Nonlinear σ models for triangular quantum antiferromagnets

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We expand the long-wavelength action of two-dimensional quantum Heisenberg antiferromagnets on a triangular lattice around the Néel-ordered classical ground state and map it onto a nonlinear σ model, where the field is an SO(3) element, describing the local orientation of the magnetizations on the three sublattices. No topological information on the microscopic value of the spin S is seen to subsist in the mapping, although some nontrivial third-order terms are found.

I. INTRODUCTION

Many authors[1–4] have recently shown that the long wavelength action of two-dimensional (2D)-quantum antiferromagnets does not contain any topological contribution able to distinguish between microscopically different values of the spin S.[5] One may ask whether this conclusion is generic in two dimensions to any approach to the problem from the ordered phase side, or whether it depends on the lattice to which the antiferromagnet is attached. In order to shed some light on this question and to complete our previous study of the square lattice (Ref. 4, hereafter referred as I), we investigate in this paper the case of the triangular lattice.

The reason for this particular choice is twofold. First of all, since the classical ground state in this case has three sublattices with noncollinear spins, the local order parameter is no more a unit vector $\mathbf{n}$ but rather a rotation matrix R defining the local orientation of the three spins. It seemed to us interesting to work out the details of the mapping in this new situation. Second, the resonating valence bond (RVB) picture was originally proposed for the triangular lattice,[6] because of its nonbipartite structure, implying frustrated Heisenberg antiferromagnetic interactions. Recently, interesting analogies to the quantum Hall effect have been put forward and used to construct RVB-type wave functions.[7] The nature of the ground state on a triangular lattice is at present far more controversial than on the square lattice, although some recent variational calculations seem to favor the existence of long-range order even in the former case.[8] In view of all this, the triangular lattice appears to offer the best chances of finding some exotic physics.

We present in this work an expression of the Lagrangian density containing second- and third-order terms in derivatives of the order parameter. The second-order terms give access to the spin-wave spectrum at long wavelength. We find two different spin-wave velocities in or out of the plane spanned by the three sublattice magnetizations in equilibrium. We obtain nontrivial third-order terms describing nonlinear couplings between spin-waves. One of them has the property of remaining imaginary in Euclidean space but unfortunately, it does not correspond to the topological invariant quantizing $\pi_2(\mathrm{SO}(3))=Z$.

The paper is organized as follows: first we discuss the nature of the order parameter for a triangular antiferromagnet and the correspondent homotopy groups, which are relevant to our problem. We then show how to parametrize long-wavelength and low-energy fluctuations around the ordered phase, following the same line of thought as Haldane in his treatment of the spin chain.[9] In Sec. III we present our calculation of the action to second order, putting a large emphasis on the kinetic term, known to be responsible for the interesting topological effects in the previously studied cases. In Sec. IV we briefly quote our results for the third-order corrections to the Lagrangian density.

II. TOPOLOGICAL CONSIDERATIONS AND PARAMETRIZATION OF THE LOW-ENERGY FLUCTUATIONS

The classical ground state of the triangular antiferromagnet has three sublattices 1,2,3 with spins on each sublattice at an angle of $2\pi/3$ to those on the other two sublattices. The lattice can be divided in upright triangular plaquettes, having the site 1 as upper vertex (see Fig. 1). To fix a reference ordered state, we take the three spins on such a triangular plaquette to be directed along the vectors $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ defined by (Fig. 2)

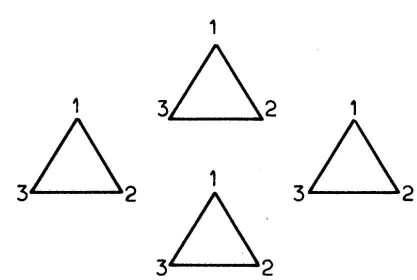

FIG. 1. The partition of the triangular lattice in triangular plaquettes.

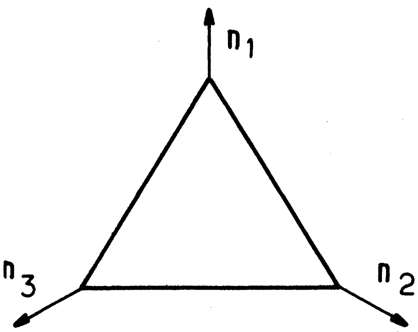

FIG. 2. Orientation of the spins on a triangular plaquette.

$$\mathbf{n}_1=(0,1,0),\quad \mathbf{n}_2=(\sqrt{3}/2,-\tfrac{1}{2},0),\quad \mathbf{n}_3=(-\sqrt{3}/2,-\tfrac{1}{2},0). \tag{1}$$

Other degenerate ground states are trivally deduced from our particular choice by a global orthogonal transformation $\mathbf{S}_i=SR\mathbf{n}_i$, where R satisfies $R\,{}^tR={}^tRR=1$.

In a smooth spatial configuration of R, we note that we may restrict ourselves to $R\in SO(3)$ since if $\det R=1$ at some point, it has to be everywhere so by continuity. Therefore we shall assume in the following our parameter space to be SO(3). Since $SO(3)\cong SU(2)/Z_2$, we have the following homotopy groups

$$\pi_1(SO(3))=Z_2,\quad \pi_2(SO(3))=0,\quad \pi_3(SO(3))=Z\ . \tag{2}$$

$\pi_1=Z_2$ implies the existence of topologically nontrivial vortices (or singular point defects in two dimensions) having a 2π circulation (4π vortices are trivial). However these objects have a singular core and are beyond our approach, which in essence applies to continuous configurations of the order parameter. $\pi_2=0$ shows that there are no "skyrmions" in this case, unlike in the O(3)-nonlinear σ model adequate for bipartite lattices (order parameter given by a unit vector $\mathbf{n}$). Finally, from $\pi_3=Z$, we see that there exist topologically distinct configurations of a SO(3)-matrix field in (2+1)-space-time dimensions. They are classified by the topological invariant[10]

$$q=1/(24\pi^2)\int dx\,dy\,dt\,\epsilon^{\mu\nu\lambda}\times\mathrm{Tr}[(R^{-1}\partial_\mu R)(R^{-1}\partial_\nu R)(R^{-1}\partial_\lambda R)]\ , \tag{3}$$

which assumes integer values for any continuous $R(x,y,t)$ tending to a constant at infinity. In contrast to the Hopf invariant, q is not related to linking properties of curves and the presence of a term θq in the action would not mean nontrivial θ statistics for any particular topological defect. But the ground-state properties of the triangular quantum antiferromagnet would certainly depend on the value of θ if q was there.

To answer this question, we need to derive microscopically an expression for the action of the triangular antiferromagnet including third-order derivatives of R. For this purpose, we write the three spins of a particular triangular plaquette as

$$\mathbf{S}_i=SR(\mathbf{n}_i+a\mathbf{L})/(1+2a\mathbf{n}_i\cdot\mathbf{L}+a^2L^2)^{1/2}\ ,\qquad i=(1,2,3)\ . \tag{4}$$

In this formula the $\mathbf{n}_i$ are the three unit vectors previously introduced, a is the nearest-neighbor distance. R is a rotation matrix, $\mathbf{L}$ a vector, both of them defined on each triangular plaquette. R and $\mathbf{L}$ together give us six degrees of freedom (three from R and three from $\mathbf{L}$), which is what we need to describe an arbitrary position of three spins. In the following we shall assume that R and $\mathbf{L}$ are slowly varying fields with furthermore $aL\ll 1$.

It may be worth commenting here upon what we are doing. The motion of the three spins attached to a same plaquette is supposed to be primarily a rigid rotation, described by R. By adding $\mathbf{L}$, we allow for some small deformation of the triad $\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3$. Finally, the denominator in (4) is required to insure a proper normalization of the spins. Our approach is indeed very similar to the one used for bipartite collinear antiferromagnets. In the last case, one can group the sites by pairs and write inside a given pair

$$\mathbf{S}_{1,2}=S(\pm\mathbf{n}+a\mathbf{L})/(1+a^2L^2)^{1/2}\ \text{ with } \mathbf{L}\cdot\mathbf{n}=0\ . \tag{5}$$

In both situations, $\mathbf{L}$ is an unstaggered quantity related to the local net magnetization (physically necessary to produce the rotations of the spins). From (4), we get to first order in $\mathbf{L}$.

$$\mathbf{S}_i\cong SR\{\mathbf{n}_i+a[\mathbf{L}-(\mathbf{L}\cdot\mathbf{n}_i)\mathbf{n}_i]\} \tag{6}$$

and thus, the net magnetization on a triangular plaquette is given by

$$\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3\cong 3aSR(T\mathbf{L})\ , \tag{7}$$

where the tensor T is defined by $T_{\alpha\beta}=\delta_{\alpha\beta}-\frac{1}{3}(\sum_i n_{i\alpha}n_{i\beta})$ (with our particular choice of the $\mathbf{n}_i$, $2T_{xx}=2T_{yy}=T_{zz}=1$, $T_{ij}=0$ otherwise). It will not come as a surprise that this vector $T\mathbf{L}$ plays an important role in the following.

We are now ready to proceed to the expansion of the action along the path outlined in I. It turns out that the calculations are greatly simplified by defining the fields $\mathbf{L}$ and R on each site rather than on groups of three sites, although the physical meaning of these fields become then less obvious. We checked that the results we obtained, using each of the definitions were equivalent, up to a canonical transformation of the fields. The main advantage of the second choice is to keep more transparent the underlying symmetries of the lattice, avoiding the somewhat arbitrary partition of the lattice in triangular plaquettes.

III. THE CALCULATION OF THE ACTION TO SECOND ORDER

With the notations of I, the action we wish to estimate reads

$$\sum_p \int dt\ \mathbf{A}(\mathbf{S}_p)\cdot\partial_t \mathbf{S}_p - J \sum_{\langle p,q\rangle} \int dt\ \mathbf{S}_p\cdot\mathbf{S}_q\ , \tag{8}$$

where the summation in the first term (kinetic one) is done on each site and in the second term (Heisenberg interaction) on each pair of nearest neighbors. $\mathbf{A}$ is the vector potential of a magnetic monopole of flux 4π and is such that $\int \mathbf{A}(\mathbf{\Omega})\cdot d\mathbf{\Omega}$ measures the area covered by a unit vector $\mathbf{\Omega}$ on the unit sphere S^2. What we have to do is to put the definition (4) into (8) and to expand in powers of $\mathbf{L}$ and space-time derivatives of R.

A. The kinetic term contribution to second order

To calculate this part of the action, we group together the three spins of a triangular plaquette and obtain to second order

$$\sum_i \mathbf{A}(\mathbf{S}_i)\cdot\partial_t\mathbf{S}_i = S\sum_i \mathbf{A}(R\mathbf{n}_i)\cdot\partial_t R\mathbf{n}_i + aS\sum_i \epsilon_{\alpha\beta\gamma}L_\alpha(R^{-1}\partial_t R)_{\gamma\gamma'}(n_{i\beta}n_{i\gamma})\ , \tag{9}$$

where we have used to get the second term the definition of the vector potential $\mathbf{A}$ $\epsilon_{\alpha\beta\gamma}\partial A_\beta/\partial\Omega_\gamma = \Omega_\alpha$ and the property of rotation matrices $\epsilon_{\alpha\beta\gamma}R_{\alpha\alpha'}R_{\beta\beta'}R_{\gamma\gamma'} = \epsilon_{\alpha'\beta'\gamma'}$. Forgetting for the time being the first term, we may assume in the second term that the fields R and $\mathbf{L}$ take their values at the same site: corrections to this approximation are pure divergence terms, vanishing in the continuum limit. Using then the fact that matrices of the type $R^{-1}\partial_\mu R$ are antisymmetric, one can write the second term under the nice form

$$3aS(T\mathbf{L})\cdot\mathbf{V}\ , \tag{10}$$

where the tensor T has been introduced in (7) and the vector $\mathbf{V}$ is contributed from the matrix $R^{-1}\partial_t R$ by

$$V_\alpha = -\tfrac{1}{2}\epsilon_{\alpha\beta\gamma}(R^{-1}\partial_t R)_{\beta\gamma}\ .$$

This is the term producing the coupling between the local magnetization and the time derivative of the spin orientation.

Coming back to the first term in (9), it would seem here *a priori* dangerous to neglect the spatial variation of R between the three sites 1,2,3 constituting the triangular plaquette. In particular, such an approximation, in the $O(3)$-nonlinear σ model, would miss the interesting topological quantities associated to "skyrmions." However, in the present case, it can be shown that this is again a correct procedure. Mathematically this comes from the observation that

$$\epsilon_{\mu\nu\lambda}(R^{-1}\partial_\mu R)(R^{-1}\partial_\nu R) = -\epsilon_{\mu\nu\lambda}\partial_\mu(R^{-1}\partial_\nu R)\ , \tag{11}$$

which means that cross products of first derivatives of rotation matrices are pure divergence terms. More fundamentally, this result could have been anticipated from the triviality of $\pi_2(\mathrm{SO}(3))$, as opposed to $\pi_2(S^2)=Z$.

We are finally left with the task of evaluating $S\int dt \sum_i \mathbf{A}(R\mathbf{n}_i)\cdot\partial_t R\mathbf{n}_i = K$, where R is a function of t only, tending to the identity at infinity. For three unit vectors of zero sum like the $\mathbf{n}_i$ defined in (1), it can be shown that K is topological invariant, classifying the homotopy group $\pi_1(\mathrm{SO}(3))$, which we know to possess two elements [see (2)]. The proof goes as follows: let $R_0(t)$ be a first continuous path satisfying the boundary conditions at infinity and $R(t)=R_0(t)+\delta R(t)$ a second one, infinitesimally near to $R_0(t)$. Taking the variation of K and integrating by parts the term involving $\partial_t(\delta R)$, it is straightforward to get

$$\delta K = S\int dt\ \epsilon_{\alpha\beta\gamma}(R_0)_{\gamma\gamma'}(\partial_t R_0)_{\alpha\alpha'}(\delta R)_{\beta\beta'}\left[\sum_i n_{i\alpha'}n_{i\beta'}n_{i\gamma'}\right]$$
$$= S\int dt\ \epsilon_{\alpha\beta\gamma}(R_0^{-1}\partial_t R_0)_{\alpha\alpha'}(R_0^{-1}\delta R)_{\beta\beta'}T_{\alpha'\beta'\gamma'}\ , \tag{12}$$

where $T_{\alpha\beta\gamma}=\sum_i n_{i\alpha}n_{i\beta}n_{i\gamma}$. This tensor $T_{\alpha\beta\gamma}$, which we shall meet again in the following, has two noteworthy features: it is symmetric with respect to permutations of α,β,γ and it satisfies

$$\sum_\gamma T_{\alpha\gamma\gamma} = \sum_i n_{i\alpha} = 0\ . \tag{13}$$

Now we take advantage of the antisymmetric properties of both matrices $R_0^{-1}\partial_t R_0$ and $R_0^{-1}\delta R$. We write $(R_0^{-1}\partial_t R_0)_{\alpha\alpha'}$ as $-\epsilon_{\alpha\alpha'\gamma'}V_{\gamma'}$, where the vector $\mathbf{V}$ has already been defined in (10). We get in this way

$$\delta K = S\int dt\ \epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\alpha'\gamma'}V_{\gamma'}(R_0^{-1}\delta R)_{\beta\beta'}T_{\alpha'\beta'\gamma}$$
$$= S\int dt[V_\gamma(R_0^{-1}\delta R)_{\beta\beta'}T_{\beta\beta'\gamma} - V_\beta(R_0^{-1}\delta R)_{\beta\beta'}T_{\gamma\gamma\beta'}]\ . \tag{14}$$

The first term is zero because $R_0^{-1}\delta R_{\beta\beta'}$ and $T_{\beta\beta'\gamma}$ are, respectively, antisymmetric and symmetric with respect to β and β', the second one also cancels because of (13).

Therefore K is a constant depending only on the homotopy class of the path $R(t)$. For trivial loops, contractible to zero, we have $K=0$ [by zero loop we mean $R(t)=1$ for all t]. The simplest example of a noncontractible loop in SO(3) constitutes of rotations about a fixed axis by an angle $\theta(t)$ varying from 0 to 2π between $t=-\infty$ and $t=+\infty$. Choosing, for instance, the rotation axis to be along $\mathbf{z}$ with the $\mathbf{n}_i$ in the plane $(\mathbf{x},\mathbf{y})$ and taking the singularity of $\mathbf{A}$ to be also along $\mathbf{z}$ (i.e., $\mathbf{A}=[(1+\cos\theta)/\sin\theta]\varphi$ in spherical variables), one finds in this case $K=6\pi S$ (three times the contribution of one spin undergoing a 2π rotation about $\mathbf{z}$ at $\theta=\pi/2$). This gives in the action a factor $(-1)^{2S}$ for such loops on each triangular plaquette. However, if R goes through such a loop at some point $\mathbf{r}$ in space, it must do the same everywhere by continuity and does not then satisfy the right boundary condition at infinity (i.e., R tends to the identity, in particular for all values of t). Therefore, for our purposes, we shall assume in the following K to be zero. Hence the result (10), divided by 3, gives us the contribution to the Lagrangian density per site coming from the kinetic term.

B. Evaluation of the interaction term and resulting expression of the Lagrangian

To obtain the contribution to the Lagrangian density per site coming from the Heisenberg interaction, we first consider a spin of a given sublattice (index i) interacting with its local environment of six spins belonging to the two other sublattices. Then we sum over the three sublattices (or equivalently the three spin orientations) and divide the result by 3. The quantity which is therefore to be calculated can be written as

$$-H=\frac{-J}{6}\sum_{i=1,3}\sum_{j\neq i}\sum_{k=1,3}\mathbf{S}_i\cdot\mathbf{S}_j^k , \tag{15}$$

where the index k is there to recall that the spins of the species j live on different sites from the central one occupied by $\mathbf{S}_i$. Performing a gradient expansion we can write $\mathbf{S}_j^k$ in (15) as

$$\mathbf{S}_j^k\cong\mathbf{S}_j+a(\mathbf{e}_{ij}^k\cdot\nabla)\mathbf{S}_j+\tfrac{1}{2}a^2(\mathbf{e}_{ij}^k\cdot\nabla)^2\mathbf{S}_j+\tfrac{1}{6}a^3(\mathbf{e}_{ij}^k\cdot\nabla)^3\mathbf{S}_j , \tag{16}$$

where $\mathbf{e}_{ij}^k$ are unit vectors directing the bonds between nearest neighbors and $\mathbf{S}_j$ is now defined at the same point as $\mathbf{S}_i$. We define the vectors $\mathbf{e}_{ij}^k$ as $\theta_{ij}\mathbf{e}^k$ with

$$\mathbf{e}^1=(1,0),\quad \mathbf{e}^2=(-\tfrac{1}{2},\sqrt{3}/2),\quad \mathbf{e}^3=(-\tfrac{1}{2},-\sqrt{3}/2) , \tag{17}$$

and $\theta_{ij}=-\text{sgn}[P(i,j)]$, $P(i,j)$ being the permutation transforming (1,2,3) into (i,j,l). The $\mathbf{e}^k$ are obtained from the $\mathbf{n}_i$ defined at the beginning by a $-\pi/2$ rotation in the $(\mathbf{x},\mathbf{y})$ plane. Like them they have a zero sum and upon summation over k the various derivative operators involved in (16) simplify into

$$\sum_k(\mathbf{e}_{ij}^k\cdot\nabla)=0,\quad \sum_k(\mathbf{e}_{ij}^k\cdot\nabla)^2=\tfrac{3}{2}(\partial_{x^2}^2+\partial_{y^2}^2) ,$$
$$\sum_k(\mathbf{e}_{ij}^k\cdot\nabla)^3=\theta_{ij}T'_{\alpha\beta\gamma}\partial_{\alpha\beta\gamma}^3 , \tag{18}$$

where the third-order tensor $T'_{\alpha\beta\gamma}=\sum_k e_\alpha^k e_\beta^k e_\gamma^k$ is equivalent to the tensor $T_{\alpha\beta\gamma}$ defined before in (12), after interchange of the coordinates x and y (or 1 and 2 indices).

Neglecting for the time being the third-order derivative terms and discarding unimportant constants, we get for H

$$\begin{aligned}-H=&-(J/2)(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)^2\\&-(Ja^2/8)(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)\cdot(\partial_{x^2}^2+\partial_{y^2}^2)(\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3)\\&+\frac{Ja^2}{8}\sum_{i=1,3}\mathbf{S}_i\cdot(\partial_{x^2}^2+\partial_{y^2}^2)\mathbf{S}_i .\end{aligned} \tag{19}$$

In this expression $\mathbf{S}_1,\mathbf{S}_2,\mathbf{S}_3$ take their values on the same sites and we may therefore use our previous result (7) for the net magnetization on a triangular plaquette to leading order, $\mathbf{S}_1+\mathbf{S}_2+\mathbf{S}_3=3aSR(T\mathbf{L})$. The second term in (19) is seen to be fourth order, in the third one we may simply replace $\mathbf{S}_i$ by $SR\mathbf{n}_i$ and we end up, to second order in powers of $\mathbf{L}$ or $R^{-1}\partial_\mu R$, with the result

$$\begin{aligned}-H=&-(9J/2)a^2S^2(T\mathbf{L})^2\\&+(3J/16)a^2S^2\\&\times\text{Tr}\{P[(R^{-1}\partial_xR)^2+(R^{-1}\partial_yR)^2]\} .\end{aligned} \tag{20}$$

In this formula P is the projection operator in spin space onto the plane spanned by $\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3$ [within our initial choice (1), $P_{xx}=P_{yy}=1$, $P_{ij}=0$ otherwise].

Using then (10), we get for the Lagrangian density Λ to second order (normalizing every quantity by the unit cell area $a^2\sqrt{3}/2$)

$$(a^2\sqrt{3}/2)\Lambda=aS(T\mathbf{L})\cdot\mathbf{V}-H . \tag{21}$$

The extremization with respect to the field $\mathbf{L}$ yields finally

$$\begin{aligned}\Lambda=(g^{-1}/2)(&-c^{-1}\text{Tr}(R^{-1}\partial_tR)^2\\&+c\,\text{Tr}\{P[(R^{-1}\partial_xR)^2+(R^{-1}\partial_yR)^2]\}) ,\end{aligned} \tag{22}$$

where we have defined a coupling constant $g^{-1}=S/6a$, a spin-wave velocity $c=(3\sqrt{3}/2)JSa$ and have rewritten V^2 as $-\frac{1}{2}\text{Tr}(R^{-1}\partial_tR)^2$.

Were it not for the presence of the anisotropic tensor P in the spatial derivative term, Λ would be the Lagrangian of a standard nonlinear σ model with an order parameter in SO(3). P contains the information on the initial anisotropy of the spin orientation in the ordered state. To make this point more explicit, we represent the matrices $M=R^{-1}\partial_\mu R$ by a triplet (A_μ,B_μ,C_μ) with $A_\mu=M_{12}$, $B_\mu=M_{13}$, $C_\mu=M_{23}$. The A components describe variations of the spin orientation *in* the plane defined by the three sublattice local magnetizations, whereas the B and C components correspond to fluctuations *out* of the plane. In terms of A, B, and C, Λ becomes

$$\begin{aligned}\Lambda=g^{-1}\{&c^{-1}(A_t^2+B_t^2+C_t^2)\\&-c[A_x^2+A_y^2+\tfrac{1}{2}(B_x^2+C_x^2)+\tfrac{1}{2}(B_y^2+C_y^2)]\} .\end{aligned} \tag{23}$$

In the linear regime around the ordered state defined in (1), our result gives a spin-wave velocity equal to c in the plane (x,y) and to $c/\sqrt{2}$ out of this plane. The first part of this result is in agreement with the spin-wave calculations as performed for instance in Ref. 11, where authors considered XY-like anisotropy in the Heisenberg interaction. On the other hand we have not found in the literature any calculation of the B and C modes.

IV. THIRD-ORDER TERMS IN THE LAGRANGIAN

We now turn to a brief discussion of the third-order terms appearing in the Lagrangian. We note $\mathbf{S}_i^1$ and $\mathbf{S}_i^2$ respectively, the first- and second-order terms in the expansion of $\mathbf{S}_i/S$ as given by (4) in powers of $\mathbf{L}$,

$$\begin{aligned}&\mathbf{S}_i^1=aR[\mathbf{L}-(\mathbf{L}\cdot\mathbf{n}_i)\mathbf{n}_i] ,\\&\mathbf{S}_i^2=a^2R\{\mathbf{n}_i[-L^2/2+\tfrac{3}{2}(\mathbf{L}\cdot\mathbf{n}_i)^2]-\mathbf{L}(\mathbf{L}\cdot\mathbf{n}_i)\}\end{aligned} \tag{24}$$

It can be shown that the contribution to third-order from the kinetic term, with the conventions of (9), is

$$S\sum_i[(R^{-1}\partial_tR\mathbf{n}_i)\cdot(\mathbf{S}_i^2\wedge\mathbf{n}_i)+\tfrac{1}{2}\partial_t\mathbf{S}_i^1\cdot(\mathbf{S}_i^1\wedge R\mathbf{n}_i)] . \tag{25}$$

After some algebraic manipulations, this expression is found to be equal to

$$\tfrac{3}{2}a^2SL_\alpha L_\beta V_\gamma T_{\alpha\beta\gamma} \ , \tag{26}$$

where the tensor $T_{\alpha\beta\gamma}$ has been previously introduced in (12).

As for the Heisenberg interaction $-H$, we get from (18) and (19)

$$\begin{aligned}-JS^2(\mathbf{S}_1^1+\mathbf{S}_2^1+\mathbf{S}_3^1)\cdot(\mathbf{S}_1^2+\mathbf{S}_2^2+\mathbf{S}_3^2)\\ +(J/4)a^2S^2\sum_i \mathbf{S}_i^1\cdot(\partial^2_{x^2}+\partial^2_{y^2})R\,\mathbf{n}_i\\ +(J/16)a^3S^2T'_{\alpha\beta\gamma}\mathbf{n}_1\cdot(R^{-1}\partial^3_{\alpha\beta\gamma}R)(\mathbf{n}_2-\mathbf{n}_3)\ .\end{aligned} \tag{27}$$

By expliciting each term, (27) can be written in the form

$$\begin{aligned}Ja^3S^2\{-\tfrac{9}{4}T_{\alpha\beta\gamma}L_\alpha L_\beta L_\gamma\\ -\tfrac{1}{4}L_\alpha[(R^{-1}\partial_xR)^2+(R^{-1}\partial_yR)^2]_{\beta\gamma}T_{\alpha\beta\gamma}\\ +(\sqrt{3}/12)T'_{\alpha\beta\gamma}(R^{-1}\partial^3_{\alpha\beta\gamma}R)_{21}\}\ .\end{aligned} \tag{28}$$

To deduce the Lagrangian density to third-order in space-time derivatives of R, it is enough to put in (26) and (28) the leading expression of $\mathbf{L}$ since by construction, it extremizes the Lagrangian to second order. By doing so, it is seen that the terms with three time derivatives cancel exactly, leaving us with the following corrections to our previous result (22) for Λ,

$$\begin{aligned}(S/\sqrt{3})V_\alpha[(R^{-1}\partial_xR)^2+(R^{-1}\partial_yR)^2]_{\beta\gamma}T_{\alpha\beta\gamma}\\ +(J/6)aS^2T'_{\alpha\beta\gamma}(R^{-1}\partial^3_{\alpha\beta\gamma}R)_{21}\ .\end{aligned} \tag{29}$$

It is interesting to note that the spatial isotropy is lost at this order: indeed $T'_{\alpha\beta\gamma}\partial^3_{\alpha\beta\gamma}=\frac{3}{4}(\partial^3_{x^3}-3\,\partial^3_{xy^2})$. On the other hand, we find also a term linear in S, involving one time derivative and two space derivatives. This means that this term gives an imaginary part in the action, after analytic continuation in Euclidean space, as a topological term would do. But our result does not correspond to the desired quantity q of (3), which is completely antisymmetric with respect to the three variables x,y,t. In terms of the A,B,C components defined in (23), the third-order corrections read

$$\begin{aligned}g^{-1}(a6\sqrt{3})\{-[B_t(B_x^2+B_y^2-C_x^2-C_y^2)\\ -2C_t(C_yB_y+C_xB_x)]\\ +c[A_x(A_x^2+B_x^2+C_x^2)\\ -3A_y(A_xA_y+B_xB_y+C_xC_y)\\ +B_x\partial_xC_x-3B_y\partial_xC_y]\}\ .\end{aligned} \tag{30}$$

We note that most of the terms involve three powers of A, B, or C and are intrinsically nonlinear. They do not affect the spin-wave dispersion relations, except for the last two terms in (30), which are quadratic and yield a coupling between the B and C components. This is quite satisfying physically, because we do not expect on symmetry grounds any odd powers of the wave vector $\mathbf{k}$ in the spin-wave dispersion relations.

V. CONCLUSION

To summarize this work, we have found no additional topological term in the action of the triangular quantum antiferromagnet like in previous studies of the square lattice. Since these two systems are very far from each other in the family of two-dimensional antiferromagnets, our result seems to support the view that such topological effects, if they exist at all in 2D, cannot be obtained by an approach from the ordered phase side.

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