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Integer charge transport in Josephson junctions

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We describe a macroscopic demonstration of quantized adiabatic transport coefficients using superconductors coupled by Josephson junctions. The transport coefficients relate charges and currents to changes in potential differences and loop fluxes. They have geometrical significance and quantized average related to first Chern characters.

Recently, one of us (J.A.)[1] discussed the adiabatic transport coefficients of coherent electron states with applications to mesoscopic normal electron systems. In multiply connected systems, the transport coefficients for the charge transported around one loop due to a slow enough change of the magnetic flux in a second loop have geometrical significance and have quantized averages related to first Chern characters[2] (i.e., the quantized integrals of curvature associated with eigenfunction spaces). In normal electron systems it is difficult to estimate how slow the changes must be to justify the adiabatic approximation. In the present work we consider a macroscopic implementation using superconductors coupled by Josephson junctions, where the adiabatic restriction might be expected to be more easily satisfied. In addition, the experimental feasibility can be estimated from recent experimental work on the macroscopic quantum description.[3]

We consider two classes of examples. In the first we take a simple circuit of two superconductors connected by two junctions, and study the transport of charge between the superconductors as the chemical potential between them is varied. For this network simple arguments lead to explicit results for the quantized averages. We find that *exactly* a unit charge is transferred between the superconductors each time the potential difference is raised by the charging energy q^2/C. This result is related to recent work on charging effects on the tunneling between small conductors:[4] our result is a fully coherent quantum version of this phenomenon. More surprisingly, we find that on averaging over the magnetic flux contained in the loop the charge is carried *entirely* by the stronger link.

The second class of networks we briefly consider involves more complex geometries, but now the quantized transport coefficients relate charges carried around loops to flux changes in other loops, analogous to the situation of Ref. 1.

Consider an array of superconductors coupled by Josephson junctions. A chemical potential μ_v is associated with the vth superconductor and a gauge potential a_e is associated with the eth junction with a_e the integral of the vector potential across the junction. We allow the vectors of chemical and vector potentials $\boldsymbol{\mu}$ and $\mathbf{a}$ to depend adiabatically on time. It is also useful to define loop fluxes Φ_l equal to the sum of the a_e around each loop.

The response of the system to $\boldsymbol{\mu}$ and $\mathbf{a}$ is in two parts: a "persistent response" that is determined by $\boldsymbol{\mu}$ and $\mathbf{a}$ (and the state of the system) alone and a "transport response" that is proportional to the rates $\dot{\mathbf{a}}$ and $\dot{\boldsymbol{\mu}}$. The coefficients of proportionality are the transport coefficients. They have geometric and topological significance, as is often the case for adiabatic transport.[2]

First we consider two finite superconductors coupled by two Josephson junctions as in Fig. 1. The system is described, quantum mechanically, by the Mathieu equation[5]

$$H(\mu,a_1,a_2)=\frac{q^2}{2C}N^2+N\mu+J_1\cos[\theta-(q/\hbar c)a_1] +J_2\cos[\theta-(q/\hbar c)a_2]\,. \qquad (1)$$

$N\equiv -i\partial_\theta$ is the number transfer operator, θ is the super-

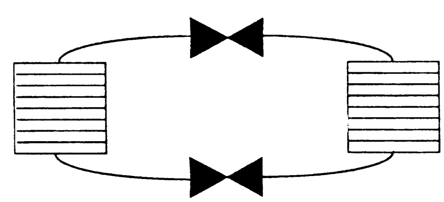

FIG. 1. Two superconductors, one at chemical potential 0 and one at μ, coupled by two Josephson junctions: J_1 and J_2. The vector potentials, a_1 and a_2, across the junctions are related to the flux Φ_1 through the figure by $\Phi_1=a_2-a_1$.

conducting phase difference, and $q=2e$ is the pair charge. C is the capacitance and $J_{1,2}$ the coupling of the junctions.

We regard $H(\mu,a_1,a_2)$ as a quantum Hamiltonian depending on the three parameters μ, a_1, and a_2 and we shall focus on the charge transported between the two superconductors as the parameters μ, a_1, and a_2 are varied adiabatically. It is convenient, for notational purposes, to treat the three parameters on equal footing so we let $\mathbf{x}$ be the triplet (μ,a_1,a_2). Similarly ∂_j, $j=0, 1, 2$ denotes the components of ∇_x.

Let $\psi(x)$ be a nondegenerate eigenfunction of $H(x)$ and $E(x)$ the associated energy. It was pointed out by Berry[6] that for a closed loop in parameter space, the adiabatic evolution gives rise to a geometric phase: the adiabatic holonomy. It is also known that in the case of the Hall effect the adiabatic holonomy is related to the Hall conductance. This view leads to topological interpretation of the (integer) quantum Hall effect, relating the integers to first Chern characters.[2] Our purpose here is to extend this approach to junctions. Thus, first we want to identify the transport coefficients related to the adiabatic holonomy and second to determine the Chern characters.

The answer to the first question comes from focusing on the right observables. The observables $\nabla_x H$ turn out to have transport coefficients related to the adiabatic holonomy, as we shall presently see. For the case at hand

$$\nabla H=\left[N,\frac{1}{c}I_1\frac{1}{c}I_2\right],$$

where $I_{1,2}=(qJ_{1,2}/\hbar)\sin[\theta-(q/\hbar c)a_{1,2}]$ are the Josephson currents through the junctions.

An easy computation from the time-dependent Schrödinger equation gives

$$\begin{aligned}\langle\psi|(\partial_j H)|\psi\rangle&=\partial_j E+\sum_k \dot{x}_k(t)\omega_{kj}(x)\,,\\ \omega_{kj}(x)&\equiv i\hbar(\langle\partial_k\psi|\partial_j\psi\rangle-\langle\partial_j\psi|\partial_k\psi\rangle)\,.\end{aligned}\tag{2}$$

This is the basic equation of the theory of adiabatic response. It may be viewed as a generalization of the Feynman-Hellman theorem to the adiabatic, time-dependent Hamiltonian. The term $(\partial_j E)$ is the persistent response in the sense that it is the response when $\dot{\mathbf{x}}\equiv 0$; $\omega_{kj}(x)$ are the *adiabatic curvatures* and are identified as the transport coefficients since the response is proportional to their product with the rate of change of the parameters $\dot{\mathbf{x}}(t)$. Note that $\omega_{kj}(x)$ is, by Eq. (2), an antisymmetric matrix. For the case at hand $\omega_{12}(x)$ has the physical significance of conductance for it relates currents to voltages, whereas $\omega_{01}(x)$ and $\omega_{02}(x)$ relate charges to voltages (or equivalently, currents to voltage rates).

The three operators $\nabla_x H$ are related: the Heisenberg equation for n is the equation for charge conservation

$$q\dot{N}=q(i/\hbar)[H(x),N]=I_1+I_2\,.\tag{3}$$

As well shall show below, this simple fact, together with Eq. (2), determines the Chern characters and so the average transport coefficients. This concludes the geometric part of the problem.

The topological aspect comes from combining the geometric aspect, together with the fact that parameter space can be viewed as a torus. This is clearly the case for the $a_{1,2}$ since $H(x)$ is periodic in $a_{1,2}$ with period of the quantum flux $2\pi(\hbar c/q)$. The Hamiltonian $H(x)$ is, of course, not periodic in μ (it is linear). The basic period in μ is determined by

$$e^{-i\theta}H(\mu,a_1,a_2)e^{i\theta}=H(\mu+q^2/C,a_1,a_2)+(q^2/2C+\mu)\,,\tag{4}$$

q^2/C can be viewed as a period because the last term on the right-hand side of Eq. (4) is a c number. It follows that

$$E(\mu,a_1,a_2)=E(\mu+q^2/C,a_1,a_2)+(q^2/2C)+\mu\,.\tag{5}$$

This structure makes the ω_{kj} periodic up to a complete derivative.

It is natural to consider paths in x space that connect points that differ by a period. In particular, for an adiabatic path where μ increases by a period. Equations (2) and (5) combine to give

$$\langle\psi|N|\psi\rangle(\mu+q^2/C,a_1,a_2)-\langle\psi|N|\psi\rangle(\mu,a_1,a_2)=1\,.\tag{6}$$

So, a single quasiparticle is transported between the two superconductors. Note that the individual terms, $\langle\psi|N|\psi\rangle(x)$, on the left-hand side of Eq. (6), need not be integers, only the difference is.

Equation (6) can now be used to relate $\omega_{01}(x)$ and $\omega_{02}(x)$. This is because the transported charge can be computed also by integrating Eq. (3) over time, and then using Eq. (2). One finds

$$\int_\mu^{\mu+(q^2/C)}d\mu'[\omega_{01}(x)+\omega_{02}(x)]=q/c\tag{7}$$

and we have used the fact $\partial_1 E+\partial_2 E=0$ since $E(x)$ is only a function of a_1-a_2, the flux in the loop.

Equation (7) is, of course, not sufficient in order to determine $\omega_{01}(x)$ and $\omega_{02}(x)$. However, as we now show, combined with a topological deformation argument, it determines the periods $\langle\omega_{01}\rangle$ and $\langle\omega_{02}\rangle$.

The periods $\langle\omega_{jk}\rangle$ are defined by

$$\langle\omega_{jk}\rangle\equiv\int dx_j dx_k\,\omega_{jk}(x)\,,\tag{8}$$

where the integration is over a single period, and so $\langle\omega_{jk}\rangle$ are closely related to averages. The periods are first Chern numbers and are quantized to be integer multiples of $2\pi\hbar$. Notice that $\langle\omega_{jk}\rangle$ is a function of x_l with j, k, and l distinct.

It is a property of the Mathieu equation (1), that for $J_1\neq J_2$, no eigenvalues cross.[7] [The eigenvalues are considered as functions of (μ,a_1,a_2)]. The $\langle\omega_{jk}\rangle$, being first Chern characters, can only jump at crossings. Since there are none, we can deform the subdominant J to zero without affecting the $\langle\omega_{jk}\rangle$. Combining this with Eq. (7) gives

$$\langle\omega_{01}\rangle(a_2)=\begin{cases}0 & \text{if } J_1<J_2\,,\\ 2\pi\hbar & \text{if } J_1>J_2\,.\end{cases}\tag{9}$$

This says that as μ increases by a period, (q^2/C), the single charge transported between the two superconductors, goes, on the average (over a_1), through the dominant junction. This is remarkable for the $(0,2\pi\hbar)$ dichotomy hold even if J_1 and J_2 are very close.

It is clear that an analogous result to Eq. (9) holds for $\langle\omega_{02}\rangle$. By a different, but related, argument, one shows that $\langle\omega_{12}\rangle=0$ so that all the Chern characters for this equation can be determined without explicitly solving the eigenvalue problem (1). The method we have used to derive Eq. (9) is related to a Galilean invariance argument, used in Ref. 8 to determine the Chern numbers for commensurate and incommensurate problems. We may extend the argument to determine all the Chern numbers associated to two superconductors coupled by an arbitrary number of junctions. Similar results hold for the tight-binding model associated with Fig. 1.

For more general networks of V vertices connected by E edges such as Fig. 2 the Hamiltonian is the natural generalization of (1),

$$H(\mu,\mathbf{a})=\tfrac{1}{2}(\mathbf{N},C^{-1},\mathbf{N})+(\mu,\mathbf{N}) + \sum_{R=1}^{E} J_i\cos[\theta_e-(e/\hbar c)a_e]\,, \qquad (10)$$

with $\mathbf{N}$ the vector of number operators for the vertices, C the $V\times V$ capacitance matrix with $\infty>C>0$ and $(\mu,\mathbf{N})=\sum_{i=1}^{V}\mu_i N_i$. Also, θ_e is the phase difference $\theta_v-\theta_u$ between vertices connected by the edge e. The fluxes in each loop are defined by $\Phi_l=\sum_{\text{loop } l}a_e$ and as above are the important magnetic variables. The Hamiltonian is periodic in the fluxes a_e with period of one flux quantum. The adiabatic curvatures ω_{12} for the loops associated to the fluxes Φ_1 and Φ_2, are transport coefficients relating the loop current I in loop 1 to flux changes $\dot{\Phi}_2$ in loop 2. The quantized values for the periods $\langle\omega_{12}\rangle=\int\omega_{12}d\Phi_1 d\Phi_2$ correspond to quantized values for the charge carried around loop 1 due to a change of flux by one flux quantum in loop 2, when averaged over a period of the flux in loop 1:

$$\overline{Q_1}=\int\overline{I_1 dt}=c\int\overline{\omega_{12}d\Phi_2}=\text{integer}\times e\,, \qquad (11)$$

where the bar corresponds to an average over Φ_1. The integer in Eq. (11) must be calculated for each network, and set of parameters (e.g., Φ_3 in Fig. 2). If all the μ_i are equal, the Hamiltonian (10) is real and the integers are zero. This case corresponds to a charge-conjugation symmetry. It is necessary to break this symmetry by imposing different μ_i to get a nonzero integer.

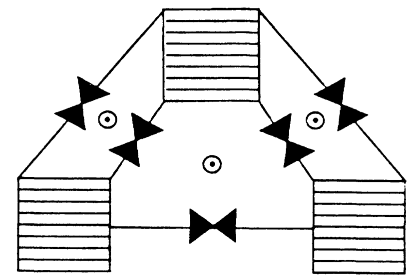

FIG. 2. More complicated network of three superconductors, five junctions and three fluxes.

We hope that these effects may be observed experimentally. Here we briefly discuss requirements for them to be observable in principle, without inquiring about the practical feasibility, although we are encouraged by recent experiments observing single electron charging phenomena.[4] The theory requires that the rate of change Ω in the variables is *slow* enough so that the adiabatic approximation holds, but *fast* enough that the macroscopic quantum description (1) or (10) is a good description (e.g., incoherence effects from coupling to microscopic degrees of freedom are small). The scale of the energy gaps in (1) or (10) limits the maximum rate of change and is of order the "Josephson plasma frequency" $\omega_p\sim\sqrt{JC/e^2}$. In experiments by Devoret, Martinis, and Clarke[3] $(\omega_p/2\pi)$ was of order 5×10^9 Hz. We can bound the incoherence rate $\Gamma=\tau^{-1}$ with τ the quantum coherence time by the lowest observed tunneling rate out of the lowest level in a current biased junction, which is well described by the macroscopic Hamiltonian. This gives $\Gamma\lesssim2\times10^3$ Hz leaving a large range 10^9 Hz $\gtrsim\Omega\gtrsim10^3$ Hz for the approximations to be good.

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