Metastable states of Ising spin glasses and random ferromagnets

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Using a recursive method, we have calculated and compared the number of single-spin-flip metastable states for one-dimensional (1D) (chain) and 2D (strip) Ising spin glasses and random ferromagnets at zero temperature. For the 1D case, an extensive study was made of the distribution of the metastable states $N_s(\varepsilon,m)$ with regard to their magnetization m per spin and energy ε per spin, and the effect of an applied magnetic field on this distribution was investigated. The distributions of metastable states in energy and in magnetization for the 2D systems (strips of width up to 6 and length 10^4 spins) are qualitatively similar to those of the chains. Our results suggest that the ground state of the spin glass is extremely sensitive to changes in the magnetic field. The number of low-energy metastable states depends on the energy above the ground-state energy ε_0 as $(1/N)\overline{\ln N_s(\varepsilon)}\sim(\varepsilon-\varepsilon_0)^{1/\lambda}$. For $d=1$, $1/\lambda$ is known exactly. For $d=2$ random ferromagnets, the low-energy metastable states have different character for wide and narrow strips: In wide strips, they arise primarily from flipping two-spin clusters relative to the ground state, while for narrow strips (open at both ends) they are due to flipping all spins to the left (or right) of a line. The value of the exponent $1/\lambda$ predicted from this picture agrees with the numerical value at each width. For $d=2$ spin glasses, our results suggest that $1/\lambda$ is greater than 1/2.

I. INTRODUCTION

The dynamical properties of spin glasses and random ferromagnets are governed by the existence of exponentially large numbers of local energy minima. The number and distribution of these metastable states have been studied for the infinite-range Ising spin glass by Tanaka and Edwards[1] and Bray and Moore.[2] Bray and Moore also considered expansions about the infinite-range limit in powers of the inverse of the coordination number z; near the most probable energy, their $1/z$ expansion gives very good results for both $d=1$ and 2 despite the small values of z. Short-range Ising models have been studied extensively in $d=1$ at zero temperature[3-7] by both analytical and numerical methods, but only a few results for two- and three-dimensional Ising spin glasses and random ferromagnets[7,8] are available.

We use a recursive method (described in Sec. II) to calculate numerically the logarithm of the number $N_s(\varepsilon,m)$ of single-spin-flip metastable states at arbitrary energy ε and magnetization m. In Sec. III the method is applied to Ising spin glass (SG) and random ferromagnet (RFM) chains, and the results are compared with known analytical solutions. Our numerical results agree well with previous analytical results where the two overlap, and they provide additional information on the distribution of metastable states in the energy-magnetization plane. The effect of an external magnetic field on the metastable states of the chains is also investigated. The recursive method is easily generalized to deal with strips of spins rather than chains, and is applied to the 2D Ising SG and RFM in Sec. IV; the results are discussed within the context of the zero-temperature scaling argument of Ettelaie and Moore.[9]

II. RECURSIVE METHOD FOR CALCULATION OF THE NUMBER OF METASTABLE STATES

We outline a recursive algorithm which enables the calculation of the number of metastable states at zero temperature of an Ising spin glass (or random ferromagnet) chain. For a chain of length L, the 2×2 matrix $N_L(S_L,S_{L-1})$ gives the number of metastable states of the chain for the four configurations of the last two spins, S_L and S_{L-1}. The number of metastable states for a chain of length $L+1$ is:

$$N_{L+1}(S_{L+1},S_L)=\sum_{S_{L-1}} N_L(S_L,S_{L-1})\Theta(H_L S_L)\ , \quad (1)$$

where

$$\Theta(H_L S_L)=\begin{cases}1 & \text{if } H_L S_L>0\ ,\\ 0 & \text{otherwise}\end{cases}$$

and $H_L=J_{L-1,L}S_{L-1}+J_{L,L+1}S_{L+1}$ is the local field at spin S_L. At each stage of iteration, the total number of

metastable states is given by the sum of all the elements of the matrix $N_L(S_L,S_{L-1})$. In order to extract the quantities of interest, the matrix $N_L(S_L,S_{L-1})$ is normalized at each step so that the sum of the elements is unity. This normalization factor is then averaged over all steps to give the number of metastable states. If a_i is the normalization constant at each step i, the logarithm of the number of metastable states per spin, averaged over the bond distribution, is

$$\frac{1}{N}\overline{\ln N_s}=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\ln a_i \ . \tag{2}$$

The above algorithm can be generalized to give the number $N_s(\varepsilon,m)$ of metastable states with energy per spin ε and magnetization per spin m by introduction of the weight factor $\exp(\beta E+\beta_1 M)$; we describe the generalization for the energy E—the magnetization M is treated in the same manner. The recursion formula becomes

$$N_{L+1}(S_{L+1},S_L;\beta)=\sum_{S_{L-1}} N_L(S_L,S_{L-1};\beta)\Theta(H_L S_L)e^{\beta E_L} , \tag{3}$$

where $E_L=\frac{1}{2}H_L S_L$. Taking the derivative with respect to β gives

$$D_{L+1}=\sum_{S_{L-1}}(D_L+E_L N_L)\Theta(H_L S_L)e^{\beta E_L} , \tag{4}$$

where the derivative matrix is defined by

$$D_L(S_L,S_{L-1};\beta)\equiv\frac{\partial}{\partial\beta}N_L(S_L,S_{L-1};\beta) \ .$$

The normalization of the matrices after each iteration is given by

$$N'_{L+1}=\frac{N_{L+1}}{a} , \tag{5a}$$

$$D'_{L+1}=\frac{D_{L+1}}{a}-\frac{b}{a^2}N_{L+1} , \tag{5b}$$

where a and b are the sums of the elements of N_{L+1} and D_{L+1}, respectively.

In general, the number of metastable states for a given value of β and given length L of the chain is a sum over all energies

$$N_L(\beta)=\sum_E N_L(E)e^{\beta E} . \tag{6}$$

The summand will be sharply peaked at some energy E' so that $N_L(\beta)$ can be approximated by

$$N_L(\beta)\sim N_L(E')e^{\beta E'} . \tag{7}$$

Thus we have

$$\frac{1}{N}\overline{\ln N_s(\varepsilon)}+\beta\varepsilon=\overline{\ln a} \tag{8a}$$

and

$$\varepsilon=\overline{b/a} \tag{8b}$$

from which

$$\frac{1}{N}\overline{\ln N_s(\varepsilon)}=\overline{\ln a}-\beta\overline{b/a} \ . \tag{8c}$$

The number of metastable states at energy ε can be calculated by appropriate averaging of the normalization factor at each step. In practice, the first 2000 or so values must be excluded from the averaging, since the initial values of the elements of the matrix N_L are chosen arbitrarily; the recursion relation is iterated at least a further 10^4 times to give the averaged quantities.

III. RESULTS FOR ISING SG AND RFM CHAINS

Each metastable state is characterized by its energy ε per spin and its magnetization m per spin. For each choice of weighting parameters (β,β_1) our algorithm yields the number of metastable states $N_s(\varepsilon,m)$ with energy and magnetization (ε,m). A plot of these metastable states in the m-ε plane indicates the allowed values of ε and m for the metastable states. Figure 1 shows the distribution of metastable states in the m-ε plane for the SG with bonds chosen from a Gaussian distribution. There are no metastable states with magnetization $|m|$ above some maximum value $m_{\max}$, and none with energies below the ground-state energy ε_0 or above the maximum energy $\varepsilon_{\max}$. For states with nonzero magnetization, the range of allowed energies is more constrained. The numerical results (a chain of length 10^5 sites was used) for the five quantities ε_0, $\varepsilon_{\max}$, $m_{\max}$, $\varepsilon_{m.p.}$ (the energy at which the density of metastable states is a maximum) and the number of metastable states at the energy $\varepsilon_{m.p.}$ were $\varepsilon_0=-0.7984$, $\varepsilon_{\max}=-0.5766$, $m_{\max}=0.4416$, $\varepsilon_{m.p.}=-0.687$, and $(1/N)\overline{\ln N_s(\varepsilon_{m.p.})}=0.231$; these values are in good agreement with the exact analytical values,[3-6] with errors at most a few parts in 10^3. Bray and Moore[2] have obtained the first two terms in the expansion (in powers of $1/z$, where z is the coordination number of the hypercubic lattice) of the function $(1/N)\overline{\ln N_s(E)}$ for short-range Ising spin glasses. For $z=2$, their expansion gives $(1/N)\overline{\ln N_s(\varepsilon_{m.p.})}=0.233$

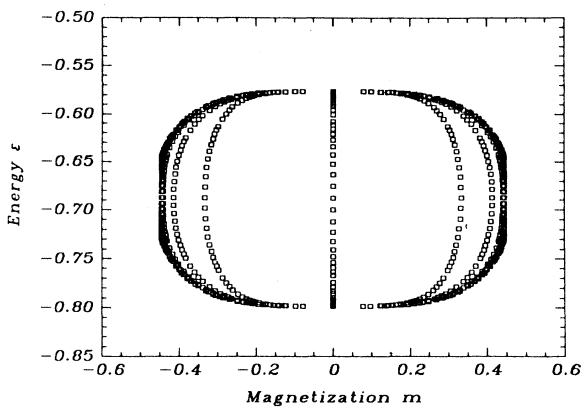

FIG. 1. Distribution of metastable states in the m-ε plane for the 1D Gaussian spin glass in zero applied field. Each point represents a single pair of parameter values (β,β_1).

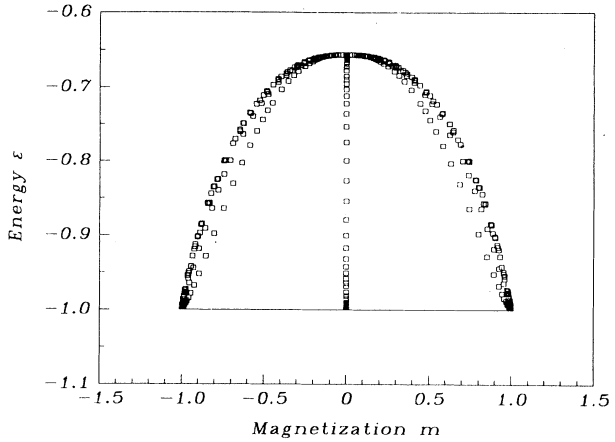

FIG. 2. Distribution of metastable states in the m-ε plane for the 1D random ferromagnet, with bond distribution $U(0,2)$ (bonds uniformly distributed in the range 0 to 2), in zero field.

with $\varepsilon_{m.p.} = -0.697$, in good agreement with the exact values despite the small value of z; the prediction for $\varepsilon_{\max}$ (-0.414) is not as good, but still surprisingly close.

The surface defined by the function $(1/N)\ln N_s(\varepsilon,m)$ has the shape of a truncated semiellipsoid for the spin glass, with the truncations at $m=\pm m_{\max}$. A cut through this surface along $m=0$ reproduces the results of Ettelaie and Moore[5] for the logarithm of the number of metastable states as a function of energy. A similar cut along $\varepsilon=\varepsilon_{m.p.}$ yields a curve of the number of metastable states at a given magnetization m, a result derived analytically by Derrida and Gardner[6] for a symmetric bond distribution and zero field.

Figure 2 shows the distribution of metastable states (in the m-ε plane) for the random ferromagnet with bonds chosen from the uniform distribution $U(0,2)$. The low-energy bound ε_0 is independent of the magnetization since the energy per spin to flip a single, large domain is negligible in the thermodynamic limit.

The recursive method allows one to treat arbitrary bond and field distributions and may, for example, be used to compare the SG with the RFM both in the presence and absence of an applied magnetic field. Figures 3 and 4 show the allowed values of energy ε and magnetization m for selected values of the field H. Note that, for the SG, the discontinuity in the magnetization remains for nonzero field. Although the applied field breaks the time-reversal symmetry $S_i \rightarrow -S_i$, the distribution of metastable states in the m-ε plane retains some symmetry due to the symmetry in the bond distribution; for a symmetric bond distribution $P(-J)=P(J)$, the maximum and minimum magnetizations and energies for each field are located symmetrically about their most probable values. The dependence of the most probable number of metastable states on the applied field is shown in Fig. 5 for various bond distributions.

Figures 6(a)–6(c) show the field dependence of the maximum, minimum and most probable values of the energy ε and magnetization m. The dependence of the magnetizations on field is simple: the magnetizations m increase linearly with field for small fields, then saturate

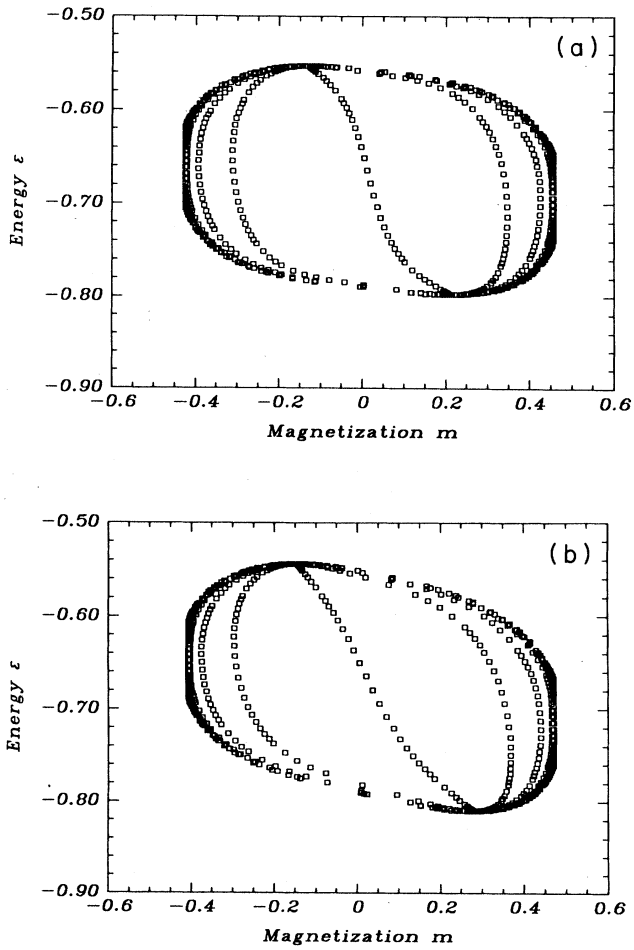

FIG. 3. Distribution of metastable states in the m-ε plane for the 1D Gaussian spin glass in applied fields $H=0.05$ (top) and $H=0.1$ (bottom).

at $m=1$ for high fields, corresponding to increasing alignment of the spins with the applied field. For the SG with uniform distribution $U(-1,1)$ of bonds, the magnetization saturates at $H=2.0$ since this field is sufficient to overcome the strongest of negative local fields; for the Gaussian distribution, arbitrarily large bonds can occur and the magnetization does not saturate. For both spin-glass distributions, at high fields the energy ε becomes linear with the field and equal to the field energy. There are no metastable states of negative magnetization for fields greater than $H \simeq 0.79$ for the Gaussian SG, $H \simeq 0.54$ for the SG with bonds chosen from $U(-1,1)$ and $H \simeq 1.1$ for the RFM with bond distribution $U(0,2)$. A notable feature of all three energy vs field curves in Fig. 6 is the initial "spreading" of the maximum and minimum energies for small fields; that is, the difference between the maximum and minimum allowed energies initially increases (widens) for small fields before decreasing (narrowing) again at higher fields. This broadening is explained by the following picture of the effect of an applied field.

Application of a small magnetic field has several

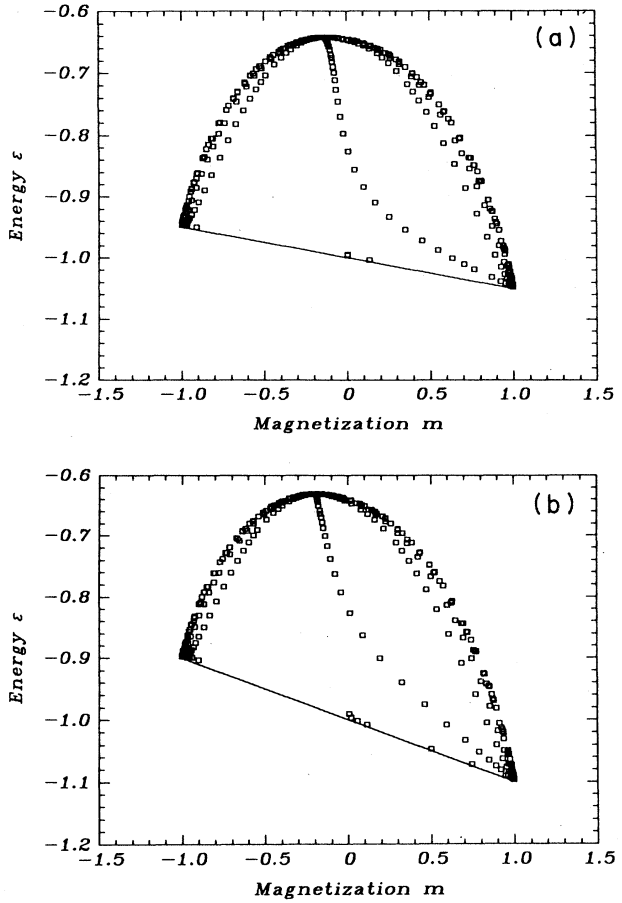

FIG. 4. Distribution of metastable states in the m-ε plane for the 1D random ferromagnet, with bond distribution $U(0,2)$, in applied fields $H=0.05$ (top) and $H=0.1$ (bottom).

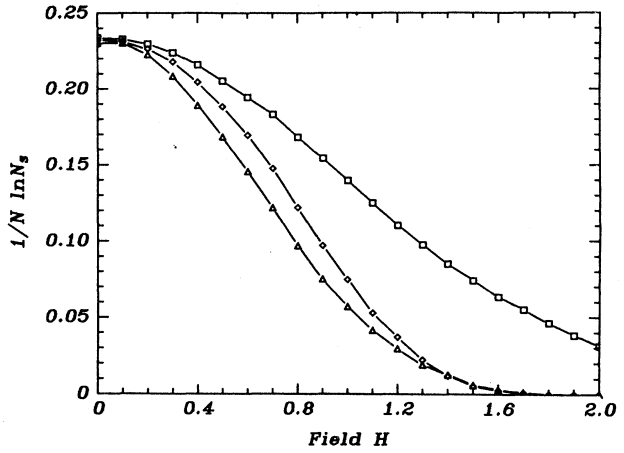

FIG. 5. Logarithm of the most probable number of metastable states vs the applied field for the 1D spin glass with Gaussian bonds (□), the 1D spin glass with bond distribution $U(-1,1)$ (△) and the 1D random ferromagnet with bond distribution $U(0,2)$ (◇).

effects. (i) It changes the energy of states, proportionally to their magnetization. A small applied field "shears" the distribution of metastable states in the m-ε plane (as shown in Figs. 3 and 4), and consequently broadens the energy range. (ii) It affects the stability of states; states which were metastable in zero field may no longer be stable in the new field and states which were previously unstable may now be metastable. One expects that as the field is increased, states with the most negative magnetization will become unstable while new states with positive magnetization will appear. (iii) It causes single spins to flip; although the spin configuration is changed, the state is not, for the flip reverses on decreasing the field. On the other hand, the reversal of one spin may set off a cascade of flips; such processes are included in effect (ii), for the changes are generally not reversible. We expect effect (i) to dominate at low fields. As a test, for the Gaussian spin glass we added the field energies to the zero-field data for fields of $H=0.05$ and 0.10; reasonable agreement was found with the actual values, although it is certain that effects (ii) and (iii) above must be taken into account for a full description. For higher fields, this picture breaks down and the other effects cannot be neglected. The energies of the spin-glass states will cross as the field is changed and the ground state at one value of the field differs appreciably from the ground state at nearby values. The implication is that the ground state of the SG chain is sensitive to arbitrarily small changes in the applied field; that is, small changes in H give rise to large-scale rearrangements of the spin configurations. This result, obtained here for the chain at $T=0$, is expected to carry over to higher dimensions. The spin glass has been previously argued to be extremely sensitive to changes in the temperature[10,11] and in the bonds.[12]

The energy distribution near the ground state can be described by an exponent $1/\lambda$ defined by

$$\frac{1}{N}\overline{\ln N_s(\varepsilon)} \sim (\varepsilon-\varepsilon_0)^{1/\lambda}, \quad \varepsilon \to \varepsilon_0 . \tag{9}$$

The value of $1/\lambda$ can be obtained directly from a log-log plot or indirectly by plotting $\ln\alpha(\varepsilon)$ vs $\ln\beta$ where

$$\frac{1}{N}\overline{\ln N_s(\varepsilon)} = \alpha(\varepsilon)\ln 2 . \tag{10}$$

For the SG chain of length 10^6 sites with Gaussian bonds we obtained $1/\lambda=0.495\pm0.005$ while for the RFM chain with uniform distribution of bonds $1/\lambda=0.491\pm0.001$. Both values agree reasonably with the analytical value of $1/\lambda=\frac{1}{2}$,[5] but our method of determining the exponent may have a small systematic error.

In the case of $d=1$ chains, the distribution of metastable states is easily obtained analytically using standard methods of statistical mechanics.[13] This is due to the fact that for any continuous distribution of bonds, the chain is divided into blocks of spins separated by weak links.[3] Each block can be up or down and hence acts like a block spin. All configurations of these block spins correspond to metastable states. The calculation of the distribution of these states in energy is equivalent to calculating the entropy per site as a function of the energy per site of a chain with an effective distribution of bonds given by the distribution of weak links. In this way, the SG or RFM

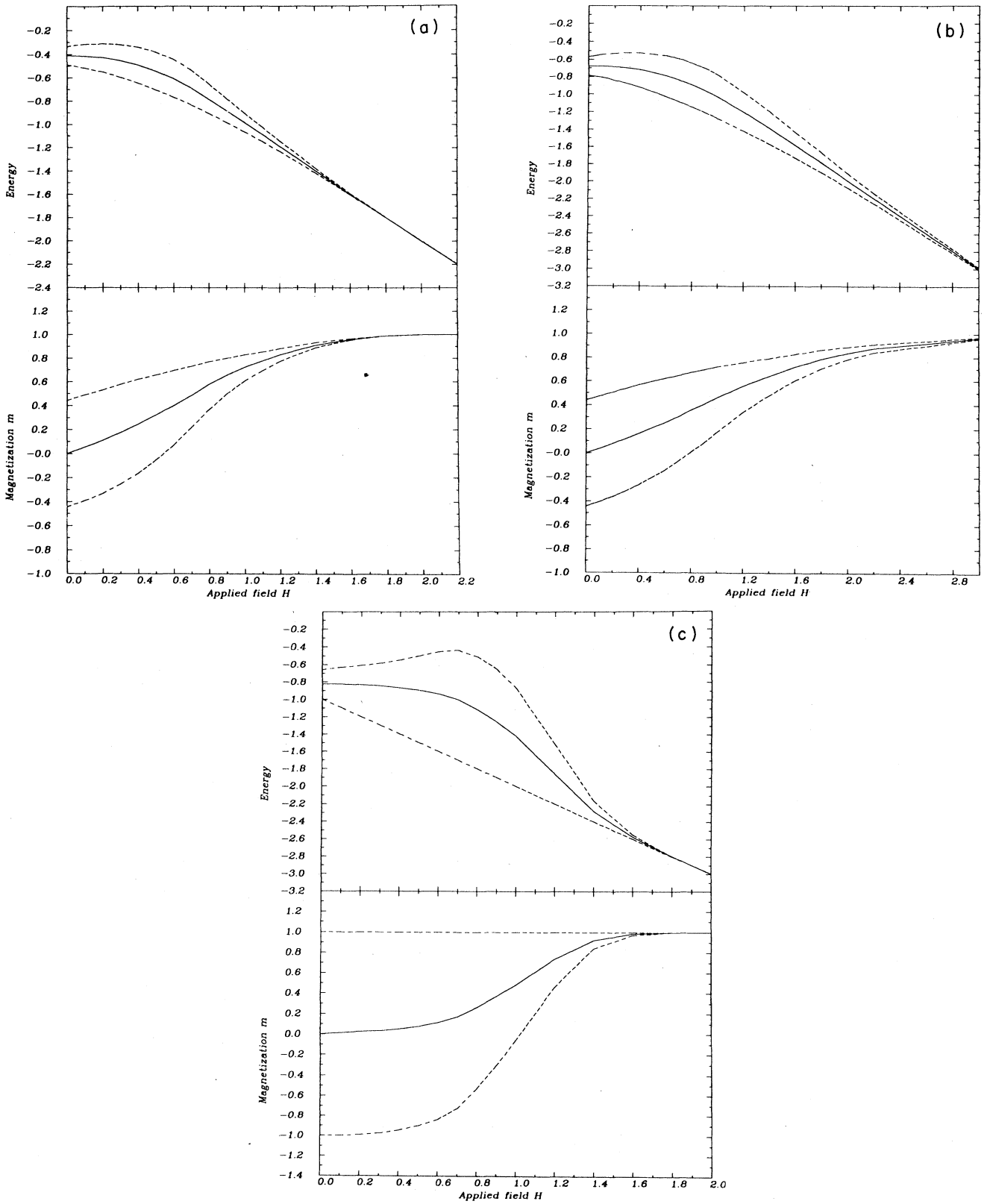

FIG. 6. Upper graphs: the maximum, minimum, and most probable values of the energy (ε_{max}, ε_0, $\varepsilon_{m.p.}$ in the text) as functions of the applied field H. Lower graphs: the maximum, minimum, and most probable values of the magnetization m. The solid (dashed) lines correspond to the most probable (maximum and minimum) values. The systems are chains: (a) $U(-1,1)$ spin glass, (b) Gaussian spin glass, and (c) $U(0,2)$ random ferromagnet.

can be viewed as a system of weakly interacting two-level systems. The energy $\tilde{\varepsilon}=\varepsilon-\varepsilon_0$ per site above the ground-state energy ε_0 is

$$\tilde{\varepsilon}=-\frac{1}{N}\sum_i (|J_i|\tanh\beta|J_i|-|J_i|)$$
$$=\frac{1}{N}\sum_i 2|J_i|\frac{e^{-2\beta|J_i|}}{1+e^{-2\beta|J_i|}} \tag{11}$$

and the entropy s per site is

$$s=\frac{1}{N}\sum_i [\ln\cosh(\beta J_i)-\beta J_i\tanh(\beta J_i)]$$
$$=\beta\tilde{\varepsilon}+\frac{1}{N}\sum_i \ln(1+e^{-2\beta|J_i|}) , \tag{12}$$

where the summation is over the $N/3$ weak bonds J_i, and β is a Lagrange multiplier used to fix the energy. These expressions were also obtained by Ettelaie and Moore,[5,9] by a steepest-descent method.

As $\beta\to\infty$, $\varepsilon\to\varepsilon_0$ and the energy $\tilde{\varepsilon}$ near the ground-state energy behaves as

$$\tilde{\varepsilon}\sim\int_{-\infty}^{+\infty} P_w(J)|J|e^{-2\beta|J|}dJ , \tag{13}$$

where the summation has been replaced by an integration over the probability distribution $P_w(J)$ of the weak bonds. The exponent $1/\lambda$ is determined by the form of the weak-link distribution $P_w(J)$ near $J=0$. If the distribution behaves as $|J|^p$ near $J=0$, the energy $\tilde{\varepsilon}$ behaves as

$$\tilde{\varepsilon}\sim\int dJ|J|^p|J|e^{-2\beta|J|}\sim\frac{1}{\beta^{p+2}} \tag{14}$$

from which the behavior of the entropy s is

$$s\sim\beta\tilde{\varepsilon}\sim\left[\frac{1}{\beta}\right]^{1+p}\sim\tilde{\varepsilon}^{(1+p)/(2+p)} , \tag{15}$$

and the exponent is

$$1/\lambda=(1+p)/(2+p) . \tag{16}$$

The above expression predicts the exponent $\frac{1}{2}$ if the bond distribution $P(J)$ does not vanish as $J\to 0$, in agreement with the numerical results for the SG and RFM chains studied above. As a further test of Eq. (16), we have studied the case $p=1$ using a triangular distribution $P_w(J)$ generated by adding two random numbers each distributed as $U(0,1)$, as well as the case $p=\infty$, for which we used a uniform distribution $U(1,2)$. The numerical results for the energy exponent $1/\lambda$ were 0.66 ± 0.01 for the case $p=1$ and 1.0 ± 0.1 for $p=\infty$, in agreement with Eq. (16).

The argument leading to Eq. (16) was made for $d=1$, but the result can be applied to higher d if one can view the system as composed of weakly interacting subsystems; a knowledge of the analogue of the weak-link distribution is required to predict the exponent. In the next section, we find the effective weak-link distribution for the RFM in two dimensions; it provides a full understanding of the RFM distribution of states at low energies.

IV. RESULTS FOR ISING SG AND RFM STRIPS

For both the SG and the RFM the algorithm was applied to strips of length $L=10^4$ and width up to $W=6$ spins. The first 2000 iterations along the length were discarded as before, and helical boundary conditions were used in the direction of the width. Although a full investigation of the distribution of metastable states in the m-ε plane was prohibited by the large amount of computer time required, selected regions were probed to give the pertinent data. For the SG strips, the distributions of metastable states with respect to magnetization are similar to the 1D distributions: there are exponentially large numbers of metastable states up to a cutoff magnetization $m_{\max}$ above which there are no metastable states. From the data for strips of width 2 to 5 inclusive [shown in Fig. 7(a)], the values of $m_{\max}$ extrapolate for infinite width to $m_{\max}=0.46$ for the Gaussian SG and $m_{\max}=0.49$ for the $U(-1,1)$ SG; both values are only slightly larger than the 1D result $m_{\max}=0.44$. The persistence of the discontinuity in the magnetization to the two-dimensional case is remarkable. From Fig. 7(b), the exponent for the number of metastable states at the maximum magnetization is approximately $(1/N)\ln N_s(m_{\max})=0.03$, about half the value 0.063 obtained by Derrida and Gardner[6] for the one-dimensional case.

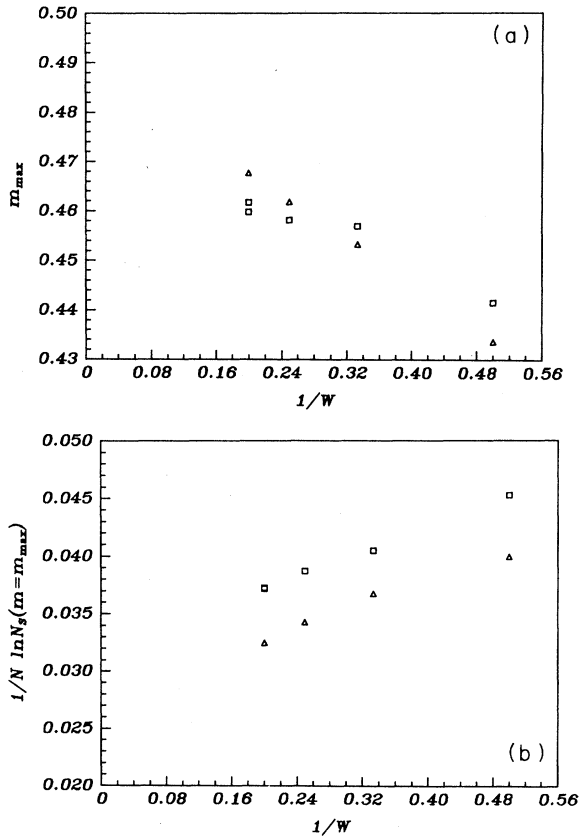

FIG. 7. (a) Maximum magnetization $m_{\max}$ as function of the inverse width $1/W$ for two-dimensional Gaussian (□) and $U(-1,1)$ (△) spin glasses. (b) Similar plot of the logarithm of the number of metastable states at $m_{\max}$.

The energy distribution of metastable states was investigated, with special attention to the region near the ground state. The energy distributions for the two-dimensional systems are qualitatively similar to those for the chains, with one notable difference: while the distributions for the chains are symmetric about their maxima, this is no longer the case for the 2D systems. The asymmetry in $\ln N_s(E)$ is very slight for both SG systems [Gaussian bond and uniform $U(-1,1)$ bond], with states of low energy being slightly favored over those of high energy, whereas the RFM exhibits a marked asymmetry with many more high-energy states than low-energy states. That the SG and RFM differ in this respect is not unexpected, since they are no longer related by a simple gauge transformation as in one dimension. Figure 8 shows results for two systems of width 6 and length 10^4 for the SG with Gaussian bonds and the RFM with bonds chosen from $U(0,2)$.

The ground-state energy of the 2D Gaussian SG is estimated to be -1.308 ± 0.004, in agreement with previous estimates.[14] The ground-state energy of the 2D SG with bonds chosen from $U(-1,1)$ is -0.791 ± 0.001. For both SG's the number of metastable states is $(1/N)\overline{\ln N_s}=0.215$; for the RFM, the number is slightly less: $(1/N)\overline{\ln N_s}=0.198$. In contrast, Cieplak and Gawron[7] found the values $1/4.5413=0.2202$ and $1/4.4307=0.2257$ for the Gaussian SG and the corresponding ferromagnetic system (with Gaussian couplings $|J|$), respectively. The most probable and the maximum energies are $\varepsilon_{m.p.}=-0.992$ and $\varepsilon_{\max}=-0.669$ for the Gaussian SG, and $\varepsilon_{m.p.}=-0.585$ and $\varepsilon_{\max}=-0.376$ for the $U(-1,1)$ SG. These results for the Gaussian SG are in remarkable agreement with the predictions of the $1/z$ expansion of Bray and Moore[2] near $\varepsilon=\varepsilon_{m.p.}$. The result of the $1/z$ expansion with $z=4$ is plotted as the solid line in Fig. 8(a); predicted values for various quantities are $\varepsilon_{m.p.}=-0.995$, $(1/N)\ln N_s(\varepsilon_{m.p.})=0.216$, and $\varepsilon_{\max}=-0.579$. As in the 1D case, the $1/z$ expansion works very well, and best, near $\varepsilon=\varepsilon_{m.p.}$.

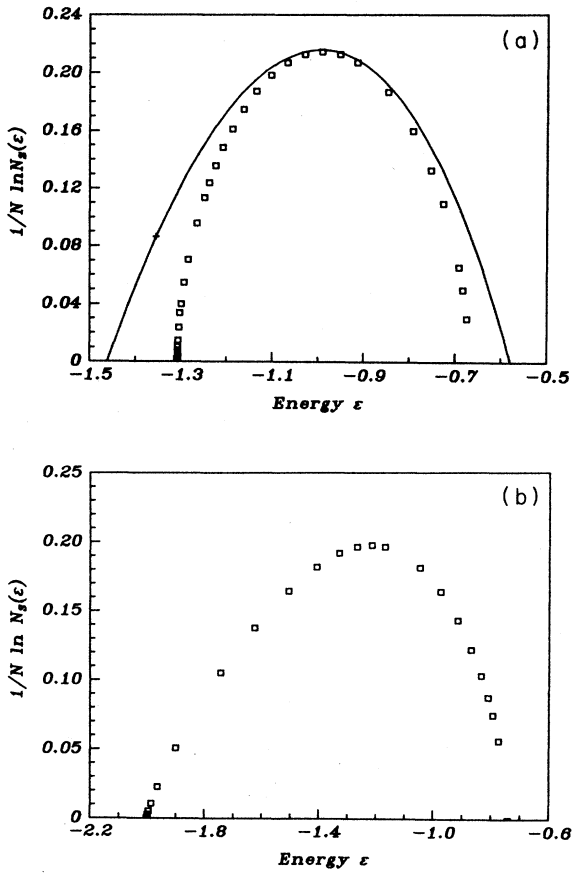

FIG. 8. Distribution of the metastable states in energy for (a) the two-dimensional Gaussian spin glass and (b) the two-dimensional random ferromagnet with bonds in $U(0,2)$. Results shown are for systems of width 6 and length 10^4. The solid line in (a) shows the prediction of the $1/z$ expansion[2] for $z=4$, and the critical energy ε_c is indicated on the curve.

The asymmetry in the energy distribution of the RFM is understood as follows. The excitation energy above the ground state is determined by the bonds on the perimeter of the domains, including the domains within domains. Low-energy states come from flipping small numbers of small domains and high-energy states result from the flipping of a few large domains or many small ones (or a combination of both). Hence there are more high-energy states than low-energy ones.

A zero-temperature scaling argument for spin glasses[9,15] gives a scaling relation for the logarithm of the number of metastable states as a function of energy $\tilde{\varepsilon}$. Two possibilities for the behavior of $\overline{\ln N_s(\varepsilon)}$ at low energies have been suggested:[9]

$$\frac{1}{N}\overline{\ln N_s(\varepsilon)}\sim\sqrt{\tilde{\varepsilon}} \tag{17}$$

and

$$\frac{1}{N}\overline{\ln N_s(\varepsilon)}\sim\tilde{\varepsilon}^{\,1/(1-y/d)}, \tag{18}$$

where d is the dimension of the system and y is the zero-temperature scaling exponent found to have values of -1, -0.3, and 0.2 for one-, two- and three-dimensional Ising spin glasses.[16] The first relation is based on the assumption that a spin glass can be regarded as consisting of a set of weakly interacting two-level systems with $P(J)$ finite at $J=0$; it gives the value $\frac{1}{2}$ for the exponent $1/\lambda$ which describes the density of low-energy metastable states. The second relation, based on a domain-wall argument for systems which do not order at $T>0$, suggests a value of $1/\lambda=0.86$ for the $d=2$ SG.

Our numerical results for the exponent $1/\lambda$, for both the SG and RFM, are plotted as a function of the width W in Fig. 9. We discuss these results first for the spin glass and then for the random ferromagnet.

The low-energy metastable states in the SG are expected to differ from the ground state and from each other by very large clusters of spins[9] on the order of the size of the system. Hence one expects the exponent for the SG strip to reach its limiting value only for systems large in both directions. Numerically, the value of $1/\lambda$ increases

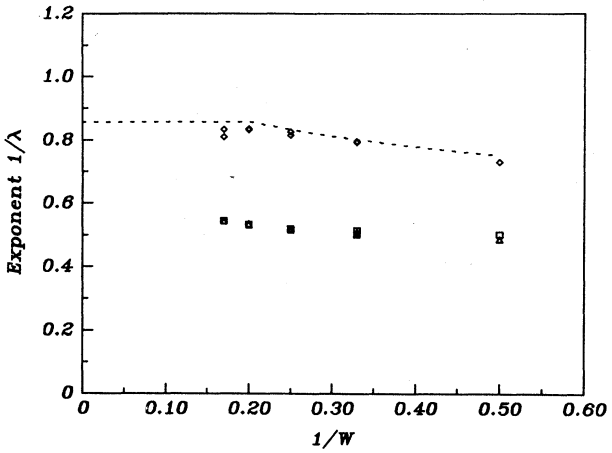

FIG. 9. The energy exponent $1/\lambda$ vs inverse width $1/W$ for the Gaussian spin glass (□), the $U(-1,1)$ spin glass (△), and the $U(0,2)$ random ferromagnet (◇). The dashed line is the prediction of the two-level picture (random ferromagnet); for clarity, W is treated as a continuous variable.

monotonically but slowly with the width of the strip; for width $W=6$ the value obtained is $1/\lambda=0.54$, but the limiting value may be considerably larger. Much larger widths than those permitted by our computing resources are required to extrapolate to infinite width. The explicit form of Eq. (17) is inconsistent with our numerical result (that $1/\lambda$ is greater than 0.5 for the SG). Perhaps the weakly interacting two-level picture is valid, but the weak-link distribution vanishes at $J=0$ [that is $P_w(0)=0$]; unfortunately, we lack a theory of the low-energy excitations of the spin glass. The assumption of weak interaction is questionable, since the low-energy domains are large for the $d=2$ spin glass, but arguments favoring it are given by Fisher and Huse.[11] The value predicted by Eq. (18) is not confirmed by our results, but neither can we rule it out (because our strips are too narrow).

The situation is much clearer for the random ferromagnet. First, since the RFM orders at finite temperature in $d=2$, the second relation Eq. (18) is not applicable. In the following, we derive the exponent for all widths; we obtain the weak-link distribution (by considering the low-energy excitations of the system) and then use Eq. (16). We start by quoting a general result: if the bond distribution has nonvanishing weight at $J=0$, the number of domains which are flipped at an energy cost E is proportional to $E^{N_{bb}-1}$, where N_{bb} is the number of broken bonds; the effective weak-bond distribution then behaves as $P_w(J)\sim J^p$ with $p=N_{bb}-1$. The low-energy excitations have different character for wide and narrow strips. For wide strips, the dominant contribution to the low-lying excitations obviously comes from flipping the smallest possible domains, those of two spins. For the square lattice, six bonds are broken, and the exponent $1/\lambda$ for the RFM with bonds in $U(0,1)$ should be $(1+5)/(2+5)\sim 0.86$ for large widths, in reasonable agreement with the limiting value found numerically. For narrow strips, the low-energy excitations arise not from flipping two-spin clusters, but rather from flipping all spins to the right (or left) of a line (a domain wall) normal to the length—recall that our systems are open at both ends. With helical boundary conditions, the number of bonds broken is $N_{bb}=W+1$ where W is the width of the strip, and the exponent is therefore predicted to be $1/\lambda=(W+1)/(W+2)$ for small widths. The crossover between the large- and small-width values occurs at width 5 where the number of bonds broken is 6 for both kinds of domains; the numerical results confirm this transition from one- to two-dimensional behavior. As a further test, runs with periodic (rather than helical) boundary conditions in the width were carried out; for these boundary conditions, the number of bonds broken by flipping all spins to the right (or left) of a line normal to the length is equal to the width W, and our argument predicts $1/\lambda=W/(W+1)$ for widths less than or equal to 6. The numerical values are $1/\lambda=0.69$ for $W=2$ and $1/\lambda=0.76$ for $W=3$. Thus the picture of weakly interacting two-level systems explains the low-energy excitations of the RFM.

This picture may well be valid for both the SG and the RFM regardless of dimensionality, that is regardless of whether the system orders at finite temperature. Our numerical results support the picture for systems (the 1D SG and the 1D RFM) which do not order at finite T and for a system (the 2D RFM) which orders at finite T; for the 2D SG (where there is no ordering at finite T), our results are inconclusive. Further work on the density of low-energy excitations in 2D and 3D spin glasses is desirable, and might confirm the validity of the picture for these systems as well.

V. SUMMARY

We have devised an algorithm to obtain the number of metastable states, and their distribution in energy and magnetization, of Ising chains and strips. Previous (analytical) studies were unable to treat arbitrary bond distributions, or to treat nonzero external magnetic field, or to find the full energy-magnetization distribution. A reasonably complete study was made of one-dimensional spin glasses and random ferromagnets for lengths up to 10^4 sites (for some quantities 10^5 sites). Two-dimensional spin glasses and random ferromagnets were studied for lengths up to 10^4 sites and widths up to 6. The number of metastable states is exponential for both the SG [Gaussian and $U(-1,1)$ distributions] and the RFM; the exponent for the latter is only 10% smaller, confirming that at least this one measure of the complexity of the energy surface is qualitatively the same in the SG and RFM. The magnetization distribution of the two-dimensional spin glass is discontinuous (as in one dimension). For the random ferromagnet, the density of low-energy states is fully explained by the weakly interacting, two-level picture; for the spin glass, no conclusive results were obtained for the density of low-energy states other than that the exponent is greater than $\frac{1}{2}$.

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