-2q)^2$. Consequently, one of the operators becomes marginal when the following condition is satisfied:

TABLE I. Long-range fixed points and the corresponding scaling exponents to leading order in ϵ and x. $D(\xi)$ is given by Eq. (12) in the text, and ξ represents the ratio x/ϵ. The remaining symbols are $W^*=2^6 3^2 w^{*2}K_d$, $U^*=2^6u^*K_d$, $R^*=4\sigma r^*$, and $f=3-2q$.

	Fixed point	$\lambda_r=1/\nu$	λ_1	λ_2
Gaussian	$W^*=U^*=R^*=0$	σ	ϵ	$\epsilon+x$
Pure	$W^*=\frac{\epsilon q}{3-q}$; $U^*=0$ $R^*=(q-2)\epsilon/(3-q)$	$\sigma+\frac{q-2}{3-q}\frac{\epsilon}{2}$	$-\epsilon$	$x+\frac{\epsilon}{3-q}$
Unphysical	$W^*=0$; $U^*=-(\epsilon+x)$ $R^*=-2(\epsilon+x)$	$\sigma-\frac{\epsilon+x}{2}$	$-2\epsilon-3x$	$-3(\epsilon+x)$
Random	$W^*=(2\epsilon+3x)q/f$ $U^*=[\epsilon+(3-q)x]/f$ $R^*=[2(q-1)\epsilon+qx]/f$	$\sigma-\frac{\epsilon+x}{2}$	$-\frac{\epsilon}{2}\left[1\pm\frac{\sqrt{D(\xi)}}{3-2q}\right]$	

$$D(\xi)=(3-2q)^2 \ . \tag{14}$$

Equation (14) has two real roots, i.e.,

$$\xi_1^1=-\tfrac{2}{3} \text{ and } \xi_2^1=-\frac{1}{3-q} \ . \tag{15}$$

Note that for $q<\frac{3}{2}$ one has $\xi_2^1>\xi_1^1$, whereas $\xi_2^1<\xi_1^1$ for $q>\frac{3}{2}$. Thus, for ξ between ξ_1^1 and ξ_2^1 and $q<2$, and similarly when $2<q<3$ and for ξ within the two intervals (ξ_2^1,ξ_2^0) or (ξ_1^0,ξ_1^1), both scaling exponents $\lambda_{1,2}^R$ are real and negative, while for ξ between ξ_2^0 and ξ_1^0 they are complex with negative real parts, implying that the random fixed point is stable for all values of ξ between ξ_2^1 and ξ_1^1. These regions of stability are shown in Fig. 3. On the other hand, we find (cf. Table I) that at exactly the same values as given by Eqs. (15), i.e., for $\xi=\xi_1^1$ and $\xi=\xi_2^1$, the fixed-point values of the parameters w_R^2 and u_R change their signs (assuming $\epsilon>0$) so that at least one of them is negative in the region between ξ_1^1 and ξ_2^1. The negativity of one of the parameters w_R^2 and u_R in the entire region of stability of the random fixed point leads to the conclusion that this fixed point is not accessible in any physical system described by model (1), since w must be a real, and u a real and positive parameter.

A formally similar analysis can be done for $q>3$ (see Fig. 3), although we expect that at some critical value $q=q_c$, which is close to three,[13] a first-order phase transition takes over in the pure Potts model. In particular, the pure fixed point becomes unphysical for $q>3$. In this case there are two disconnected regions along the ξ axis where the random fixed point is stable. In particular, $\lambda_{1,2}^R$ are real and negative for $\xi_1^0<\xi<\xi_1^1$ and for $\xi_2^1<\xi<\xi_2^0$, while they are both complex with negative real parts for $\xi>\xi_2^0$ and for $\xi<\xi_1^0$. However, in this case $u_R>0$ for $\xi>\xi_2^1$, while $w_R^2>0$ for $\xi<\xi_1^1$, i.e., there is no connected region where both parameters are positive. For negative values of ϵ, at least one of the eigenvalues in Eq. (11) is positive for all values of q, so that the random fixed point is unstable.

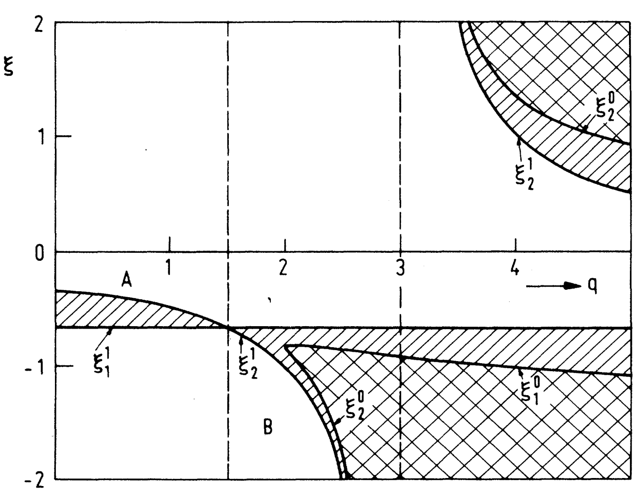

FIG. 3. Regions of stability of the random fixed point drawn for $\epsilon>0$ in the (q,ξ) plane $(\xi=x/\epsilon)$. The scaling exponents of Eq. (11) are real and negative within the single-hatched area, and complex with negative real parts in the cross-hatched regions. Both parameters u_R and w_R^2 are positive only in regions A $(\xi_2^1<\xi<0;\ q<\frac{3}{2})$ and B $(\xi<\xi_2^1;\ \frac{3}{2}<q<3)$.

In deriving the above results we have assumed that the short-range interaction term in (1) can be ignored compared with the term representing long-range interactions. The validity of the long-range expansion for $\sigma<2$ can be justified by retaining both terms in the Hamiltonian.[2,3] Then a fixed point of the corresponding recursion relations can be determined by the following expressions:

$$\frac{w^{2*}}{(c^*+b^*)^3}=w_R^2, \quad \frac{u^*}{(c^*+b^*)^2}=u_R \ , \tag{16}$$

where on the right-hand side are the fixed-point values of the corresponding parameters at the random fixed point given in Table I. Thus, these equations are satisfied if we set $c^*=0$ and $b^*=1$, as we have assumed above. At the long-range random fixed point we find that the new parameter c is governed by the scaling exponent λ_c [defined via $c(l)=c(0)\exp(\lambda_c l)$], which is given by

$$\lambda_c=\sigma-2-\frac{\epsilon+x}{2}B(\sigma) \ , \tag{17}$$

with

$$B(\sigma)=\frac{\Gamma\left[\frac{\sigma}{2}+1\right]\Gamma\left[\frac{\sigma}{2}\right]}{\Gamma(\sigma+2)} \times\left[\frac{\sigma}{2}+\left[\frac{\sigma}{2}-1\right]\left[\frac{\sigma}{2}+1\right]\right] \ . \tag{18}$$

Hence, for $\sigma<2$ we have that λ_c^R is negative, leading to the conclusion that the short-range interactions are irrelevant to the critical behavior for this range of values of σ.

III. DISCUSSION AND CONCLUSIONS

The absence of a stable physical fixed point for model (1), which has been explicitly demonstrated here within a double ϵ,x expansion to leading order, applies both to the case of percolation $(q\to 1)$ as well as to other realizations for $q>1$. Obviously, a different interpretation of this conclusion is needed in these two physically different situations. The absence of a stable physical fixed point in the presence of disorder for $q>1$ might reflect the existence of a first-order phase transition occurring in the pure Potts model due to the presence of instantons, which appear in the resummation of the perturbation series to all orders.[14] A first-order phase transition certainly occurs for $q>3$, where the pure fixed point also becomes unphysical. This problem does not exist in the limit $q\to 1$, where the transition is second order in the pure Potts model.[14] Thus, our results can be interpreted as a smearing[10] of the percolation transition due to the long-range correlations in the occupation probabilities.

It should be noted that our conclusions are also applicable to the case of extended defects in the presence of long-range interactions. In that case the correlation function of the random variable $\phi(p)$ will be given by[10]

$$[\phi(p)\phi(k)]=u\delta^{(d)}(p+k)\delta^{(\epsilon_d)}(p_{\parallel}) , \tag{19}$$

where $d=3\sigma-\epsilon$ is the dimensionality of the system, while $\epsilon_d=\sigma+x$ is the dimensionality of the defect. The anisotropy of correlations in general leads to two different exponents $\eta_{\parallel}$ and $\eta_{\perp}$ along and perpendicular to the direction of the defect, respectively. In particular, the diagram which contains the defect line [see Fig. 1(a)] contributes to $\eta_{\parallel}$ only, while the bubble in Fig. 1(a) contributes equally to both $\eta_{\parallel}$ and $\eta_{\perp}$, and its contribution is identical with the first term in the brackets in Eq. (5). We find

$$\eta_{\parallel}-\eta_{\perp}=-2^5uK_d\frac{\sigma}{2}\frac{\epsilon+x}{\sigma-2} . \tag{20}$$

Therefore, in analogy with Eq. (5) we may conclude that to linear order in ϵ and x both exponents retain the classical value, i.e.,

$$\eta_{\perp}=\eta_{\parallel}=2-\sigma , \tag{21}$$

thus rendering the recursion relations isotropic as in the present model. In fact, in the presence of long-range interactions the fractal dimensionality of the percolation cluster, which is defined as $d_F=d-\beta/\nu$, is independent of the range of correlations in the occupation probabilities. We find that $d_F^{\mathrm{LR}}=(d+\sigma)/2$ at the long-range fixed point, in contrast to the case of short-range interactions with algebraically decaying correlations,[9] where one has $d_F^{\mathrm{SR}}=\frac{12}{11}+d/2-a/22$ at the random fixed point.

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