PHYSICAL REVIEW B VOLUME 39, NUMBER 1 1 JANUARY 1989

Random resistor network with an exponentially wide distribution of bond conductances

Stéphane Tyč and B. I. Halperin
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138
(Received 11 August 1988)

We use a percolation analysis to study the conductivity of a random resistor network with bond conductances $g_i = g_0 \exp(\lambda x_i)$, where x_i is a random variable. In the limit $\lambda \to \infty$, we may write the network conductivity as $\sigma = C a^{2-d} g_c \lambda^{-y}$ where a is the lattice constant, y a critical exponent, C a constant, and g_c the percolation conductance. We derive rigorous bounds to σ and we present evidence that supports the hypothesis that $y = 0$ for all two-dimensional lattices. Numerical results for a $d = 3$ simple-cubic lattice are presented.

Random conductance networks have been widely used to study a variety of physical phenomena in idealized disordered media. For instance, some transport properties of amorphous materials and some spin-wave properties of disordered ferromagnets can be mapped into a resistor network problem.[1,2]

A problem often studied consists in computing the conductivity of a network where the conductance assigned to each link is chosen independently from a specified random distribution. When the distribution of conductances is narrow and the correlations are short ranged, the effective-medium theory (EMT) is a very successful approach.[2,3] When the distribution of conductances is wide on a logarithmic scale, however, an alternate approach based on percolation theory has proved more useful.[4-11]

In the present Rapid Communication we shall refine the previous percolation analyses to develop a more accurate estimate of the network conductivity, in the limit of a wide distribution of conductances. To do this, we will examine some rigorous bounds to the network conductivity and we will present the results of some numerical simulations.

Our results can be explained most readily if one considers conductances on a regular lattice in d dimensions, with lattice constant a. The conductance on bond i will be written in the form

$$g_i = g_0 \exp(\lambda x_i) , \tag{1}$$

where g_0 is a constant, and x_i is a random variable with probability distribution $D(x_i)$. We assume that there are no correlations between the values of x_i on different bonds, and that the distribution $D(x)$ is the same for all bonds.

A percolation conductance g_c may be defined so that the fraction of bonds with $g_i \geq g_c$ is equal to the threshold probability p_c for bond percolation on the lattice in question. In our model we have $g_c = g_0 \exp(\lambda x_c)$, where

$$\int_{x_c}^{\infty} D(x) dx = p_c . \tag{2}$$

We assume that $D(x)$ is smooth in the vicinity of x_c and that $D(x_c) \neq 0$. The limit of a wide distribution is achieved if we take $\lambda \to \infty$. According to the analysis of Refs. 4 and 5, for example, we may write

$$\lim_{\lambda \to \infty} \lambda^{-1} \ln(\sigma a^{d-2}/g_0) = x_c , \tag{3}$$

so that to a first approximation $\sigma \propto g_c$. In this limit, in fact, we may construct a more accurate estimate of the conductivity. We shall demonstrate, below, that there exist rigorous upper and lower bounds to the conductivity, in the limit $\lambda \to \infty$, which may be written in the form

$$\sigma_{\pm} = C_{\pm} \frac{g_c}{a^{d-2}} \left(\frac{D(x_c)}{\lambda} \right)^{y_{\pm}} , \tag{4}$$

where $C_{\pm}$ are constants, which depend on the particular lattice. The bounding exponents $y_{\pm}$ are given, in turn, by the formulas

$$y_+ = 1 - s , \tag{5a}$$

$$y_- = t - 1 , \tag{5b}$$

where t and s are the exponents for the conductivity at the percolation threshold of a mixture of unit normal resistors with insulating or superconducting bonds, respectively. The exponents are defined by $\sigma \propto (p - p_c)^t$ and $\sigma \propto (p_c - p)^{-s}$, where p is the fraction of bonds with the higher conductance. It is believed that these exponents depend only on the dimensionality of the lattice, and the numerical values for $d = 2$ and $d = 3$ are known reasonably well from previous investigations.[6] Equation (3) is implied by (4).

It seems natural to hypothesize that the conductivity for our model in the limit $\lambda \to \infty$ has the actual asymptotic form[7]

$$\sigma \sim C \frac{g_c}{a^{d-2}} \left(\frac{D(x_c)}{\lambda} \right)^{y} , \tag{6}$$

where C is again a constant, which depends on the lattice in question, while the exponent y depends only on the dimensionality. If this assumption is correct, then our rigorous bounds imply

$$y_+ \leq y \leq y_- . \tag{7}$$

Using the values of t and s from previous numerical investigations, these bounds may be written

$$-0.3 \leq y \leq 0.3 \quad \text{for } d = 2 , \tag{8a}$$

$$0.2 \leq y \leq 1.0 \quad \text{for } d = 3 . \tag{8b}$$

Rigorous bounds which have previously appeared in the

literature may be expressed in the form $-s \leq y \leq t$.[8] Equations (5) and (7) represent a considerable improvement over these bounds.

As will be discussed below, for a square lattice, if $D(x)$ is symmetric about x_c, then the network conductivity is exactly given by (6) with $y=0$ and $C=1$. We have performed numerical simulations on a *triangular* lattice with linear sizes up to $L=250$ lattice constants, with $\lambda/D(x_c)$ in the range between 5 and 25, and we find that $|y_{\text{triangle}}| \leq 0.02$, which is clearly consistent with the conjecture $y=0$ for all lattices in $d=2$.

In three dimensions, we have simulated a simple-cubic lattice with linear sizes in the range $L=4$–20. The estimate of y obtained from these simulations is $y \simeq 0.6$, which is close to the center of the range delineated by the bounds of (8b). Very recently, Le Doussal[9] has shown that for hierarchical lattices, the exponent y is given by

$$y=(d-2)\nu, \tag{9}$$

where ν is the exponent which describes the divergence of the characteristic length ξ at the percolation threshold. Le Doussal has suggested that (9) applies also in Euclidean space for $d \leq 6$. This is consistent with our results for $d=2$. Since the value of ν is believed to be $\simeq 0.88$ in $d=3$,[6] however, Eq. (9) would imply $y=0.88$ for $d=3$, which is close to the upper bound of (8b), and larger than the apparent results of our numerical simulation.

A relation equivalent to Eq. (9) for $d=3$ was proposed in 1973 by Ambegaokar, Cochran, and Kurkijärvi,[7] in the context of a continuum model, as a consequence of the assumption of a scaling form for the conductivity in a finite system (see also Shklovskii and Efros[10]). The value of ν was not well known at the time, but on the basis of simulations of this continuum model, with up to 2000 sites present, Kurkijärvi concluded that the exponents y and ν had a value 0.6 ± 0.25 in $d=3$.[11] This result coincides with our result for y on the simple-cubic lattice. Wilke has studied the size dependence of the average percolation threshold $\bar{p}_c(L)$ on a simple-cubic lattice for sizes up to $L=200$ with various boundary conditions.[12] If one defines an effective value of ν_{eff} as $-d(\ln L)/d\ln|\bar{p}_c - p_c|$ one finds a large correction to ν, with a value $\nu_{\text{eff}} \simeq 0.6$ for $L \simeq 15$. Thus, our numerical results may in fact be consistent with $y=\nu=0.88$ in the limit $L=\infty$. Further work is clearly necessary on this point.

The model defined by (1) can be generalized slightly if we consider a probability distribution $P(g_i)$ for the bond conductances g_i, which depends on a parameter λ. Then the criterion for a wide distribution for large λ is that $g_c P(g_c) \ll 1$, and that $gP(g)$ is slowly varying for a large range $M^{-1} g_c < g < M g_c$ with $M \gg 1$. Without loss of generality, we may define the parameter λ such that

$$\lambda^{-1} = g_c P(g_c). \tag{10}$$

All our previous discussions apply to this model, with the substitution $D(x_c)=1$ [for the model of (1), we have $gP(g)=\lambda^{-1}D(x)$, but it is always possible to rescale x so that $D(x_c)=1$].

A simple estimate of the conductivity of a random network is the critical-conductor approximation (CCA), in which all the conductances of the network are replaced by the critical conductance g_c.[13] This leads to a conductivity of the form of (6) with $y=0$ in all d. Although $y=0$ is outside the rigorous bounds for $d=3$, it appears to be correct for $d=2$. The value of C predicted by the CCA is $C=1$ for a square- or a simple-cubic lattice, and $C=\sqrt{3}$ in the triangular case. By contrast, our numerical simulations yield $C=0.97$ for the triangular lattice (smaller that the CCA by a factor of 1.8) if we set $y=0$ in our fit to the data. Of course the most important dependence on the width of the distribution [i.e., the exponential factor $\exp(\lambda x_c)$], is given correctly by the CCA, in any dimension $d \geq 2$.

We now derive our rigorous bounds for the conductivity, using the following well-known theorems.[14]

Theorem 1: If the conductance of any bond or set of bonds in a network is increased, the conductance (or conductivity) of the entire sample must either increase or remain the same.

Theorem 2: If each resistance in a random network is replaced by its expectation value, the new resistivity will be larger than the expectation value of the resistivity of the original network.

Theorem 3: If each conductance in a random network is replaced by its expectation value, the new conductivity will be larger than the expectation value of the conductivity of the original network.

Upper bound for the conductivity. We study a random conductance network with a distribution $P(g)$ that is wide in the sense defined above with $g_c P(g_c) = 1/\lambda$. We consider an infinite system, where we may assume $\sigma = \langle\sigma\rangle = \langle\sigma^{-1}\rangle^{-1}$ with probability unity. Let us pick a realization of the infinite network, and let us also choose a constant $K>1$. We will clearly obtain an upper bound for the conductivity if we replace all the conductances with value greater than Kg_c by infinite conductances (Theorem 1). We can also replace all the remaining conductances by their expectation value $\bar{g}_+$ (Theorem 3). Thus, the new network obtained is the superconducting-normal percolation network (SP), whose conductivity is given asymptotically for $p \to p_c$ by

$$\sigma_{\text{SP}} \sim C_{\text{SP}} \bar{g}_+ (p_c - p)^{-s} a^{2-d}, \tag{11}$$

where C_{SP} is a constant that depends on the lattice. The values of $\bar{g}_+$ and $(p-p_c)$ are given asymptotically for $\lambda \to \infty$ by

$$\bar{g}_+ = \int_0^{Kg_c} gP(g)\,dg \Big/ \int_0^{Kg_c} P(g)\,dg \simeq \frac{g_c K}{\lambda(1-p)}, \tag{12}$$

$$p_c - p = \int_{g_c}^{Kg_c} P(g)\,dg \simeq \lambda^{-1} \ln K. \tag{13}$$

Hence, for large λ, we have

$$\sigma \leq C_{\text{SP}} g_c a^{2-d} \lambda^{s-1} (1-p_c)^{-1} K/(\ln K)^s. \tag{14}$$

The upper bound may be optimized by choosing $K=e^s$.

The lower bound is derived in a similar manner. This time, we remove all the conductances with $g_i < g_c/K$, and we replace the remaining conductances by a conductance $\bar{g}_-$ which is the inverse of the mean value of the resistance of the remaining bonds (Theorem 2). Writing the con-

ductivity of a random network of normal conductors $\bar{g}-$ and insulators near the percolation threshold in the asymptotic form $C_{CP}\bar{g}-(p-p_c)^t a^{2-d}$, we obtain the lower bound

$$\sigma \geq C_{CP} g_c a^{2-d} \lambda^{1-t} p_c (\ln K)^t / K . \quad (15)$$

This bound is optimized by the choice $K=e^t$.

Let us now investigate the exponent y for lattices in two dimensions. For the square lattice one can prove[1,15] that $\sigma=g_c$ is exact, provided that $D(x)$ is symmetric with respect to x_c. The proof uses the self-duality of the square network. In general, one defines the dual of a planar two-terminal network by applying the usual definition of the node mesh duality to the network where the two terminals are connected by an external conductance.[1,15] Each bond in the dual network corresponds to a bond in the original network and is assigned a conductance g_i^* such that $g_i g_i^* = h^2$ where h is a constant. Under these conditions it can be shown that $GG^* = h^2$, where G and G^* are the conductances of the original network and of its dual between their respective terminals. The duality transformation takes the variable x_i into $-x_i + cst$. If we choose h equal to g_c, and $D(x)$ is symmetric about x_c, then the transformation leaves $D(x)$ unchanged, so that $G=G^*=g_c$. Thus, $\sigma=g_c$, and $y=0$ for the square lattice.

Using the fact that the triangle lattice is dual to the honeycomb lattice, and arguing that the exponent y cannot depend on the details of the distribution $D(x)$, one can show that the values of y must be related by $y_{\text{triangle}} = -y_{\text{honeycomb}}$. However the duality argument does not determine the value of y in either case, and we turn instead to a numerical solution.

In all the simulations we have taken $D(x)$ to be uniform on the interval $0<x<1$. However, the results of the numerical simulations should apply in the much more general case of a distribution $D(x)$ smooth at x_c with $D(x_c)\neq 0$. For the triangular lattice, we take the exact result for $p_c = 1-2\sin(\pi/18)$.[16] We use the bond propagation algorithm of Frank and Lobb[17] and compute the conductance of lattices of linear size 50 (with 900 realizations) to 250 (with 9 realizations). The choice of λ must obey two constraints: First, λ must be big enough so that the distribution is wide and that the percolation arguments go through. This implies roughly that $p-p_c \propto \ln(K)/\lambda \ll 1$ where K is some constant of order 10 or more. Second, we must also have λ small enough so that the correlation length $\xi_\lambda \propto (p-p_c)^{-\nu} \propto (\lambda/\ln K)^\nu$ is small compared to L the size of the system, where ν is the correlation length exponent for classical percolation. This leaves us with a narrow window for the values of λ; we chose to simulate $\lambda = 10, 15, 20, 25$. For each size L and each value of λ, we compute the average of the conductivity. We use three different definitions of the average: The arithmetic, the geometric, and the harmonic mean. For each average of the conductivity $\bar{\sigma}(L,\lambda)$, we get a value $y(L)$ (shown in Fig. 1) by fitting $\ln[\bar{\sigma}(L,\lambda)] = \lambda x_c - y\ln\lambda + b$. The numerical results are well fitted by $\sigma(\lambda) = 0.56\sigma_0 \exp(\lambda x_c)$, where σ_0 is the value for $\lambda=0$. This is accurate within 3% as long as $\lambda \geq 10$ and ξ_λ is small compared to L. In any case, we are able to put numerical bounds on $|y_{\text{triangle}}| \leq 0.02$.

In three dimensions we have simulated a cubic lattice with the same distribution as in the triangular lattice and $p_c = 0.25$.[12] In this case we used the transfer matrix algorithm[18] to solve for cubes of linear size L. We went from $L=4$ and 2000 realizations to $L=20$ and 25 realizations. For each average we have extracted y in two different ways. First, we can fit a straight line $\langle \ln\sigma\rangle = \lambda x_c - y\ln\lambda + b$ for each size and thus define $y(L)$ (cf. Fig. 2). This is done by minimizing the χ^2 and the weights are given by the root-mean-square fluctuation of $\langle \ln\sigma\rangle$. The error bars coming out of this fit are fairly large, which is an indication that L and λ are not quite big enough to have reached the asymptotic regime. This analysis is consistent with a value of $y \simeq 0.6 \pm 0.1$, which is in agreement with the rigorous bounds (8b). Second, we have tried to fit the results by assuming that the average conductivity in the region of interest has a scaling form, with additional finite-size corrections $\propto L^{-1}$:

$$\langle \ln[\sigma/(g_0 a^{d-2})]\rangle = \lambda x_c - y\ln\lambda + f(r) + h(r)/L , \quad (16)$$

where $r = L/\xi$ and $\xi = [\lambda/D(x_c)]^\nu$ with $\nu = 0.88$. We assume that for the range of interest we can write

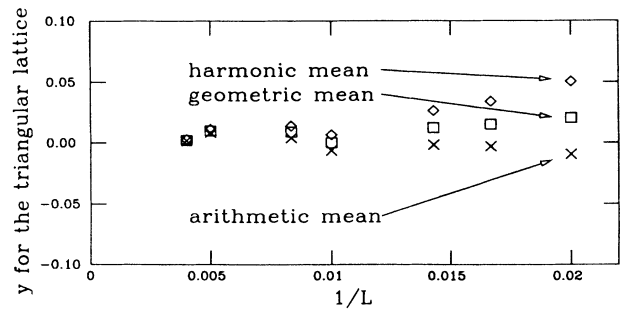

FIG. 1. Effective y as a function of inverse size L^{-1} for the triangular lattice.

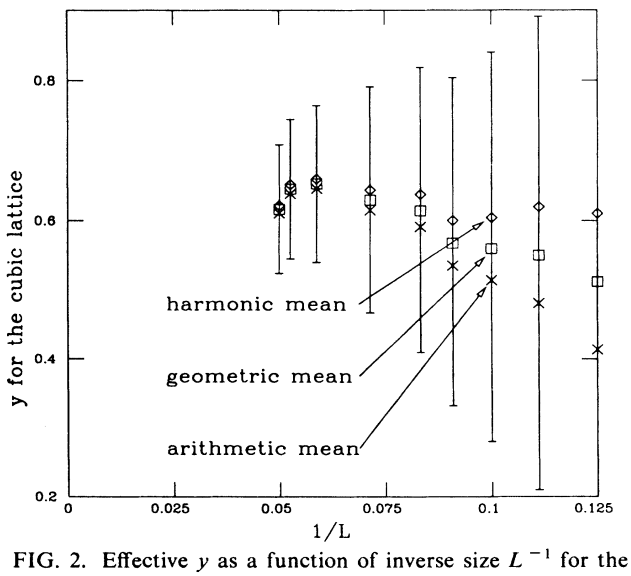

FIG. 2. Effective y as a function of inverse size L^{-1} for the cubic lattice, the error bars are plotted for the geometric mean only.

$f(r)=a+br$ and $h(r)=c+dr$. We then perform a five-parameter fit and it yields $y=0.57\pm0.1$, $y=0.60\pm0.1$, $y=0.62\pm0.1$, for $\ln(\langle g\rangle)$, $\langle \ln g\rangle$, and $\ln(\langle 1/g\rangle^{-1})$. The four other parameters vary typically by factors of two between the different averages.

We wish to thank S. Chakravarty and Chris Lobb for stimulating discussions and for providing us with references. We also thank P. Le Doussal for discussing his work prior to publication. This work was supported by the National Science Foundation through the Harvard Materials Research Laboratory and Grant No. DMR 85-14638.

Note added in proof. It has been brought to our attention that the rigorous bounds given by Eq. (5) have been previously derived by Charlaix, Guyon, and Roux.[19]

[1]J. Bernasconi, W. R. Schneider, and H. J. Wiesmann, Phys. Rev. B **16**, 5250 (1977).
[2]S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).
[3]D. A. G. Bruggeman, Ann. Phys. (Leipz.) **24**, 636 (1935).
[4]V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).
[5]B. I. Shklovskii and A. L. Efros, Zh. Eksp. Teor. Fiz. **60**, 867 (1971) [Sov. Phys. JETP **33**, 468 (1971)].
[6]D. Stauffer, *Introduction to Percolation Theory* (Taylor and Francis, London, 1985).
[7]V. Ambegaokar, S. Cochran, and J. Kurkijärvi, Phys. Rev. B **8**, 3682 (1973).
[8]M. Pollak and B. I. Halperin, Solid State Commun. **13**, 869 (1973).
[9]P. Le Doussal, following paper, Phys. Rev. B **39**, 881 (1989).
[10]B. I. Shklovskii and A. L. Efros, Usp. Fiz. Nauk **117**, 401 (1975) [Sov. Phys. Usp. **18**, 845 (1975)].
[11]J. Kurkijärvi, Phys. Rev. B **9**, 770 (1974).
[12]S. Wilke, Phys. Lett. **96A**, 344 (1983).
[13]D. Berman, B. G. Orr, H. M. Jaeger, and A. M. Goldman, Phys. Rev. B **33**, 4301 (1986).
[14]See, e.g., S. Feng, B. I. Halperin, and P. Sen, Phys. Rev. B **35**, 197 (1987).
[15]J. Marchant and R. Gabillard, C. R. Acad. Sci. Ser. B **281**, 261 (1975).
[16]M. F. Sykes and J. W. Essam, J. Math. Phys. **5**, 1117 (1964).
[17]C. J. Lobb and D. J. Frank, Phys. Rev. B **30**, 4090 (1984).
[18]B. Derrida and J. Vannimenus, J. Phys. A **15**, L557 (1982).
[19]E. Charlaix, E. Guyon, and S. Roux, in *Transport in Pourous Media 2* (Riedel, Dordrecht, 1987), p. 31.