Large currents in random resistor networks

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The distribution of currents in random resistor networks is studied numerically, and the results are compared with recent predictions. The configuration of resistors in the neighborhood of large currents is shown to have a "funnel" shape.

I. INTRODUCTION

This report presents numerical results on the distribution of currents in random resistor networks and compares them with recent theoretical predictions.[1,2] We study $L \times L$ square networks composed of conductors of two sizes, 1 and G, with $0<G<1$. The fraction of large conductors in each network is p; the fraction of small conductors is $1-p$. We consider values of p away from the percolation threshold.

One motivation for studying the distribution of large currents in random resistor networks is to understand the electrical and mechanical breakdown properties of composite materials. The random fuse network, introduced by de Arcangelis *et al.*[3] and studied by Duxbury, Beale, and Leath,[4] is a simple model in which to study breakdown phenomena. The random fuse network is obtained from a random resistor network by introducing a critical current at which the conductance of any resistor irreversibly drops to zero. As the voltage across the network is increased, breakdown first occurs at the resistor carrying the largest current. After the breakdown of the first resistor, there is new distribution of currents. The distribution of large currents thus plays a central role in understanding the random fuse model.

Li and Duxbury[2] (LD) studied random resistor networks with bond dilution, $G=0$, and showed that the expected largest current in a network, $\langle i_{\max} \rangle$, increases with the size of the network L as

$$\langle i_{\max} \rangle \sim i_{\rm av}(\ln L)^{\alpha} , \tag{1}$$

where $i_{\rm av}$ is the average current in the lattice. The exponent α depends on the dimension d of the network and is bounded by $\frac{1}{2}(d-1) \le \alpha \le 1$. For $d=2$ they find

$$\alpha = 1 \quad (G=0) . \tag{2}$$

Machta and Guyer[1] (MG) studied the case where all resistors in the network are finite, $0<G<1$. They found the same form, Eq. (1), for the largest current, but the value of α is now a function of G and d. For two-dimensional networks α is given by

$$\alpha = \tfrac{1}{2}[1-(4/\pi)\tan^{-1}(G^{1/2})] \quad (0<G<1) . \tag{3}$$

It should be noted that α is not continuous at $G=0$.

In this paper we test the prediction for α given in Eq. (3) and some of the assumptions which underly this prediction. Rather than considering the maximum current in each realization of a network we examine the closely related question of the distribution of large currents.

II. THEORY

We begin by reviewing the theoretical ideas of Refs. 1 and 2. The first assumption is that the magnitude of the current in a given bond is determined by the configuration of the resistors in the neighborhood of that bond. The large currents in a network are found in neighborhoods which have the ability to focus current from a large cross section into a single bond. The probability of finding a given local configuration decreases exponentially in the number of specified resistors. Thus, the large currents in the network are almost always found in those configurations which are most effective in focusing current for a given number of specified resistors. We refer to these neighborhoods as critical defects, these are the sites where breakdown will be initiated in the random fuse model. For two-dimensional networks with $0<G<1$, MG show that the critical defect is an hourglass or funnel-shaped region with large conductors along the axis of the average current and small conductors along the axis perpendicular to the average current, see Fig. 1. The largest currents occur in resistors near the center of the funnel. For the site dilution case, $G=0$, LD show that the critical defect is a line of insulators with a conducting hole in the center; see Fig. 1. In both cases, the magnitude of the current at the center of the defect can be obtained using continuum methods.

The current $i(l)$ in the bond at the center of a funnel of size l behaves as

$$i(l) \sim i_{\rm ave} l^{1-\nu} \quad \text{(funnel defect)} \tag{4}$$

with ν, the smallest eigenvalue of Laplace's equation with boundary conditions prescribed by current conservation and Ohm's law, given by

$$\nu = (4/\pi)\tan^{-1}(G^{1/2}) . \tag{5}$$

On the other hand, LD used conformal mapping methods to show that the current $i(l)$ in the central bond of a line defect behaves as

$$i(l) \sim i_{\rm ave} l/2 \quad \text{(line defect)} . \tag{6}$$

The next ingredient needed for obtaining the distribu-

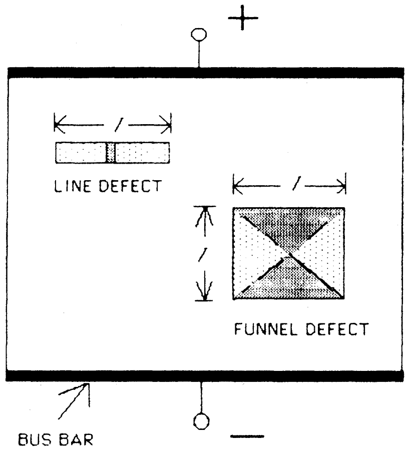

FIG. 1. A square network of size L with a funnel defect and a line defect depicted in it. The gray region is of conductivity one; the dotted region is of conductivity G; and the white region is of average conductivity. The funnel defect is $l\times l$; the line defect is $l\times 1$ with a single unit conductor in the middle. The horizontal bus bars maintain the voltage drop across the network.

tion of current values is the statistics of defect sizes. The probability that an $l\times l$ region is configured as a funnel is given by $[p(1-p)]^{l^2/2}$. This is roughly the cumulative distribution function $f(l)$ for a given bond to be at the center of a funnel of size l or larger,

$$f(l)=[p(1-p)]^{l^2/2} . \tag{7}$$

Assuming that the large currents are almost always found at the center of critical defects, $f(l)$ is equal to the cumulative current distribution

$$\int_{i(l)}^{\infty} di\, \rho(i)=f(l) , \tag{8}$$

where $\rho(i)$ is the probability density for the bond currents and l and $i(l)$ related by Eq. (4). Differentiating both sides with respect to $i(l)$ and setting $i_{\text{ave}}=1$ yields

$$\rho(i)\sim -f'(l)\frac{dl}{di}$$
$$\sim -\frac{l^{1+\nu}}{1-\nu}[p(1-p)]^{l^2/2}\ln[p(1-p)] . \tag{9}$$

Taking the logarithm of both sides and using Eq. (4) yields

$$\ln\rho(i)\sim -Ai^{1/\alpha}+(1/\alpha-1)\ln i+\text{const} \tag{10}$$

with

$$\alpha=(1-\nu)/2=\tfrac{1}{2}[1-(4/\pi)\tan^{-1}(G^{1/2})] \quad (0<G<1)$$

and

$$A=-\tfrac{1}{2}\ln[p(1-p)]$$

or, to leading order,

$$\rho(i)\sim\exp(-Ai^{1/\alpha}) \quad \text{as } i\to\infty . \tag{11}$$

The $G=0$ case may be treated in the same way except that now

$$f(l)=(1-p)^l . \tag{12}$$

Repeating the same steps as before we obtain,

$$\rho(i)\sim\exp(-Bi^{\alpha}) \quad \text{as } i\to\infty \tag{13}$$

with

$$\alpha=1$$

and

$$B=-2\ln(1-p) .$$

For a two-dimensional network, α approaches 0.5 as G approaches zero. However, $\alpha=1$ at $G=0$. MG suggested that the discontinuity in α at $G=0$ reflects a crossover from the dominance of funnel to linear critical defects. For small l and G, the small conductors in a line defect behave as insulators and Eq. (6) holds. However, as l increases, the current in the center hole approaches an upper bound, i_c, which can be shown to be

$$i_c=i_{\text{ave}}/G . \tag{14}$$

Since line defects are more probable than the two-dimensional funnel defects, one expects line defects to dominate for $i\ll i_c$ leading to exponential decay for $\rho(i)$ while for $i\gg i_c$ we expect to see the asymptotic behavior, Eq. (11), characterized by the exponent α given in Eq. (3). For $i\approx i_c$ we expect to see an effective exponent between α and 1.

III. NUMERICAL RESULTS

To test the prediction of Eq. (11), we chose three values of G: 0.25, 0.5, and 0.75. For each value, we computed the current in each conductor for 100 networks. Each network was 100×100 with $p=0.8$. A voltage drop across each network was maintained by two horizontal bus bars on opposite edges such that the average current per vertical conductor is unity (see Fig. 1). Periodic boundary conditions were used in the transverse direction to minimize edge effects. We used Kirchoff's current law and the conjugate gradient method[4,5] to compute node voltages using a Cyber 205. The solution was considered accurate when the fractional deviation from current conservation was less than 2×10^{-3} at each node. The probability distribution $\rho(i)$ is displayed in Fig. 2. If G were 1, the distribution would consist of two sharp peaks, at $i=0$ corresponding to horizontal bonds; and at $i=1$ corresponding to vertical bonds. As G decreases, the peaks broaden and overlap.

In Fig. 3, for each value of G, we plot $\ln[\ln(1/\rho)]$ versus $\ln i$ for the tail of the distribution. For $G=0.75$, we grouped currents into regular intervals of size 0.01 and considered the largest 0.032% of the currents. For $G=0.5$, we chose interval size of 0.03 and the largest 0.043% currents; for $G=0.25$, we chose interval size of 0.08 and the largest 0.032% currents. The effective exponent is determined from the slope of each plot, and compared with the theoretical prediction in Fig. 4. The numerical value is close to the theory for all G with increasing deviations above the theoretical curve as G decreases. This is in accord with the ideas concerning a crossover to $\alpha=1$ as G decreases. The agreement with the theory is perhaps too good since the range of i explored in the simulation is less than i_c. For $G=0.25$, $1.76<i<2.08$ and $i_c=4$; for $G=0.75$, $1.14<i<1.18$ and $i_c=1.33$.

We tested the assumption that current magnitudes are determined by the local arrangement of the resistors by rearranging the network and then recomputing the currents. We interchanged the 9×9 neighborhood surrounding a large current with a randomly chosen 9×9 neighborhood thus creating a new network having the same local environment for the bond with a large current but a different global environment. For $G=0.75$ we ex-

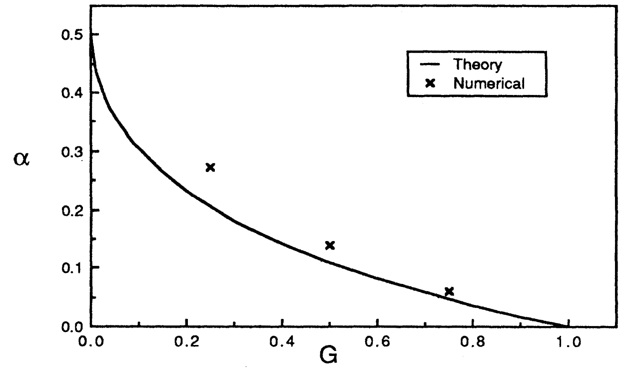

FIG. 4. A comparison of α obtained from Fig. 3 to the theoretical result (solid line), Eq. (3).

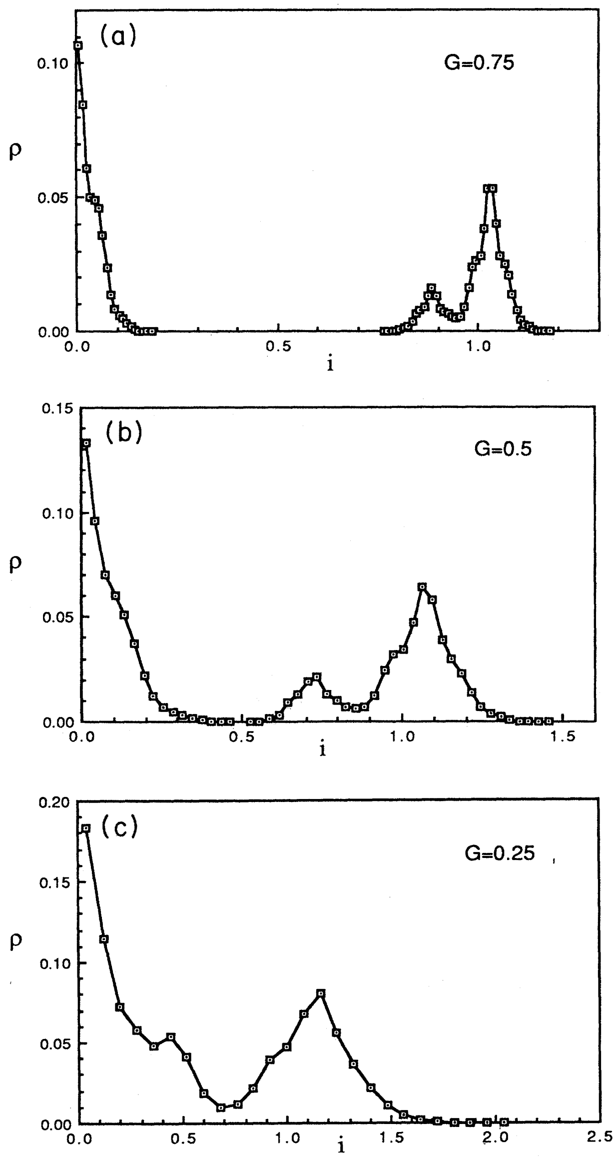

FIG. 2. Distribution of currents in an ensemble of 100 configurations of 100×100 networks with a given G, where ρ is the density of currents of magnitude i. (a) $G=0.75$, (b) $G=0.5$, (c) $G=0.25$.

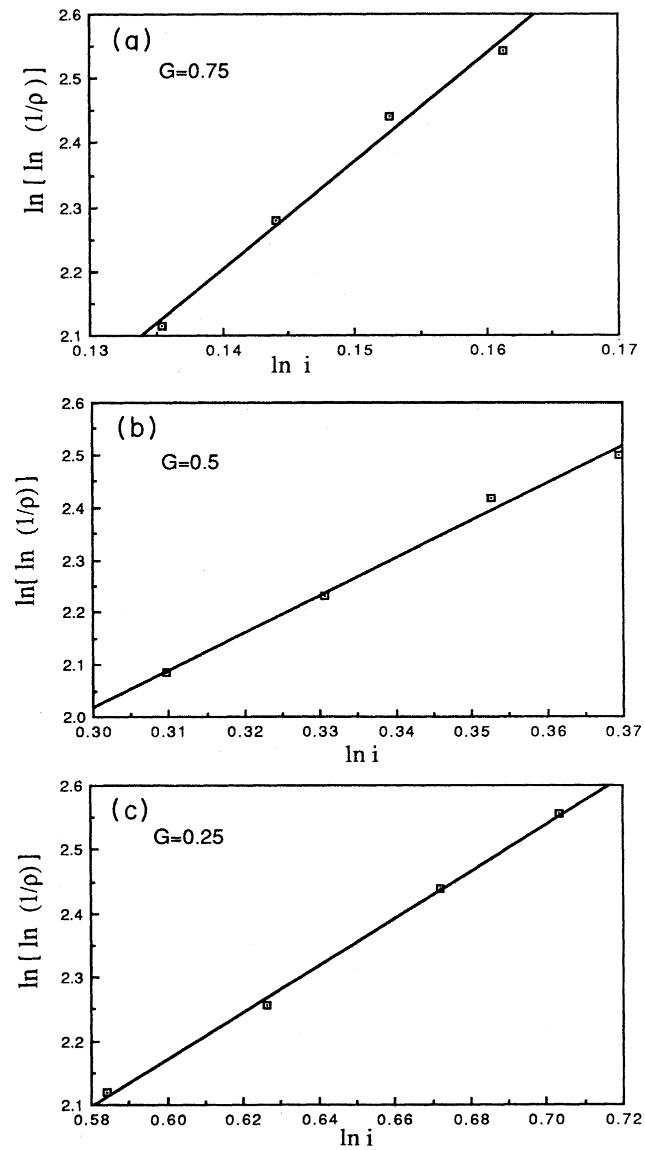

FIG. 3. $\ln[\ln(1/\rho)]$ vs $\ln i$ at a given G. The slope of this plot gives $1/\alpha$, see Eq. (11). (a) $G=0.75$, (b) $G=0.5$, (c) $G=0.25$.

TABLE I. Funnel defect detected numerically. The numbers in the array are the ensemble averaged fraction of large conductors, F, at each vertical bond site in the 9×9 neighborhood surrounding the large current which is located at the center. The average current flow is in the vertical direction. F is shown in underlined boldface if $F>p$; and in italics if $F<p$ where $p=0.8$ is the average number of large conductors in the network.

0.76	**0.84**	**0.85**	**0.91**	**0.91**	**0.85**	**0.85**	**0.85**	**0.82**
0.74	**0.80**	*0.77*	**0.89**	**0.93**	**0.84**	**0.84**	**0.82**	*0.78*
0.79	*0.79*	**0.88**	**0.86**	**0.98**	**0.96**	**0.86**	**0.82**	*0.77*
0.66	*0.64*	*0.74*	**0.81**	**1.0**	**0.82**	*0.66*	*0.68*	*0.77*
0.78	*0.68*	*0.46*	*0.0*	**1.0**	*0.0*	*0.32*	*0.69*	*0.76*
0.75	*0.74*	*0.75*	**0.84**	**1.0**	**0.80**	*0.74*	*0.77*	*0.79*
0.81	**0.80**	**0.85**	**0.91**	**0.99**	**0.89**	**0.80**	*0.73*	**0.85**
0.85	*0.79*	*0.73*	**0.88**	**0.93**	**0.90**	**0.89**	**0.86**	*0.78*
0.81	**0.84**	**0.84**	**0.88**	**0.97**	**0.85**	**0.82**	**0.82**	*0.78*

amined four large currents with five transpositions each and observed a maximum change in the magnitude of the large current of 1.5%. For comparison, the median vertical current is more than 12% below the large currents explored here. Having established that it is the local environment which determines the magnitude of large currents we can now study this environment in more detail.

Although the individual environments of large currents appear to be rather random, we can detect a funnel pattern by superimposing an ensemble of neighborhoods containing large currents.[6] We computed the currents for ten 100×100 configurations with $G=0.75$ and $p=0.8$. We examined the 9×9 neighborhoods surrounding the ten largest currents in each configuration. With the bond carrying the large current chosen as the origin we superimposed these neighborhoods and calculated the fraction of large conductors at each bond position relative to the origin. Neighborhoods which included the bus bars were omitted from the ensemble. The fraction of large conductors at each vertical bond site, F, was displayed in Table I. Those bonds which have a fraction greater than p (which is 0.8) reside in a large conductivity region. The funnel shape is roughly apparent.

In conclusion, we have studied the asymptotic properties of the distribution of currents in random resistor networks. The results are consistent with recent theoretical ideas and show that large currents in the network result from funnel-shaped defects.

ACKNOWLEDGMENTS

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