PHYSICAL REVIEW B VOLUME 39, NUMBER 11 15 APRIL 1989-I

dc and ac soliton conductivity of disordered charge-density-wave systems and long Josephson junctions

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(Received 19 September 1988)

A model of a commensurate charge-density-wave system with randomly distributed charge impurities, based on a driven damped or overdamped sine-Gordon equation with additional random terms, and a similar model of a driven damped or overdamped randomly inhomogeneous Josephson junction are considered. A fundamental assumption is that the system contains phase solitons trapped by an effective random potential. With the increase of the dc drive, the solitons are released gradually. The corresponding *I-V* characteristics are found with regard to dissipative and radiative losses. A frequency dependence of the ac conductivity is also found.

It is generally believed that the nonlinear conductivity of one-dimensional metals is accounted for by the action of commensurability[1] and impurity[2] pinning on a charge-density wave (CDW). An interesting problem is a joint effect of the two factors.[3] Evolution of a phase misfit of a commensurate CDW in a system with charged impurities is governed by a perturbed sine-Gordon (SG) equation, which is a slight generalization of that put forward by Fukuyama:[3]

$$\phi_{tt}+\gamma\phi_t-\phi_{xx}+\sin\phi+f=\epsilon_0\sum_n\delta(x-x_n)\cos\left[\frac{\phi}{M}+2k_Fx_n\right], \quad (1)$$

where $M\phi$ is the phase misfit, M is a commensurability index, γ is a dissipative constant, f is an external drive (dc voltage), ϵ_0 is a constant of coupling of the CDW to impurities, and x_n are their coordinates. In a commensurate CDW system, charge is carried by phase solitons.[4] If the parameters γ, f, and ϵ_0 in (1) are small, a soliton is close in form to the unperturbed SG kink,

$$\phi_k(x,t)=4\arctan(\exp\{\sigma[x-\xi(t)](1-V^2)^{-1/2}\}), \quad (2)$$

where $\sigma=\pm1$, $\xi(t)$, and V are the kink's polarity, coordinate, and velocity ($V^2<1$). According to Ref. 3, in a real one-dimensional metal a mean distance l between the impurities is small compared to the soliton's size, i.e., in our notation, $l\ll1$. The coordinate x_n of an individual impurity may be regarded as a random quantity.[2] Therefore, being interested in the soliton's dynamics, one may replace Eq. (1) by the equation

$$\phi_{tt}+\gamma\phi_t-\phi_{xx}+\sin\phi+f=\zeta_1(x)\sin\frac{\phi}{M}+\zeta_2(x)\cos\frac{\phi}{M}, \quad (3)$$

$\zeta_{1,2}(x)$ being random Gaussian functions subject to the correlations

$$\langle\zeta_{1,2}(x)\rangle=\langle\zeta_1(x)\zeta_2(x')\rangle=0,$$
$$\langle\zeta_1(x)\zeta_1(x')\rangle=\langle\zeta_2(x)\zeta_2(x')\rangle=\epsilon^2\delta(x-x'), \quad (4)$$

where $\epsilon^2\equiv\epsilon_0^2/2l$ will also be regarded as a small parameter [a different but similar continuum approximation of Eq. (1) was employed by Fukuyama[3]]. The model (3) and (4), with $\zeta_2\equiv0$ and $M=1$, was considered earlier[5] as a model of a dc-driven damped long Josephson junction (JJ) with the maximum supercurrent density subject to a random spatial modulation.

According to Refs. 6 and 7 in real one-dimensional metals in which the commensurability takes place (e.g., $NbSe_3$ and TaS), dissipation is very strong, so that an appropriate model for them is the overdamped sine-Gordon (OSG) equation

$$\gamma\phi_t-\phi_{xx}+\sin\phi+f=\zeta_1(x)\sin\frac{\phi}{M}+\zeta_2(x)\cos\frac{\phi}{M}. \quad (5)$$

In the JJ theory, this model [with $\zeta_2(x)\equiv0$ and $M=1$] describes a randomly inhomogeneous junction of the superconductor–normal-metal–superconductor type, i.e., two bulk superconductors separated by a thin layer of a normal metal.

A few comments about the commensurability index M. As is well known, it may take values ≥3. At the same time, $M=1$ and $M=2$ may be generated by an external spatially periodic potential (ionic superlattice) imposed on the CDW system.[8–11] An argument can be made in favor of the presence of such a superlattice in the one-dimensional metals KCP (Refs. 8 and 11) and $NbSe_3$.[9] $M=2$ is also possible when interaction of two sorts of phonons with a CDW must be taken into account.[12] In the present work, all the values $M\geq1$ will be admitted.

A known model of the nonlinear conductivity of the CDW systems developed by Maki[13] is based on a quantum-mechanical calculation of the rate of production of kink-antikink pairs in electric field. In that model, impurities do not play a crucial role. The aim of the present paper is to propose another model based on electric-field depinning of kinks trapped by a random potential relief (a conductivity model based on depinning of an *incommensurate* CDW is well elaborated.[14,15]) In the spirit of the perturbation theory for SG solitons,[16] it is easy to find an effective kink's potential corresponding to the models (3) and (5):

$$U(\xi)=\int_{-\infty}^{+\infty}dx\sum_{k=1,2}\zeta_k(x)U_k(\xi-x)\,,$$
$$U_1(z)\equiv M\{1+\cos[M^{-1}\phi_k(z)]\}\,, \quad (6)$$
$$U_2(z)\equiv M\{1-\sin[M^{-1}\phi_k(z)]\}\,,$$

where ϕ_k is the wave form (2) with $V=0$.

The present model is based on the assumption that at $f=0$ the system contains "ready-made" kinks with a density n_0 [in the SG model (3) they may be of both polarities, while in the OSC model (5) they must be unipolar]. With the increase of f, at some $f=f_{\rm cr}$ a trapped kink escapes at a point $\xi=\xi_{\rm cr}$ where $U''(\xi)=0$. The basic ingredient of the model is to calculate a share $\mu(f)$ of the points $\xi_{\rm cr}$ for which $f_{\rm cr}<f$ (for a given f, kinks trapped in a vicinity of these points have escaped already). Proceeding from the probability-density functional for the Gaussian fields

$$P[\zeta_{1,2}(x)]\sim\exp\left[\int_{-\infty}^{+\infty}(2\epsilon^2)^{-1}[\zeta_1^2(x)+\zeta_2^2(x)]dx\right]\,,$$

one can easily find the probability distribution for the values $|f_{\rm cr}|$ corresponding to the potential (6):

$$p(f_{\rm cr})=2(2\pi/I_1)^{1/2}\epsilon^{-1}\exp(-2\pi^2f_{\rm cr}^2/I_1\epsilon^2)\,, \quad (7)$$

$$I_j\equiv\int_{-\infty}^{+\infty}dx\sum_{k=1,2}[d^jU_k(x)/dx^j]^2\quad(j=1,2,3)\,. \quad (8)$$

It is now straightforward to find the above-mentioned $\mu(f)$:

$$\mu(f)=\int_0^f p(f_{\rm cr})df_{\rm cr}=\mathrm{erf}(\sqrt{2}\pi f/\sqrt{I_1}\epsilon)\,. \quad (9)$$

Further analysis differs for the two models (3) and (5).

The OSG model. In this case it is necessary to take account of the fact that a released kink may be trapped by a vicinity of another point $\xi_{\rm cr}$ with $f_{\rm cr}>f$. So, to find the density of free kinks which contribute to the conductivity, one must know the maximum density $n_t(f)$ of the kinks that may be trapped at a given f. $n_t(f)$ is proportional to a density ν of the points $\xi_{\rm cr}$. It is easy to find $\nu=\pi^{-1}(I_3/I_2)^{1/2}$, $I_{2,3}$ being defined in Eq. (8). The problem may be considered in the one-kink approximation provided $n_0\ll\nu$, i.e., $n_0\ll 1$. In this case, a range of concern is $f\gg\epsilon$, where one can easily obtain, from the expressions (7) and (9),

$$n_t(f)\approx(2\pi^3)^{-1/2}\sqrt{I_1}\nu\epsilon f^{-1}\exp(-2\pi^2f^2/I_1\epsilon^2)\,. \quad (10)$$

The points $\xi_{\rm cr}$ with $f_{\rm cr}>mf$ can trap m kinks, but a corresponding many-kink correction to the expression (10) is negligible.

The system becomes conductive at

$$f^2=f_0^2\approx(I_1/2\pi^2)\epsilon^2\ln n_0^{-1}\,, \quad (11)$$

when $n_t(f)=n_0$ (the assumption $n_0\ll 1$ was strengthened to $\ln n_0^{-1}\gg 1$). At $f>f_0$, the density of free kinks is $n(f)=n_0-n_t(f)$. The mean velocity of a free kink in the overdamped model is

$$V(f)=\pi f/4\gamma\,. \quad (12)$$

Thus at $f>f_0$ the dc current-voltage characteristic (CVC), i.e., a dependence of the current j on the voltage f, takes the form

$$j\equiv qV(f)[n_0-n(f)]\approx qn_0(\pi f/4\gamma)\{1-\exp[-(4\pi^2f_0/I_1\epsilon^2)(f-f_0)]\}\,, \quad (13)$$

where f_0 is defined in (11), and $q=2e/M$ is the kink's electric charge. In the range $0<f-f_0\lesssim I_1\epsilon^2/4\pi^2f_0$, the conductivity

$$\rho(f)\equiv dj/df\approx(\pi qn_0/2\gamma)(\ln n_0^{-1})\exp[-4\pi^2f_0(f-f_0)/I_1\epsilon^2] \quad (14)$$

differs strongly from the usual $\rho_0=\pi qn_0/4\gamma$.

SG model. Let us proceed to model (3) with $\gamma\ll\epsilon$. If the trapped kinks are distributed uniformly along the system at $f=0$, at $f>0$ the share of free kinks is equal to $\mu(f)$ defined in (9). Let us formulate conditions which guarantee that the released kinks will not be trapped again. A potential hill of a height U_0 will trap a free kink if[16] $f^2\lesssim\gamma^2U_0$. As is seen from Eq. (9), of basic concern are the values $f\sim\epsilon$. So, to avoid repeated capture of the released kinks, it is necessary to demand that the values taken by $|\zeta_{1,2}(x)|$ be limited by some ζ_m such that $\gamma^2\zeta_m\ll\epsilon^2$. Because of the above assumption $\gamma\ll\epsilon$, this limitation is not significant.

The velocity of a free kink in the SG model is[16]

$$V(f)=[1+(4\gamma/\pi f)^2]^{-1/2}\,. \quad (15)$$

A CVC determined by Eqs. (9) and (15) takes the form [cf. Eq. (13)]

$$j=qV(f)n_0\mu(f)\approx qn_0\mathrm{erf}(\sqrt{2}\pi f/\sqrt{I_1}\epsilon) \quad (16)$$

in the range $f\sim\epsilon$. A full CVC is hysteretic (Fig. 1): The branch (16) (lower in Fig. 1) is observed if f increases from zero, and the usual (upper) branch corresponding to $\mu(f)\equiv 1$ is observed if f decreases from the values $\gg\epsilon$. If f increases along the lower branch up to some $f_1\sim\epsilon$, and then turns back, one will observe an intermediate branch

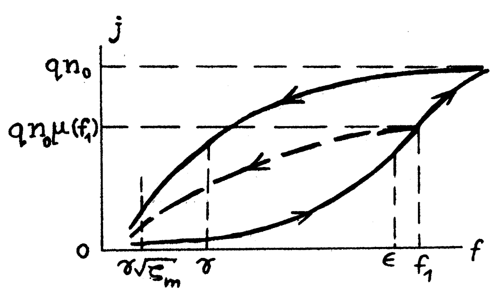

FIG. 1. The hysteretic CVC (*I-V* characteristic) of the SG model (3). The arrows indicate the sense of different branches.

(dashed curve in Fig. 1) corresponding to $\mu = \mu(f_1)$. The CVC terminates at $f \sim \gamma(\zeta_m)^{1/2}$, where, according to that stated above, the kinks will be trapped by maxima of the random potential.

It was implied that collisions between free kinks and trapped ones do not release the latter. A detailed analysis demonstrates that this is true under the above assumption $\epsilon \gg \gamma$ if the kinks are unipolar; if both polarities are present, one needs $\epsilon \gg \gamma^{2/5}$.

Radiative effects. If the γ is sufficiently small, the solitonic CVC of the SG model may be affected essentially by radiative losses (emission of linear waves from a kink scattered by inhomogeneities). Application of the perturbation theory for emission problems elaborated in detail in Ref. 17 yields the following results. In the range $V^2 \approx (\pi f/4\gamma)^2 \ll 1$ (see Ref. 15), the energy emission rate per a kink W is exponentially small (see also Ref. 5):

$$W \lesssim \exp(-\pi/V). \tag{17}$$

In the opposite range $1 - V^2 \approx (4\gamma/\pi f)^2 \ll 1$; for $M=1$,

$$W \approx 2\epsilon^2(1-V^2)^{1/2} \tag{18}$$

[if $\zeta_2 \equiv 0$, W differs from Eq. (18) by the multiplier $\frac{1}{3}$]. It is interesting to note that almost all the energy (18) is emitted backwards, and characteristic wave numbers of the emitted radiation are $\sim -(1-V^2)^{-1/2}$.

As one sees in comparing the two asymptotic expressions (17) and (18), the function $W(V)$ is nonmonotone. It has been demonstrated in Ref. 18 that in this case the dependence $V(f)$, determined by the energy balance equation

$$W(V) + 8\gamma V^2(1-V^2)^{-1/2} = 2\pi f V,$$

is hysteretic in the range $\epsilon\sqrt{\gamma} \lesssim f \lesssim \epsilon^2$, provided $\epsilon \gg \sqrt{\gamma}$ (a similar hysteresis has been revealed earlier[5] in the JJ theory). If the radiative hysteresis takes place, both branches of the CVC shown in Fig. 1 suffer additional splitting (see Fig. 5 in Ref. 18). The values $f \lesssim \epsilon^2$ crucial for the radiative hysteresis are much smaller than $f \sim \epsilon$ crucial in Fig. 1, so that two hystereses are well distinguishable.

In the case $M > 1$ dependence $W(V)$ is principally different: In the limit $1 - V^2 \rightarrow 0$ it attains the finite value $W_M = \epsilon^2 \sin^2(\pi/M)$. Note that $W_1 = 0$ in accordance with Eq. (18). A preliminary investigation[19] has demonstrated that, at least at $M=2$, the dependence $W(V)$ is monotone, i.e., the radiative hysteresis does not take place. Nevertheless, under the same assumption $\epsilon \gg \sqrt{\gamma}$, which is necessary for that hysteresis at $M=1$, in the case $M > 1$ the dependence $V(f)$ differs significantly from Eq. (15) in the range $\gamma \lesssim f \lesssim \epsilon^2$. This will result in appreciable alterations of CVC. In particular, at $f \sim \epsilon^2$, the conductivity becomes larger by a factor $\sim (\epsilon^2/\gamma)^2$.

All the results obtained are applicable, with evident modifications, to long JJ's. However, in the JJ theory the quantities f and $V(f)n(f)$ have the sense opposite to that in the CDW theory: f is the bias current density, and $V(f)n(f)$ is proportional to the dc voltage across the junction. [In the JJ theory, the kink (2) represents a magnetic-flux quantum.] In a real JJ, an important role may also belong to random inhomogeneities of the junction's inductance[20] and capacity. They are described by the additional terms $[\zeta_3(x)]_x \phi_x + \zeta_4(x)\phi_{tt}$ on the right-hand side of Eq. (3) with $M=1$, $\zeta_2 = 0$. Here $\zeta_{3,4}$ are random Gaussian functions subject to correlations analogous to (4), with ϵ replaced by some ϵ_3, ϵ_4. Formulas (6) to (16) are all directly applicable to this version of the SG model with the only modification being that in Eq. (8) the summation index k must take the values 1 and 3, where

$$U_3(x) \equiv -4(\epsilon_3/\epsilon)\mathrm{sech}^2 x.$$

However, in the limit $1 - V^2 \ll 1$ the energy emission rate differs drastically from (18):

$$W \approx \tfrac{2}{3}(4\epsilon_3^2 + \epsilon_4^2)(1-V^2)^{-3/2}.$$

This means that $1 - V^2 \sim f^{-2/3}$ at $f \rightarrow \infty$, instead of $1 - V^2 \sim f^{-2}$ ensuing from Eq. (15).

ac conductivity. If ac drive is applied to the system, a trapped kink oscillates in a vicinity of a local minimum ξ_0 of the potential (6) according to

$$\ddot{\xi} + \gamma\dot{\xi} + (\kappa/8)(\xi - \xi_0) = (\pi/4)F\exp(i\omega t), \tag{19}$$

where F and ω are the amplitude and frequency of the drive, and $\kappa \equiv U''(\xi_0)$ (in the overdamped model the term $\ddot{\xi}$ is absent). A solution to Eq. (19) is

$$\xi = \Xi \exp(i\omega t), \qquad \Xi = (\pi/4)(\kappa/8 - \omega^2 + i\gamma\omega)^{-1}F. \tag{20}$$

The ac current can be defined as follows: $j(t) \equiv J\exp(i\omega t) = n_0\langle\dot{\xi}\rangle$, where averaging is realized according to Eq. (4). The probability density for a distribution of the values κ is similar to (7):

$$p(\kappa) = 2(2\pi I_2)^{-1/2}\epsilon^{-1}\exp(-\kappa^2/2I_2\epsilon^2). \tag{21}$$

An expression for the ac conductivity $\rho(\omega) \equiv |J/F|$ can be obtained from Eqs. (20) and (21) in the two limit cases: For $\omega^2 \ll \epsilon$,

$$\rho(\omega) \approx (2\pi/I_2)^{1/2}(n_0\omega/\epsilon)\ln[\epsilon^2/\omega^2(\gamma^2+\omega^2)],$$

and for $\omega^2 \gg \epsilon$,

$$\rho(\omega) \approx (\pi/4)n_0\omega^{-1}.$$

In fact, the latter expression pertains to the homogeneous system. The full dependence $\rho(\omega)$ is shown schematically

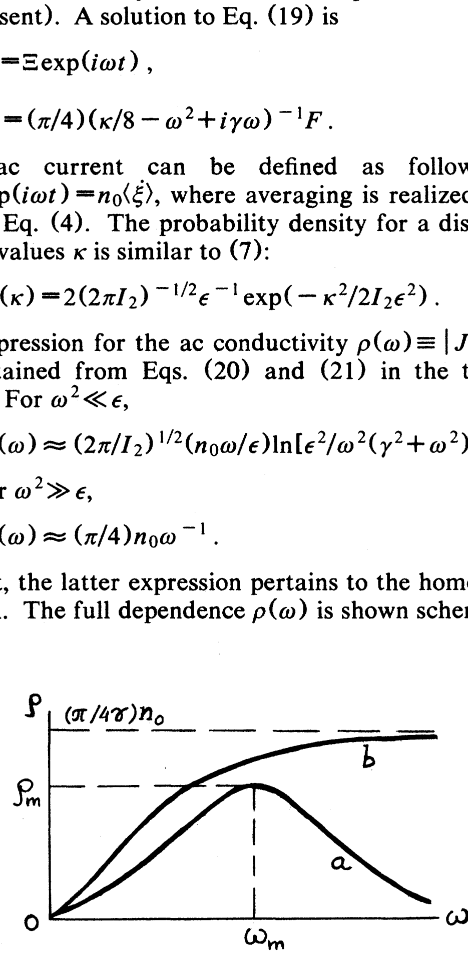

FIG. 2. The dynamical conductivity ρ vs the ac frequency ω: (a) the SG model and (b) the OSG model.

in Fig. 2 (curve a). A maximum value $\rho_m \sim n_0\epsilon^{-1/2}$ is attained at $\omega = \omega_m \sim \sqrt{\epsilon}$. In the overdamped model,

$$\rho(\omega) \approx (2\pi/I_2)^{1/2}(n_0\omega/\epsilon)\ln[\epsilon^2/(\omega\gamma)^2]$$

at $\omega \ll \epsilon$, and $\rho(\omega) \approx (\pi/4\gamma)n_0$ at $\omega\gamma \gg \epsilon$ (curve b in Fig. 2). It is easy to demonstrate that, in contrast with a model of the homogeneous overdamped system,[7] a contribution of the continuous spectrum $\rho(\omega)$ is negligible in both the SG and OSG cases.

In conclusion, investigation of the underlying model (1) with random x_n and $l \gg 1$ for $M=1$ (Ref. 18) and $M>1$ (Ref. 19) has yielded results which are qualitatively analogous to those reported in the present paper. So, the general consequences of the idea that kinks trapped initially by an effective disordered potential escape gradually with the increase of the dc drive, or oscillate under the action of the ac drive, seem insensitive to details of a model. It is also noteworthy that the same models (3) and (5) are applicable, with slight modifications, to a number of other objects, e.g, a disordered quasi-one-dimensional ferromagnet.[21]

I am indebted to I. V. Krive, A. A. Nepomnyashchy, A. S. Rozhavsky, and A. F. Volkov for valuable discussions.

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