Field theory of the two-dimensional Ising model: Conformal invariance, order and disorder, and bosonization

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In this paper we review the critical Ising model by using the properties of conformal invariance. We use the known mapping of the Ising model to a theory of Majorana fermions and recognize that the self-duality property appears as a double degeneracy of the periodic ground state. The Ising model is doubled obtaining an Ashkin-Teller model (two Ising models coupled with a four-spin coupling) at the decoupling point. Using the bosonization technique this model is mapped onto the Gaussian model at a particular value of the temperature ($K=1/\pi$). This allows us to give the expressions for products of *order, disorder,* and *energy* operators of both Ising models in terms of operators in the Gaussian model. We compute several correlation functions of order, disorder, energy and spinor operators and show that they reproduce the operator product expansions predicted by conformal invariance. We explicitly discuss the physics by which the correlation functions obtained from the Gaussian model (a theory with conformal anomaly $c=1$) reproduce those of the Ising model ($c=\frac{1}{2}$). This provides a proof of Kadanoff and Brown's equivalences and conjectures. We provide a *simple* prescription to compute *all* n-point correlation functions at criticality, including mixed correlations of order, disorder, energy, and spinor operators. A bosonized form of the continuum limit of the transfer matrix for the critical Ising model is constructed. It is nonlocal and contains both spin-wave and vortex operators of the Gaussian model.

I. INTRODUCTION AND MOTIVATIONS

The Ising model is perhaps one of the best-known examples of statistical mechanical systems that undergo a second-order phase transition. Its appeal stems from the fact that it is very simple to formulate and its exact solution for zero external magnetic field was given by Onsager[1] over 40 years ago. For all its beauty, however, Onsager's solution involves a highly sophisticated method. The solution was first simplified by the introduction of fermionic variables by Kaufman.[2] Later, Schultz, Mattis, and Lieb in a remarkable paper used this method of introducing fermions to cast the solution in a very elegant and simplified manner.[3]

The formidable work of McCoy and Wu[4] is to this day perhaps the most thorough treatise on the Ising model. One of the problems that is first realized is that the solution is easily cast in terms of fermionic variables, but the correlation functions of order and disorder operators involve some formidable algebra involving the calculation of Toeplitz determinants.[4] It is this mathematical intractability that prevents the computation of n-point functions of order and disorder operators. Field theoretical approaches succeeded in computing the two-point function of order operators.[5,6] Luther and Peschel used the equivalence of two decoupled Ising models to an $S=\frac{1}{2}$ XY quantum chain and computed some correlation functions by using bosonization.[7]

From the operator algebra, Kadanoff and Ceva[8] were able to compute some correlation functions of order and disorder operator on a line. Later, Zuber and Itzykson,[9] by doubling the Ising model and using bosonization, computed the two- and four-point functions of the order operators by taking the "square roots" in the doubled Ising model. This procedure was also advocated by Schroer and Truong.[10] Kadanoff and Brown,[11] by using the result of Zuber and Itzykson[9] and building upon universality arguments and the operator product expansion, give an identification between products of operators of the two Ising models at the decoupling point of the Ashkin-Teller model, and spin-wave and vortex operators in a Gaussian model at a particular value of the temperature. Currently most of the relevant information on the correlation functions of the Ising model is fairly well known but a great deal of mathematical sophistication is required to obtain these correlation functions (see Ref. 4).

More recently, the remarkable work of Friedan, Qiu, and Shenker[12] (FQS) and Belavin, Polyakov, and Zamolodchikov[13] pointed out that conformal invariance severely restricts the operator content and completely determines the operator product expansion (OPE) of a large number of critical systems in two dimensions. The Ising model is perhaps the simplest theory in the classification of FQS.[12] In this series of theories, the correlation functions of scaling operators are completely determined from the knowledge of the "degenerate" or "null states" in the representations of the Virasoro (conformal) algebra.[12,13] From these states a set of differential equations can be obtained that the correlation functions of scaling operators must satisfy. The solution to these differential equations determines completely the OPE of the theory. Needless to say, the difficulty for solving these equations grows with the order of the correlation functions. Recently some efforts to solve these complicated equations have been reported.[14]

The purpose of this paper is to reconcile some aspects of what is known about the Ising model from the conformal invariance[12,13] point of view and the "doubling" of the Ising model to compute the correlation functions by taking square roots. In particular, the doubled (critical) Ising model is equivalent to the Gaussian model,[11] which is a theory with conformal anomaly $c=1$ and an *infinite* number of primary scaling operators,[12,13] whereas the Ising model has $c=\frac{1}{2}$ and a *finite* number of scaling operators (energy, order and disorder, and spinor). The spectrum of these theories is radically different. It certainly is not clear why or whether the correlation functions obtained from the doubling procedure should yield the operator product expansions of the Ising model.

This question cannot be answered with the correlation functions computed in Refs. 9 and 10. Mixed correlation functions are needed to obtain the full OPE, and these have not yet been obtained by these techniques. Furthermore, we remark that the seminal work of Kadanoff and Brown *combined* results from doubling and bosonization[9] with the OPE of the Ising model known previously. The missing link is of course whether the doubling approach leads to the correct OPE. Understanding these points is clearly interesting since it may point to a possible construction of theories with $C<1$ by suitable conditions on a $C=1$ theory.

Section II is a brief review of the transfer matrix approach to the field-theoretical description.

Section III studies the conformal properties (Virasoro algebra) and analyzes the ground states in the periodic and antiperiodic sectors. By resorting to a doubling of the Ising model, it is shown that the ground state in the periodic sector is double degenerate (with conformal weight $h=\frac{1}{16}$), this degeneracy being a consequence of the order and disorder duality that is hidden in the fermionic representation of the model. Part of this section is fairly standard and most of its material is known, but is included to reach the nonexperts.

Section IV represents the bulk of the results. Here, we construct the correlation functions at criticality by doubling the Ising model and using bosonization. We compute many mixed correlation functions and show that they lead to the correct OPE of the Ising model. We explain why physically this must be the case when only part of the full spectrum of the Gaussian model is "projected" by taking square roots when the underlying field satisfies particular boundary conditions. This provides a proof of the conjectures and identifications of Kadanoff and Brown. A brief discussion is given of the "twisted" sector of the Ashkin-Teller model and boundary conditions. In this section we also construct a *bosonic* Hamiltonian for the Ising model by bosonization of one Majorana fermion providing an example of a recently proposed construction in conformal field theory.[15,16]

The results are summarized at the end of the paper. Two appendixes are devoted to technical details.

II. THE FIELD THEORY

The equivalence between the two-dimensional Ising model and a $(1+1)$-dimensional quantum field theory is better exposed in the transfer-matrix approach.[3] In the extreme anisotropic limit, when the lattice spacing in the "time" direction approaches zero and the coupling along this direction (K_τ) and the perpendicular direction (K_x) are properly adjusted, the transfer matrix can be written as

$$\hat{T}=e^{-\tau\hat{H}}\sim 1-\tau\hat{H} \ ,$$

with $\hat{H}$ the quantum spin-chain Hamiltonian

$$\hat{H}=-\lambda\sum_i \sigma_i^z-\sum_i \sigma_i^x\sigma_{i+1}^x \ , \tag{1}$$

with σ^x, σ^y, and σ^z the $S=\frac{1}{2}$ Pauli matrices and

$$\lambda^{-1}=\sinh(2K_\tau)\sinh(2K_x) \ . \tag{2}$$

The critical curve corresponds to $\lambda=1$, and in the extreme anisotropic limit $K_\tau\to\infty$.

Introducing the operators

$$\begin{aligned} \tfrac{1}{2}(\sigma_j^x+i\sigma_j^y)&=a_j^\dagger \ , \\ \tfrac{1}{2}(\sigma_j^x-i\sigma_j^y)&=a_j \ , \\ \sigma_j^z&=2a_j^\dagger a_j-1 \ . \end{aligned} \tag{3}$$

The Hamiltonian $\hat{H}$ in Eq. (3) is written as a quadratic form in terms of the above operators. However, this Hamiltonian cannot be diagonalized because the a and $a^\dagger$ obey mixed commutation relations

$$\{a_i^\dagger,a_i\}=1, \quad [a_i^\dagger,a_j]=[a_i,a_j]=[a_i^\dagger,a_j]=0, \quad i\neq j \ . \tag{4}$$

As usual this situation is remedied by introducing the Jordan-Wigner[3] transformation and defining

$$c_j=\left[\exp\left[i\pi\sum_{k=0}^{j-1}a_k^\dagger a_k\right]\right]a_j \ , \tag{5a}$$

$$c_j^\dagger=a_j^\dagger\left[\exp\left[-i\pi\sum_{k=0}^{j-1}a_k^\dagger a_k\right]\right] \ . \tag{5b}$$

From Eqs. (4), (5a), and (5b) it is straightforward to show that

$$c_j^\dagger c_j=a_j^\dagger a_j \ , \tag{6a}$$

$$a_j=\left[\exp\left[-i\pi\sum_{k=0}^{j-1}c_k^\dagger c_k\right]\right]c_j \ , \tag{6b}$$

$$a_j^\dagger=c_j^\dagger\left[\exp\left[i\pi\sum_{k=0}^{j-1}c_k^\dagger c_k\right]\right] \ , \tag{6c}$$

$$\{c_i^\dagger,c_j\}=\delta ij, \quad \{c_i^\dagger,c_j^\dagger\}=\{c_i,c_j\}=0 \ . \tag{6d}$$

The Hamiltonian in Eq. (3) written in terms of these fermionic variables reads

$$H=-2\lambda\sum_i^N c_i^\dagger c_i-\sum_i^N (c_i^\dagger-c_i)(c_{i+1}^\dagger+c_{i+1}) \ , \tag{7}$$

where we have dropped a constant term and a surface contribution since we are ultimately interested in the limit $N\to\infty$ (N is the number of lattice sites). Defining the real spinors

$$\psi_L(j)=\frac{(-1)^j}{\sqrt{2a}}(c_j^\dagger e^{-i\pi/4}+c_j e^{i\pi/4}) , \tag{8a}$$

$$\psi_R(j)=\frac{(-1)^j}{\sqrt{2a}}(c_j^\dagger e^{i\pi/4}+c_j e^{-i\pi/4}) , \tag{8b}$$

with a the lattice spacing, these spinors satisfy the anticommutation relations

$$\{\psi_R(i),\psi_R(j)\}=\{\psi_L(i),\psi_L(j)\}=\frac{\delta ij}{a} , \tag{9a}$$

$$\{\psi_R(j),\psi_L(i)\}=0 . \tag{9b}$$

The fermions ψ_R and ψ_L are the Kauffmann-Onsager[2,3] fermions. Rescaling the Hamiltonian as $H\to(1/2a)H$ (the $1/a$ restores units of energy and the $\frac{1}{2}$ is necessary for a relativistic dispersion relation), a short calculation yields the equations of motion in the limit $a\to 0$ ($ja\to x$)

$$i\dot\psi_R(x)=-[H,\psi_R(x)]=i\left[\frac{\lambda-1}{a}\right]\psi_L-i\frac{\partial\psi_R(x)}{\partial x} , \tag{10a}$$

$$i\dot\psi_L(x)=-[H,\psi_L(x)]=-i\left[\frac{\lambda-1}{a}\right]\psi_R+i\frac{\partial\psi_L(x)}{\partial x} . \tag{10b}$$

In the continuum ($a\to 0$) limit these are the equations of motion obtained from ($L=Na$, N is the number of sites)

$$H=\tfrac{1}{2}\int_{-L/2}^{L/2}dx\left[\psi(x)\left[-i\gamma^5\frac{\partial}{\partial x}\psi(x)\right]+m\psi(x)\gamma^0\psi(x)\right] , \tag{11a}$$

with

$$\psi(x)=\begin{bmatrix}\psi_R(x)\\ \psi_L(x)\end{bmatrix} , \tag{11b}$$

where the anticommutation relations (9a) become

$$\{\psi_R(x),\psi_R(y)\}=\{\psi_L(x),\psi_L(y)\}=\delta(x-y)$$

and

$$m=\left[\frac{\lambda-1}{a}\right] , \tag{12}$$

and the Dirac γ matrices are in a Majorana representation in which γ^0 and γ^1 are imaginary, then

$$\gamma^0=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} ,$$

$$\gamma^1=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix} , \tag{13}$$

$$\gamma^5=\gamma^0\gamma^1=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} .$$

We therefore see that ψ is a self-conjugate Majorana fermion, real in the representation (13). At the critical point $\lambda=1$ [see Eq. (2) and the discussion following it] the theory is massless and conformally invariant. The spinors ψ_R and ψ_L are the right- and left-handed components (eigenvalue of $\gamma^5=\pm 1$, respectively). In this case the equations of motion (10a) and (10b) imply

$$\psi_L(x,t)=\psi_L(x+t) , \tag{14a}$$

$$\psi_R(x,t)=\psi_R(x-t) . \tag{14b}$$

By using Eqs. (3), (5a), and (5b), the order operator (magnetization) is written as

$$\sigma_i^x=\left[c_i\exp\left[i\pi\sum_1^{i-1}c_k^\dagger c_k\right]+c_i^\dagger\exp\left[-i\pi\sum_1^{i-1}c_k^\dagger c_k\right]\right] , \tag{15a}$$

or, alternatively,

$$\sigma_i^x=(c_i+c_i^\dagger)\exp\left[\pm i\pi\sum_{k=1}^{i-1}c_k^\dagger c_k\right] . \tag{15b}$$

Equation (15b) follows from (15a) by using $(c_k^\dagger c_k)^2=c_k^\dagger c_k$; this is also reflected in the sign ambiquity in (15b). The dual of the order operator is the disorder operator[8,10]

$$\mu(j)=\prod_{k<j}\sigma_k^z=(-1)^{j-1}\exp\left[\pm\pi\sum_{k=1}^{j-1}c_k^\dagger c_k\right] . \tag{16}$$

The duality transformation

$$\sigma_j^z=\mu(j)\mu(j+1),\quad \sigma_j^x\sigma_{j+1}^x=\mu_x(j) , \tag{17}$$

changes $\lambda\to\lambda^{-1}$ in the Hamiltonian (1) (and rescales it by a factor λ). At the critical point ($\lambda=1$), the theory is self-dual. In terms of the spinors ψ_R and ψ_L that diagonalize the continuum Hamiltonian Eq. (11) at the critical point, the order and disorder operators are written as

$$\sigma^x(j)=\sqrt{a}\,(\mp i)^{j-1}[\psi_R(j)+\psi_L(j)]\times\exp\left[\pm i\pi\sum_{k=1}^{j-1}[ia\psi_R(k)\psi_L(k)]\right] , \tag{18}$$

$$\mu(j)=(\pm i)^{j-1}\exp\left[\pm i\pi\sum_{k=1}^{j-1}[ia\psi_R(k)\psi_L(k)]\right] . \tag{19}$$

From expressions (18) and (19) we see that the order and disorder operators are nonlocal in terms of the fermion fields. Also, we notice from the above equations that σ^x and μ will be the product of functions of $x-t$ and $x+t$.

III. THE CONTINUUM LIMIT: VIRASORO ALGEBRA

At the critical point $\lambda=1$, the continuum theory is described by a free massless Majorana spinor with the Hamiltonian (11) for $m=0$. In terms of the right- and left-handed components ψ_R and ψ_L the Hamiltonian density becomes

$$\mathcal{H}=\tfrac{1}{2}\psi_R(x)\left[-i\frac{\partial}{\partial x}\psi_R(x)\right]-\tfrac{1}{2}\psi_L(x)\left[-i\frac{\partial}{\partial x}\psi_L(x)\right] . \tag{20}$$

At the critical point $\lambda=1$ ($m=0$) the theory is conformally invariant. In two dimensions the conformal group is infinite dimensional, the generators are the moments of the light-cone components of the energy-momentum tensor[12,13]

$$T(x^-)=\tfrac{1}{2}(\mathcal{H}+\mathcal{P}) , \quad (21a)$$

$$\bar{T}(x^+)=\tfrac{1}{2}(\mathcal{H}-\mathcal{P}) , \quad (21b)$$

with $\mathcal{H}$ and $\mathcal{P}$ being the Hamiltonian and momentum density, respectively, and $x^{\pm}=t\pm x$. From now on, we will concentrate on the right component $T(x^-)$. $\bar{T}(x^+)$ will have a similar expression in terms of the left spinors ψ_L. From the equation of motion (10a), we can expand the right-handed spinors in normal modes as

$$\psi_R(x^-)=\frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty} b_n \exp\left[-2\pi i\frac{n}{L}(t-x)\right] , \quad (22)$$

where we quantize in a finite interval $-L/2\le x \le L/2$. The factor $1/\sqrt{L}$ restores the canonical dimension to the field ψ. The reality condition on ψ_L implies

$$b_n^{\dagger}=b_{-n}, \quad \{b_n,b_m\}=\delta_{n+m,0} . \quad (23)$$

In order to define the theory properly we must define the boundary conditions on the field ψ_R. Since ψ is a real field, the boundary conditions can be of two types giving two independent sectors of the theory (inequivalent Hilbert spaces).

(a) Antiperiodic boundary conditions (BC's)

$$\psi_R\left[-\frac{L}{2}\right]=-\psi_R\left[\frac{L}{2}\right] , \quad (24)$$

with

$$n=(2m+1)/2, \quad m=0, \quad \pm1,\pm2,\ldots \quad (25)$$

in the mode expansion in (22). This is known as the Neveu-Schwartz (NS) sector.[17]

(b) Periodic BC's

$$\psi_R\left[-\frac{L}{2}\right]=\psi_R\left[\frac{L}{2}\right] \quad (26)$$

with

$$n=0, \ \pm1, \ \pm2, \ldots \quad (27)$$

in the expansion (22). This BC defines the Ramond (R) sector.[16] Using the mode expansion (22) we find

$$:T(x^-):=\tfrac{1}{2}:\psi_R\left[\tfrac{1}{2}\left[i\frac{\partial}{\partial t}-i\frac{\partial}{\partial x}\right]\right]\psi_R = \frac{2\pi}{L^2}\sum_{N=-\infty}^{\infty} L_N e^{(-2\pi i N x^-/L)} , \quad (28)$$

with

$$L_N=\tfrac{1}{2}\sum_{n=-\infty}^{\infty} n :b_{N-n}b_n: , \quad (29)$$

where the dots in (28) and (29) represent normal ordering with respect to the creation and annihilation operators b_n. The vacuum $|0\rangle$ is defined as

$$b_n|0\rangle=0 \text{ for } n>0 , \quad (30a)$$

$$b_n=\begin{cases}\text{annihilation for } n>0\\ \text{creation for } n<0 ,\end{cases} \quad (30b)$$

where, as usual for a fermionic theory, all states with negative energies are filled. The operators L_N in Eq. (28) are known as the Virasoro generators; these are the generators of the infinite dimensional conformal group in two dimensions.[12,13] They obey the Virasoro algebra[12,13]

$$[L_N,L_M]=(N-M)L_{N+M}+\frac{c}{12}N(N^2-1)\delta_{N+M,0} , \quad (31)$$

with $c=\frac{1}{2}$ corresponding to a free Majorana fermion.[16] The last term (central charge) in (31) arises from the normal-ordering prescription in (29). There is a similar expression for $\bar{T}(x^+)$ in terms of $\bar{L}_N$ obeying the algebra (31) with the same $c=\frac{1}{2}$.

Boundary sectors

We now study in detail both sectors (NS and R) of the theory.

(a) NS *sector.* This sector is defined by the BC's [Eq. (24)] with n in (22) being given by (25), $n=\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$. Let us consider the state

$$|\tfrac{1}{2}\rangle=b_{1/2}|0\rangle \quad (32)$$

with the vacuum state $|0\rangle$ defined in Eqs. (30a) and (30b). By using the expression of the Virasoro generators given by (29) it is straightforward to show that

$$L_0|0\rangle=L_{\pm1}|0\rangle=0 , \quad (33a)$$

$$L_0|\tfrac{1}{2}\rangle=\tfrac{1}{2}|\tfrac{1}{2}\rangle , \quad (33b)$$

$$L_N|\tfrac{1}{2}\rangle=0 \ \text{ for } N>0 . \quad (33c)$$

Equation (33a) indicates that the vacuum in this sector is invariant under the (SL) (2,C) subgroup of the full conformal group, generated by L_0 and $L_{\pm1}$, for which the central extension in the Virasoro algebra [Eq. (31)] vanishes.[12,13]

Equations (33b) and (33c) show that the state $|\frac{1}{2}\rangle$ is a highest-weight state (HWS) with conformal weight $h=\frac{1}{2}$ of the Virasoro algebra [Eq. (31)] with $c=\frac{1}{2}$. From the expression for L_N given in Eqs. (29), (33a), and (33b) it follows that

$$L_{-1}|\tfrac{1}{2}\rangle=b_{-3/2}|0\rangle=|\tfrac{3}{2}\rangle , \quad (34a)$$

$$L_{-1}^2|\tfrac{1}{2}\rangle=2b_{-5/2}|0\rangle=2|\tfrac{5}{2}\rangle , \quad (34b)$$

$$L_{-2}|\tfrac{1}{2}\rangle=\tfrac{3}{2}|\tfrac{5}{2}\rangle . \quad (34c)$$

Therefore, from Eqs. (34b) and (34c) we find the null vector[12,13]

$$|\text{null}\rangle=(L_{-2}-\tfrac{3}{4}L_{-1}^2)|\tfrac{1}{2}\rangle\equiv0 . \quad (35)$$

This is a null vector that makes the Kac determinant vanish at second level as shown in Refs. 12 and 13. This null vector is very useful since it gives rise to a series of partial (second-order) differential equations for the correlation functions of scaling (primary) operators as suggested in the above references.[12,13] Using the expansion (22) for antiperiodic BC's we find the correlation function

$$\langle 0|\psi_R(x^-)\psi_R(y^-)|0\rangle=\frac{1}{L}\frac{1}{2i\sin[\pi(x^- - y^- - i\epsilon)/L]}\ , \tag{36}$$

where the term $\epsilon\to 0^+$ is a convergence factor. It is more convenient to work on the plane where the conformal transformations are more transparent and they correspond to the analytic or antianalytic mappings of the complex plane.[12,13]

To achieve this, perform first the Euclidean continuation

$$i(t\pm x)\to w=(\tau-ix),\quad \bar{w}=(\tau+ix) \tag{37}$$

and then map the strip $0\le|\mathrm{Re}w|\le\infty$, $-(L/2)\le \mathrm{Im}w\le(L/2)$ onto the full complex plane by using the mapping[18]

$$z=e^{2\pi w/L},\quad \bar{z}=e^{2\pi\bar{w}/L}\ . \tag{38}$$

Because ψ is a conformal field of dimension $\frac{1}{2}$, its correlation function transforms under this transformation as

$$_P\langle 0|\psi_R(z)\psi_R(z^1)|0\rangle_P = \left[\frac{\partial\omega}{\partial z}\right]^{1/2}\left[\frac{\partial\omega^1}{\partial z^1}\right]^{1/2} {}_s\langle 0|\psi_R(w)\psi_R(w^1)|0\rangle_s\ , \tag{39}$$

with $|0\rangle_P$ and $|0\rangle_s$ the vacuum of the theory quantized on the plane and on the strip, respectively. A short calculation yields

$$_P\langle 0|\psi_R(z)\psi_R(z^1)|0\rangle_P=\frac{1}{2\pi}\frac{1}{z-z^1}\ . \tag{40}$$

(b) *R sector.* The field now obeys periodic BC's, and Eqs. (26) and (27) apply. In this situation, there is a subtlety since now there is a zero mode ($n=0$) in the expansion (22) corresponding to b_0.

The anticommutation relation (23) now implies

$$b_0^2=\tfrac{1}{2}\ . \tag{41}$$

This equation suggests the interpretation that this zero mode is an equal mixture of "particle" and "hole" (this will become clear in the next section). Now the vacuum state has a different character, Eqs. (30a) and (30b) for the b_n still hold, but also

$$b_0^2|\sigma\rangle=\tfrac{1}{2}|\sigma\rangle\ , \tag{42a}$$

where we have called $|\sigma\rangle$ the vacuum state in the Ramond sector. In fact, the above equation reflects the fact that the vacuum state is *double* degenerate [this will become clear later when the Dirac case is discussed—see Eq. (65) and discussion thereafter], and we choose [see discussion after Eq. (86)]

$$b_0|\sigma(\pm)\rangle=\pm\frac{1}{\sqrt{2}}|\sigma(\pm)\rangle,\quad \langle\sigma(+)|\sigma(-)\rangle=0\ . \tag{42b}$$

If we naively take L_N as given by (29), we would conclude that the state $|\sigma\rangle$ is annihilated by L_0. However, we now see that this conclusion is too hurried.

A straightforward calculation yields

$$L_N|\sigma(\pm)\rangle=0\quad \text{for } N>0\ , \tag{43a}$$

$$L_{-1}|\sigma(\pm)\rangle=-\tfrac{1}{2}b_0b_{-1}|\sigma(\pm)\rangle\ . \tag{43b}$$

Therefore, the norm [from now on we will generally call $|\sigma(\pm)\rangle\equiv|\sigma\rangle$]

$$\|L_{-1}|\sigma\rangle\|^2=\langle\sigma|[L_1,L_{-1}]|\sigma\rangle=2\langle\sigma|L_0|\sigma\rangle=\tfrac{1}{8}\ , \tag{44}$$

where we have used Eqs. (43a) and (43b) and the Virasoro algebra [Eq. (31)]. Hence, Eq. (44) implies

$$L_0|\sigma(\pm)\rangle=\tfrac{1}{16}|\sigma(\pm)\rangle\ . \tag{45}$$

Therefore, L_0 is shifted by the constant $\frac{1}{16}$ from its expression in the NS sector.

Equations (43a) and (45) imply that $|\sigma\rangle$ is an HWS with conformal weight $h=\frac{1}{16}$. To find the corresponding null vector we proceed as in the NS case

$$L_{-1}^2|\sigma\rangle=\tfrac{3}{4}b_{-2}b_0|\sigma\rangle\ , \tag{46a}$$

$$L_{-2}|\sigma\rangle=b_{-2}b_0|\sigma\rangle\ , \tag{46b}$$

$$(L_{-2}-\tfrac{4}{3}L_{-1}^2)|\sigma\rangle=|\text{null}\rangle\equiv 0\ . \tag{47}$$

Again (47) is the vector that makes the Kac determinant vanish at the second level.[12,13]

On the plane, the operators L_N generate the (analytic) conformal transformations

$$z\to z+\epsilon_N z^{N+1}\ . \tag{48}$$

Hence, the fact that L_{-1} does not annihilate the vacuum states $|\sigma(\pm)\rangle$ by (43b) implies that correlation functions in the vacuum $|\sigma\rangle$ are *not* translationally invariant on the plane.

By using Eqs. (22), (27), (30a), (30b), and (41) and following the same steps as for the NS correlation function [Eq. (40)], we find for the R sector

$$_P\langle\sigma(\pm)|\psi_R(z)\psi_R(z^1)|\sigma(\pm)\rangle_P = \frac{1}{2\pi}\frac{1}{z-z^1}\frac{1}{2}\left[\left[\frac{z}{z^1}\right]^{1/2}+\left[\frac{z^1}{z}\right]^{1/2}\right]\ . \tag{49}$$

Hence, correlation functions in the vacuum of the Ramond sector are double valued, there is a branch cut in the complex plane.

The fact that $|\sigma(\pm)\rangle$ are eigenstates of L_0 with eigenvalue $\frac{1}{16}$ can be seen as follows.

On the plane, the holomorphic component of the energy momentum tensor is

$$T(z)=-\tfrac{1}{2}\psi(z)\partial_z\psi(z)\ , \tag{50}$$

and in terms of the virasoro generators

$$T(z)=\frac{1}{2\pi}\sum_N L_N z^{-N-2}\ . \tag{51}$$

[The factor $1/2\pi$ differs from the normalization given in the references above, and it arises from the normalization

$2\pi/L^2$ on the strip, Eq. (28)]. Defining the normal-ordered expectation value

$$\langle\sigma|T(z)|\sigma\rangle=\lim_{z\to z^1}-\tfrac{1}{2}\left[\partial_{z^1}\langle\sigma|\psi(z)\psi(z^1)|\sigma\rangle-\frac{1}{2\pi}\frac{1}{(z-z^1)^2}\right] \tag{52}$$

by using the correlation function (49), we find

$$\langle\sigma|T(z)|\sigma\rangle=\frac{1}{2\pi}\frac{1}{16z^2}\ , \tag{53}$$

showing the $L_0|\sigma(\pm)\rangle=\frac{1}{16}|\sigma(\pm)\rangle$. Repeating the calculation for the NS sector using Eq. (40) we see that L_0 annihilates the vacuum in the NS sector. Proceeding in the same way as in Refs. 12 and 13 the states $|\sigma(\pm)\rangle$ are constructed as

$$|\sigma(\pm)\rangle=\lim_{z\to 0}\sigma_{\pm}(z)|0\rangle\ , \tag{54}$$

with $\sigma_{\pm}(z)$ being primary-field operator of dimension $\frac{1}{16}$, and $|0\rangle$ is the vacuum of the NS sector (annihilated by L_0).

Therefore we see that there are two operators of the same dimension ($\frac{1}{16}$) corresponding to a linear combination of order and disorder operators to be found later. Since, in general, scaling operators are products of primary-field functions of z and $\bar{z}$,

$$\sigma_{\pm}(z,\bar{z})=\sigma_{\pm}(z)\bar{\sigma}_{\pm}(\bar{z})\ , \tag{55}$$

with physical scaling dimension $\frac{1}{8}$. [The field $\bar{\sigma}(\bar{z})$ corresponds to the mapping on the plane of the field $\bar{\sigma}(x^+)$]. The order and disorder operators of the Ising model will be shown to be linear combinations of the above $\sigma_{\pm}$. Our aim is to understand these operators better.

Going back to Eqs. (18) and (19), we see that the order and disorder operators are nonlocal in terms of the fermionic fields. This situation is in the sense similar to the case of a free massless *Dirac* fermion. This theory has conformal anomaly $c=1$ and an infinite number of primary-field operators that can be constructed as nonlocal functions of bilinears of the Dirac fields.[17] This construction is better understood by using the bosonization mapping.[17,19,20] In terms of the associated bosonic fields these primary-field operators are easy to construct and are local. This bosonization procedure has been used in Refs. 7, 9, and 10 to compute correlation functions in the doubled (Askhin-Teller) Ising model as mentioned in the Introduction.

Since a Majorana fermion can be written as a combination of a Dirac fermion and its charge conjugate, and a Dirac fermion can be bosonized, we now review the properties of massless Dirac fermions and the construction of the Majorana counterpart. Another fundamental reason for doubling the number of Majorana fermions is that the double degeneracy of the ground state in the Ramond sector cannot be resolved since there is no other quantum number available. A Dirac fermion has a fermion charge associated to it that may be used to identify degenerate states.

Dirac and Majorana fermions

Dirac fermions obey the same equations of motion as in Eqs. (10a) and (10b), but they are complex; therefore, they have two degrees of freedom. When the mass vanishes, the right- and left-handed components are decoupled. The right-handed component of a Dirac spinor is expanded as in (22),

$$\chi_R(x^-)=\frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty}a_n e^{-2\pi i n(x^-/L)}\ ,$$

with

$$\{a_n^\dagger,a_{n'}\}=\delta_{n,n'},\quad \{a_n,a_{n'}\}=\{a_n^\dagger,a_{n'}^\dagger\}=0 \tag{56}$$

and $a_n^\dagger\neq a_{-n}$. With the spinor χ we can construct *two* Majorana spinors as

$$\psi_1=\frac{1}{\sqrt{2}}(\chi+\chi^\dagger)\ , \tag{57a}$$

$$\psi_2=\frac{-i}{\sqrt{2}}(\chi-\chi^\dagger)\ , \tag{57b}$$

where in the Majorana representation (13) $\chi^\dagger$ is the charge-conjugate of χ. Identifying ψ_1 with the Ising spinor ψ in Eqs. (10) and (11) (of course this choice is arbitrary—we could have chosen ψ_2, but these choices are equivalent) and comparing (55) and (22),

$$b_n=\frac{1}{\sqrt{2}}(a_n+a_{-n}^\dagger)\ . \tag{58}$$

Dirac fermions have a charge degree of freedom. This charge is conserved, and it is given by the normal-ordered expression (equivalent to the usual $:\chi^\dagger\chi:$)

$$Q_R=\tfrac{1}{2}\int_{-L/2}^{L/2}[\chi_R^\dagger(x),\chi_R(x)]dx\ . \tag{59}$$

Since we are interested in Majorana fermions with periodic and antiperiodic BC's let us study Dirac fermions in both cases.

(a) *Antiperiodic* BC (NS). The expansion in (55) is in terms of $n=\pm\frac{1}{2},\pm\frac{3}{2},\ldots$. The spectrum is symmetric around zero energy, the ground state of the right-handed sector has, as usual, all negative energy levels ($n<0$) filled up. Therefore,

$$\langle 0|Q_R|0\rangle=\tfrac{1}{2}\sum_n\langle 0|(a_n^\dagger a_n-a_n a_n^\dagger)|0\rangle=0\ , \tag{60}$$

because the spectrum is symmetric and n is a half-odd integer (no $n=0$ mode). A short calculation shows that the correlation function $\langle 0|\chi_R^\dagger(x^-)\chi_R(y^-)|0\rangle$ is given by Eq. (36) (on the strip $-L/2\le x\le L/2$) and Eq. (40) on the plane.

(b) *Periodic* BC (R). Now the expansion in Eq. (55) is in terms of $n=0,\pm1,\pm2,\ldots$. Again, the spectrum is symmetric but now there is a state with zero energy. The ground state is double degenerate, since all $n<0$ are filled, but the $n=0$ state may be empty or occupied with the same energy.

In this situation with a zero mode we have

$$\langle 0|Q_R|0\rangle=\tfrac{1}{2}\langle 0|(a_0^\dagger a_0-a_0 a_0^\dagger)|0\rangle\ , \tag{61a}$$

$$\langle 0|Q_R|0\rangle = \begin{cases} +\frac{1}{2} \text{ if } n=0 \text{ is occupied} \\ -\frac{1}{2} \text{ if } n=0 \text{ is empty} , \end{cases} \tag{61b}$$

where again the $n \neq 0$ modes cancel in the expectation value since the spectrum is symmetric.

The correlation function can be computed as before, and mapping to the plane by using Eqs. (38) and (39) we find

$$\langle \pm|\chi_R^\dagger(z)\chi_R(z^1)|\pm\rangle = \frac{1}{2\pi}\frac{1}{z-z^1}\left[\frac{z}{z^1}\right]^{\pm 1/2} \tag{62}$$

with $|\pm\rangle$ the ground state with charge $\pm\frac{1}{2}$, i.e., the $n=0$ mode is occupied (+) or empty (−), respectively. Because $|+\rangle = a_0^\dagger|-\rangle$ and $a_0^2 = (a_0^\dagger)^2 = 0$, we observe that

$$\langle \pm|\mp\rangle = 0 ,$$

$$\langle \pm|\chi_R^\dagger(z)\chi_R(z^1)|\mp\rangle = 0 .$$

Constructing the states

$$|0^\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle \pm |-\rangle)$$

and using ψ_1 in Eq. (57a) as the Ising spinor with periodic BC, it is straightforward to show that the correlation function

$$\langle 0^\pm|\psi_1(z)\psi_1(z^1)|0^\pm\rangle_p$$

is given by Eq. (49). (The same expression is obtained for ψ_2.) Hence, we reproduce *all* the properties and correlation functions of the Majorana fermions in both sectors by this procedure of writing a Majorana as a linear combination of Dirac. The advantages of going through the Dirac fermions, however, are many.

First, we recognize that the "cut" in the plane in the correlation function (49) is now understood as arising from the presence of a "fractional" charge ($Q = \pm\frac{1}{2}$) at the origin. The "vacuum" with this charge is not $SL(2,C)$ invariant (in particular, it is not translational invariant on the plane). The second advantage resides in the fact that Dirac fermions can be bosonized.[19,20]

As was argued in the discussion preceding Eq. (54) the ground state in the Ramond sector is obtained by applying a primary-field operator to the ground state in the NS sector. In the language of Dirac fermions, this operator has to change the charge of the vacuum by $\pm\frac{1}{2}$, i.e., change the BC's on the strip from antiperiodic to periodic. We are interested in the Ising (Majorana) fermions, not the Dirac fermions. To understand the relation between the vacua of the two theories, we notice that

$$\int_{-L/2}^{L/2}\left[\tfrac{1}{2}\psi_{1R}(x)\left[-i\frac{\partial}{\partial x}\psi_{1R}(x)\right]+\tfrac{1}{2}\psi_{2R}(x)\left[-i\frac{\partial}{\partial x}\psi_{iR}(x)\right]\right]dx = \int_{-L/2}^{L/2}\tfrac{1}{2}{:}\chi_R^\dagger(x)(-i\overleftrightarrow{\partial}_x)\chi_R(x){:}dx . \tag{63}$$

That is, the sum of the Hamiltonian for the two Majorana fermions constructed from χ and $\chi^\dagger$ [Eqs. (57a) and (57b)] is the normal-ordered Hamiltonian for the Dirac fermion χ. Therefore, as long as the boundary conditions on χ *do not mix* ψ_1 and ψ_2, the Hilbert space for the Dirac problem is a tensor product corresponding to the two Majoranas.

Certainly, periodic or antiperiodic BC's on χ keep ψ_1 and ψ_2 as independent fields [see discussion of BC's after Eq. (86)]. It is also straightforward to show that

$$\tfrac{1}{2}[\chi^\dagger(x),\chi(x)] = i\psi_1(x)\psi_2(x) ; \tag{64}$$

therefore, in the R sector [see Eq. (59)]

$$\langle \pm|Q|\pm\rangle = \langle \pm|ib_0d_0|\pm\rangle = \pm\tfrac{1}{2} , \tag{65}$$

with b_0 (d_0) the zero mode of ψ_1 (ψ_2) in the R sector. Since the vacuum states $|\pm\rangle$ are tensor products of the vacuum states for ψ_1 and ψ_2, the two possible values of the vacuum charge in (65) lead to the conclusions that the vacuum states for ψ_1 and ψ_2 are doubly degenerate $|\sigma_{1,2}(\pm)\rangle$ as proposed in Eq. (42b) and that the states $|\pm\rangle$ are tensor products of combinations of $|\sigma(\pm)\rangle$. This degeneracy of the Ising (Majorana) cannot be resolved since there is no "charge" associated to one Majorana fermion. We suggest that this degeneracy corresponds to the order and disorder self-duality property of the Ising model at the critical point, this will be clearly seen later [see discussion after Eqs. (86)]. As we argued before, primary-field operators creating these vacuum states in the Ramond sector out of the Neveu-Schwartz vacuum have conformal weight $\Delta = \frac{1}{16}$, hence, scaling dimension $\Delta + \bar{\Delta} = \frac{1}{8}$, ($\Delta - \bar{\Delta} = 0$). These operators are called "spin fields" in the literature.[21]

These operators are the square root of the operators that create the $|\pm\rangle$ states out of the NS vacuum for a *Dirac* field, since this vacuum is a tensor product of two Majoranas. These spin fields in the Dirac case can be constructed by bosonization as shown below.

IV. BOSONIZATION CORRELATION FUNCTIONS AND THE GAUSSIAN MODEL

The bosonization technique used in Refs. 9 and 10 are based upon those of Mandelstam and Coleman.[19] However, this approach has many drawbacks. In particular, line integrals are ill-defined and do not allow a natural splitting into holomorphic and antiholomorphic contributions to correlation functions precisely because of the surface terms in the line integrals. More importantly, this bozonization procedure does not lend itself to an understanding of the boundary conditions of the Fermi fields neglecting completely the relation between the boundary conditions and the charge quantum numbers of the vacuum discussed in the preceding section. In particular, it misses altogether the charge-raising operator. Here we use the more appropriate scheme of Banks *et al.*[20] that departs significantly from that used in Refs. 9 and 10.

In Appendix we have summarized the bosonization relations that are relevant to our treatment.[20] There are in the literature a number of references that the reader may consult for a more thorough treatment.[19,20]

As discussed in Appendix A, the bosonization is carried out in terms of a free massless bosonic field; therefore, all the operators in the interaction picture.[20] It is precisely the zero mode of this field that is the charge-raising operator.

As shown in Appendix A, the normal-ordered density for Dirac fermions (with normalization 1) is

$$:\chi_R^\dagger(x)\chi_R(x):=\frac{1}{\sqrt{\pi}}\frac{d\phi_R}{dx}\ , \tag{66}$$

and a similar relation holds for the left-handed density in terms of ϕ_L.

The normal-ordered Hamiltonian density is

$$:\chi_R^\dagger\left[-i\frac{\overleftrightarrow{\partial}}{\partial x}\right]\chi_R:+:\chi_L^\dagger\left[i\frac{\overleftrightarrow{\partial}}{\partial x}\right]\chi_L:=\left[\frac{\partial\phi_R}{\partial x}\right]^2+\left[\frac{\partial\phi_L}{\partial x}\right]^2 . \tag{67}$$

Equation (67) is the well-known result that a free massless Dirac fermion is equivalent to a free massless boson in one space dimension, both theories having conformal anomaly $c=1$. But, in fact, this simple equivalence in conjunction with Eq. (63) (and its counterpart for the left-handed component) also implies that two decoupled Ising models (each with $c=\frac{1}{2}$) are equivalent to the Gaussian model at a particular value of the temperature ($K=1/\pi$ in the notation of Kadanoff and Brown,[11] see normalization 2 in Appendix A). But, two decoupled Ising models also correspond to the Ashkin-Teller model at the point where the four-spin coupling (that couples the two Ising models) vanishes.

From the commutation relations (A4) in appendix A, we see that the state with charge $Q_R=\pm\frac{1}{2}$ in Eq. (62) is

$$|\pm\rangle=e^{\pm i\sqrt{\pi}\phi_{0R}}|0\rangle \tag{68}$$

with $|0\rangle$ the NS vacuum (see Appendix A). Therefore, the states

$$|0^\pm\rangle=\frac{1}{\sqrt{2}}(|+\rangle\pm|-\rangle)$$

can be written as

$$|0^\pm\rangle=\lim_{z\to 0}\frac{1}{\sqrt{2}}[S^+(z)\pm S^-(z)]|0\rangle \tag{69a}$$

and

$$S^\pm(z)=e^{\pm i\sqrt{\pi}\phi_R(z)}\ , \tag{69b}$$

with $\phi(z)$ the boson field on the plane[17] (see Appendix A).

As we argued above, the operators $S^\pm(z)$ are a combination of products of order and disorder operators of the two Ising models associated to the two Majorana spinors ψ_1 and ψ_2, since the state $|0^\pm\rangle$ is a tensor product of the ground states of these two Ising models in the R sector. It is straightforward to check once again that the correlation functions of ψ_1 and ψ_2 in the states $|0^\pm\rangle$ is given by Eq. (49). And, if we take the expectation value of the component $T(z)$ for the Ising model 1 (in terms of ψ_1) in this state, clearly we obtain Eq. (53) showing that $S^\pm(z)$ has conformal weight $\Delta=\frac{1}{16}$, and certainly the same result arises if we take ψ_2, since $S^\pm$ are *products* of the order and disorder operators of both. Since the energy-momentum tensor in the Dirac case is T_1+T_2 (the sum of the one for ψ_1 and ψ_2), $S^\pm$ have conformal weight $\Delta=\frac{1}{8}$ in the Dirac theory. The results of Appendix A and the relations (57a) and (57b) allow us to write (see Appendix A for the notation)

$$\psi_{1R}=\sqrt{2/L}\cos[\sqrt{\pi}(\phi+\tilde{\phi})]\ , \tag{70a}$$

$$\psi_{2R}=\sqrt{2/L}\sin[\sqrt{\pi}(\phi+\tilde{\phi})]\ . \tag{70b}$$

(The left-handed fermions have a similar expression in terms of $\phi-\tilde{\phi}$.) These are the expressions obtained by Kadanoff and Brown (in their convention, the above spinors correspond to the $c_+^{1,2}$ with $K=1/\pi$).

Although we know that the operators $S^\pm$ in Eq. (69b) correspond to a linear combination of products of order and disorder of the two Ising models, we would like to know the expression for the product of order ($\sigma_1\sigma_2$) and disorder ($\mu_1\mu_2$) in terms of $S^\pm$.

To resolve this question, however, we must go back to the theory defined on the lattice and, in particular, Eqs. (15)–(19). We now follow the steps of Zuber and Itzykson.[9] Using the anticommutation relation (9a) it is easy to show that (the label 1 refers to one Ising model)

$$\mu_1(j)=\prod_{k=1}^{j-1}[-2ia\psi_R^1(k)\psi_L^1(k)]\ ; \tag{71a}$$

therefore,

$$\mu_1(j)\mu_2(j)=\prod_{k=0}^{j-1}[2a\psi_R^1(k)\psi_R^2(k)]\prod_{k=0}^{j-1}[2a\psi_L^1(k)\psi_L^2(k)]\ , \tag{72}$$

using the fact that $(\psi)^2=1/2a$ for ψ^1 and ψ^2 the above equation can be written as

$$\mu_1(j)\mu_2(j)=\exp\left[-i\pi\sum_{k=0}^{j-1}[ia\psi_R^1(k)\psi_R^{(2)}(k)]\right]\times\exp\left[-i\pi\sum_{k=0}^{j-1}[ia\psi_L^1(k)\psi_L^2(k)]\right]\ . \tag{73}$$

At this stage we would like to take the continuum limit, but we notice that the above expression is real, and the sign in the exponential can either be positive or negative with the same result, arising from the ambiguity in Eqs. (15) and (16). We want to ensure that the reality of $\mu_1\mu_2$ survives in the continuum limit, and for this reason we symmetrize with respect to both signs in (73). The reason for this seemingly artificial procedure is simple. In taking the continuum limit and normal ordering the corresponding operators in order to remove the short-distance singularities, we cannot guarantee that the result is real. Symmetrization ensures this property. Taking the continuum limit, using Eqs. (64) and (66) and the results of Appendix

A with the normalization 2 conventions defined there, we find

$$\psi \sum_{k=0}^{J-1} ia\psi_R^1\psi_R^2 \to \tfrac{1}{2}\int_0^{x-a} 2\frac{d\phi_R}{dx}dx = [\phi_R(x-a)-\phi_R(0)] \ . \quad (74)$$

Dropping the $\phi(0)$ term, since we are interested in the x dependence, the continuum limit of the symmetrized expression for $\mu_1\mu_2$ is

$$\mu_1(x)\mu_2(x) = :\cos\phi(x-a): \ , \quad (75)$$

with $\phi=\phi_R+\phi_L$ (see Appendix A). In (75) we have normal ordered the exponential and dropped short- and long-distance cutoff constants multiplying (75).

We are now in position to obtain the results of Refs. 9–11. Using the result of the appendixes and the lattice expressions of Sec. II, it is a matter of straightforward algebra to find the correspondence between products of order and disorder operators of the two Ising models and operators in the Gaussian model (at $K=1/\pi$),

$$\sigma_1(x)\mu_2(x) = \lim_{a\to 0}\sqrt{a}\,[\psi_R^1(x)+\psi_L^1(x)]\mu_1(x)\mu_2(x) = :\cos\tilde{\phi}(x): \ , \quad (76)$$

$$\sigma_1(x)\sigma_2(x) = \lim_{a\to 0} -i\sqrt{a}\,[\psi_R^2(x)+\psi_L^2(x)]\sigma_1(x)\mu_2(x) = :\sin\phi(x): \ . \quad (77)$$

[The factor $(-i)$ in (77) is to render $\sigma_1\sigma_2$ Hermitian.] These results, in conjunction with Eqs. (57a), (57b), and (74), are in agreement with the results and tentative identifications of Ref. 11. The above results differ from those of Kadanoff and Brown by a trivial shift $\phi\to\phi+\pi/2$, $\tilde{\phi}\to\tilde{\phi}$, which is certainly a symmetry of the Gaussian model.

The energy operators are conjugate to $T-T_c$; from the continuum Hamiltonian [Eq. (11a)] we read

$$\epsilon_{(\alpha)} = i\psi_R^{(\alpha)}\psi_L^{(\alpha)} \ , \quad (78)$$

where $\alpha=1$ or 2 refers to the Ising model 1 or 2.

Using the results of the appendix, we find

$$\epsilon^{(1)}\epsilon^{(2)} = \partial_\mu\phi\partial^\mu\phi \ , \quad (79)$$

$$\frac{\epsilon^{(1)}+\epsilon^{(2)}}{2} = :\cos[2\phi(x)]: \ , \quad (80)$$

$$\frac{\epsilon^{(1)}-\epsilon^{(2)}}{2} = -:\cos[2\tilde{\phi}(x)]: \ . \quad (81)$$

Notice that $\epsilon^{(1)}+\epsilon^{(2)}$ is the energy operator of the Ashkin-Teller model at the decoupling point ($K=1/\pi$). The above results agree with those proposed in references.[22] From Eqs. (75)–(81) we infer the relation between the duality transformations in the Ising models and those of the Gaussian model.

Writing

$$\phi(x) = \phi_R(x)+\phi_L(x) \ ,$$

$$\tilde{\phi}(x) = \phi_R(x)-\phi_L(x) \ ,$$

the duality transformations are for the Ising models

$$\sigma_1\to\mu_1;\ \epsilon^{(1)}\to-\epsilon^{(1)} \ ,$$

$$\sigma_2\to\mu_2;\ \epsilon^{(2)}\to-\epsilon^{(2)} \ ,$$

and for the Gaussian fields $\phi_L\to-\phi_L$, $\phi_R\to-\phi_R-\pi/2$.

The transformations in the second-set are certainly symmetries of the Gaussian model, applying these transformations to one of the Ising models (e.g., number two) reproduces the relations between Eqs. (76) and (77) and Eqs. (80) and (81). This is a slight modification of the rule proposed by Kadanoff and Brown (differing from theirs by the factors $\pm\pi/2$), but essentially with the same physical consequences. The exponentials of ϕ and $\tilde{\phi}$ create spin waves and vortex excitations (see Appendix A). Therefore, again bosonization provides a substantive proof of Kadanoff and Brown's conjectures.[11] We are now in a position to compute the correlation functions of the $\mu_1\mu_2$, $\sigma_1\sigma_2$, etc. In the Gaussian theory, these are in fact very simple, being correlation functions of spin waves or vortex operators. But now we assert that since the vacuum of the Dirac theory is a *direct* tensor product of the vacua of the two uncoupled Majorana theories, and the Dirac Hamiltonian and spectrum map one to one onto those of the Gaussian model, we can now obtain the correlations of σ_1,μ_1, etc., by simply taking the square roots of those of $\sigma_1\sigma_2$ and $\mu_1\mu_2$ above. In fact, this happens only because in the vacuum sector of the Dirac theory, the boundary conditions are independent for both Majoranas (antiperiodic); therefore, the Hilbert space is a tensor product of those of both Ising models. Notice that this feature, *a priori* very intuitive, cannot be appreciated with the bosonization prescription used in Refs. 9 and 10, since there the issue of boundary conditions cannot be studied.

This is, however, a crucial but unnoticed point in the above references. We now proceed to take the square roots on the plane,

$$\langle 0|\sigma_1(r)\sigma_1(r')|0\rangle = \langle 0|\mu_1(r)\mu_2(r')|0\rangle = \frac{1}{\sqrt{2}}\frac{1}{|r-r'|^{1/4}} \ , \quad (82a)$$

$$\langle 0|\sigma_1(r_1)\sigma_1(r_2)\sigma_1(r_3)\sigma_1(r_4)|0\rangle = \left[\left|\frac{r_{13}r_{24}}{r_{12}r_{23}r_{34}r_{14}}\right|^{1/2} + 2\leftrightarrow 3 + 1\leftrightarrow 4\right]^{1/2} \ , \quad (82b)$$

where $r_{ij}=|r_i-r_j|=[(Z_i-Z_j)(\bar{Z}_i-\bar{Z}_j)]^{1/2}$.

We have chosen the positive sign of the square root; this arbitrary choice reflects the overall normalization of the fields. We define the fields to be real. It is straightforward to check that this choice does not introduce any ambiguities in higher-order correlation functions.

The above results are certainly not new, they coincide with those of Refs. 7, 9, and 10. However, identification [Eq. (75)] allows us to compute the two- and four-point correlation functions of disorder operators; a short calculation yields Eqs. (82a) and (82b), respectively, showing explicitly the self-duality property. However, the machinery that we set up allows us to do more than just reproduce the above results; in fact, we can compute any mixed correlation functions. For example, we note that

$$\langle 0|\sigma_1(r_1)\sigma_2(r_1)\epsilon_1(r_2)\sigma_1(r_3)\sigma_2(r_3)|0\rangle = {}_1\langle 0|\sigma_1(r_1)\epsilon_1(r_2)\sigma_1(r_3)|0\rangle_1 \times {}_2\langle 0|\sigma_2(r_1)\sigma_2(r_3)|0\rangle_2 \quad (83)$$

with $|0\rangle_{1,2}$ the vacua (in the NS sector) of the independent Ising models (and $|0\rangle = |0\rangle_1|0\rangle_2$). Using Eqs. (77), (78), (57), and (82a), and the results of the appendixes, we find

$$\begin{aligned} {}_1\langle 0|\sigma_1(r_1)\epsilon_1(r_2)\sigma_1(r_3)|0\rangle_1 &= \frac{1}{2\sqrt{2}}\frac{|r_1-r_3|^{3/4}}{|r_1-r_2||r_2-r_3|} \\ \langle 0|\sigma_1(r_1)\sigma_2(r_1)\psi_{2R}(r_2)\sigma_1(r_3)\mu_2(r_3)|0\rangle &= {}_2\langle 0|\sigma_2(r_1)\psi_2(r_2)\mu_2(r_3)|0\rangle_2\, {}_1\langle 0|\sigma_1(r_1)\sigma_1(r_3)|0\rangle_1\ ; \end{aligned} \quad (84)$$

hence,

$${}_2\langle 0|\sigma_2(r_1)\psi_{2R}(r_2)\mu_2(r_3)|0\rangle_2 = \frac{1}{2}\frac{1}{(Z_{23}Z_{12})^{1/2}}\frac{(Z_{13})^{3/8}}{(\bar{Z}_{13})^{1/8}} . \quad (85)$$

From Eqs. (82a), (84), and (85) we can read the operator product expansions

$$\sigma_1(r_1)\sigma_1(r_3) \underset{r_1\to r_3}{\sim} \frac{1}{\sqrt{2}}\left[\frac{1}{|r_1-r_3|}\right]^{1/4} + \frac{1}{2\sqrt{2}}|r_1-r_3|^{3/4}\epsilon_1(r_3)+\cdots , \quad (86a)$$

$$\psi_2(r_2)\mu_2(r_3) \underset{r_2\to r_3}{\sim} \frac{1}{\sqrt{2}}\left[\frac{1}{Z_{23}}\right]^{1/2}\sigma_2(r_3)+\cdots , \quad (86b)$$

$$\sigma_2(r_1)\mu_2(r_3) \underset{r_1\to r_3}{\sim} \frac{1}{2}\frac{(Z_{13})^{3/8}}{(\bar{Z}_{13})^{1/8}}\psi_2(r_2)+\cdots . \quad (86c)$$

Equations (86) are part of the main results of this paper. In fact, these equations prove that correlation functions computed in the Gaussian theory and "projected" to one Ising model in the way shown above lead to the correct OPE of the Ising model.[8,12,13] This, in fact, guarantees that the correlation functions constructed in this way do satisfy the differential equations obtained from the null vectors.[12,13] We would like to stress that in order to arrive at Eqs. (86) we need all the correlations [Eqs. (82)–(85)], and Eq. (85), for example cannot be obtained with the bosonization approach used in Refs. 9 and 10.

We are now in a position to determine the operator content of the ground state in the Ramond sector. First, we realize that if we generically call $|\sigma\rangle$ this state (it is double degenerate) by applying clustering decomposition to (49) when $Z \gg Z^1$, we find

$$\langle \sigma(\pm)|b_0|\sigma(\pm)\rangle = \pm\frac{1}{\sqrt{2}} ,$$

as proposed in Eq. (42b). In the above expression, b_0 is the zero mode of the Majorana fermion in the Ramond sector. From the OPE [Eqs. (86b) and (86c)], we find that

$$\langle \mu|\psi(r)|\mu\rangle = \langle \mu|b_0|\mu\rangle = 0$$

with

$$|\mu\rangle = \lim_{z,\bar{z}\to 0}\mu(z,\bar{z})|0\rangle ,$$

and similarly for σ, with μ and σ the order and disorder operators. Therefore, from (86b) we find

$$|\sigma(\pm)\rangle = \frac{1}{\sqrt{2}}(|\mu\rangle \pm |\sigma\rangle) .$$

A duality transformation changes $|\mu\rangle \leftrightarrow |\sigma\rangle$, or $|\sigma(+)\rangle \leftrightarrow |\sigma(-)\rangle$.

The next question that we must address is that of the boundary conditions. Combining two Majorana spinors (ψ_1,ψ_2) to form a Dirac spinor χ yield four possible BC for χ that do not mix the real (ψ_1) and imaginary (ψ_2) parts. These are (NS,NS), (R,R), (R,NS), and (NS,R), respectively. The first two correspond to the NS (translational invariant) and R for the Dirac fermion. But the last two imply $\chi(L) = \pm\chi^\dagger(-L)$; in the bosonized formulation, this corresponds to the twisted sector of the boson $\phi(L) = -\phi(L)$ (antiperiodic BC).[22] This twisted sector is also a sector of the Ashkin-Teller model,[23] and hence, the identification of the "twist" fields[22,23] of the Ashkin-Teller with the "spin" fields[21] of the Ising model.

The correlation functions being computed in the translational invariant vacuum (NS) can always be factored out since this vacuum is a tensor product of the NS vacua for both Ising models. Although the simplicity with which we can compute the n-point correlation functions seems to justify the considerable effort to reach this simplicity, there certainly is the genuine question of whether one can write down a *bosonic* theory for the critical Ising model without resorting to the doubling of degrees of freedom. This can be achieved by bosonizing the Majorana spinors by using Eq. (57a) and bosonizing the Dirac spinors using the results of the appendixes.

Using the results of Appendix A (with normalization 1) we find

$$\begin{aligned} :\mathcal{H}: &= \lim_{x\to y}\left[\frac{-i}{2}\psi_R(x)\partial_y\psi_R(y)+\frac{i}{2}\psi_L(x)\partial_y\psi_L(y)-\frac{1}{2\pi(x-y)^2}\right] \\ &= \frac{1}{2}\left\{\left[\frac{\partial\phi_R}{\partial x}\right]^2+\left[\frac{\partial\phi_L}{\partial x}\right]^2-\frac{2\pi}{L^2}[:\cos(2\sqrt{4\pi}\phi_R):-:\cos(2\sqrt{4\pi}\phi_L):]\right\} . \end{aligned} \quad (87)$$

The expression given by (87) has been recently *proposed* in Refs. 15 and 16; however, we showed that, in fact, it naturally arises after bosonization of the Majorana fermion. The double dots in Eq. (87) refer to normal ordering for *free* bosons, hence the above Hamiltonian must be understood in the *interaction picture.*

Unlike Ref. 17 we have taken into account the dimensional units shown in $\mathcal{H}$ as the factor $2\pi/L^2$ in front of the cosines. This is in fact arbitrary and can be changed by a different normal-ordering prescription.

There are *no coupling constants* in Eq. (87) as is natural, arising from a free Majorana fermion. We assert that despite the interaction terms in Eq. (87), the above is a fixed-point Hamiltonian.

One would hope to compute correlation functions in the bosonic theory [Eq. (87)] by invoking a perturbative expansion in terms of the cosines. This is a nontrivial task, there being no "couplings." The expansion should be summed *exactly.*

In Ref. 15 some correlation functions are computed in a *free* bosonic theory; this we find misleading and incorrect. In fact, we find after some algebra that the addition of the second Majorana fermion given by Eq. (57b) yields

$$:\mathcal{H}_1:+:\mathcal{H}_2:=\left[\frac{\partial\phi_R}{\partial x}\right]^2+\left[\frac{\partial\phi_L}{\partial x}\right]^2 =\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2+\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2, \qquad (88)$$

thus reproducing the Gaussian model. Correlation functions must be computed as described above in the doubled Ising model, appropriately projecting the results for one Ising model. Equation (88) also indicates that the Ashkin-Teller model, at the point where the four-spin coupling vanishes, is equivalent to the Gaussian model at $K=1/\pi$ (see normalization 2 in Appendix A). In particular, we can say that the procedure of computing the correlation functions in the doubled Ising model, relating those to the ones in one Ising model as shown in Eqs. (82)–(85), effectively "sums up" the series in terms of the cosines in (87).

The reader should not be confused: the Hamiltonian in Eq. (87) is *not* of the sine-Gordon type, since the relation between ϕ, and ϕ_R and ϕ_L are *nonlocal* (see Appendix A).

V. SUMMARY AND FURTHER QUESTIONS

The critical Ising model has been studied with particular attention given to its operator content, to calculating in a simple framework the correlation functions of these operators, and to provide a proof of Kadanoff and Brown's identifications and conjectures. By looking at the ground state in both sectors, antiperiodic and periodic, we showed that the ground state of the periodic sector was doubly degenerate and that this degeneracy is a consequence of the self-duality of the Ising model at the critical point. In considering two uncoupled Ising models (free Majorana fermions) the theory was found to be equivalent to the Ashkin-Teller model at the point when the four-spin coupling vanishes. The two Majorana spinors were combined to yield a free Dirac fermion. The ground state of this free Dirac (massless) theory in the periodic sector is doubly degenerate with fermionic charge (fermion number) $Q=\pm\frac{1}{2}$. We identified the operator that raises or lowers the charge of the Dirac vacuum by $\frac{1}{2}$ as a combination of *products* of order and disorder operators of the two underlying uncoupled Ising models.

Bosonization of the Dirac fermion allowed us to write the products of order or disorder operators corresponding to the two Ising models as exponentials of a free Gaussian boson, and provided the equivalence between the Ashkin-Teller model at the decoupling point to the Gaussian model at $K=1/\pi$.

By taking the continuum limit of the original lattice Hamiltonian, we constructed the product of order, disorder, and energy operators and identified all the operators proposed by Kadanoff ad Brown, proving their conjectures in the identification.

We have shown that the process of doubling the Ising model and taking square roots does reproduce *all* the operator product expansions predicted by conformal invariance, thereby ensuring that the correlation functions obtained from the doubled Ising model by projecting one of them (taking square roots) do obey the corresponding differential equations. This was shown to happen because the translational invariant ground state of the Gaussian model is a tensor product of the antiperiodic ground states of the two Ising models. This is the physical reason that allows the construction of correlation functions of a theory with conformal anomaly $c=\frac{1}{2}$ from those of a theory with $c=1$. We have extended the method to allow the computations of mixed correlation functions of any number of order, disorder, energy, and spinor operators.

We have shown that bosonization of a Majorana fermion gives rise to a bosonic Hamiltonian for the Ising model in terms of spin waves and vortices of a Gaussian model. We have not attempted to study the theory away from the critical point. It would be very interesting to try to extend the results obtained here for $T\neq T_c$, in particular the expectation values for order and disorder operators for $T\neq T_c$ and duality properties.

Note. After this work was completed we became aware of a paper by DiFrancesco, Saleur, and Zuber[24] where some of our results have been obtained. I would also like to thank Y. Goldschmidt for making me aware of the DiFrancesco, Saleur, and Zuber paper and thank Professor R. Shankar for making me aware of a paper by M. Ogilvie [Ann. Phys. (N.Y.) **136**, 273 (1981)] where a similar version of the bosonized Hamiltonian was proposed.

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APPENDIX A: BOSONIZATION

We follow Ref. 20 and bosonize in terms of a free massless bosonic field $\phi(x,t)$ satisfying

$$\Box\phi(x,t)=0 \ . \tag{A1}$$

The solutions are right- and left-going waves $\phi_R(t-x)$ and $\phi_L(x+t)$ with expansions in $-L/2\le x\le L/2$,

$$\phi_R(t-x)=\frac{\phi_{0R}}{\sqrt{\pi}}-\frac{\sqrt{\pi Q_R}}{L}x^- -\frac{i}{\sqrt{\pi}}\sum_{n\neq 0}\frac{\bar{a}_n}{n}e^{-2\pi i n(x^-/L)} \ , \tag{A2}$$

$$\phi_L(t+x)=\frac{\phi_{0L}}{\sqrt{\pi}}+\frac{\sqrt{\pi}}{L}Q_L x^+ -\frac{i}{\sqrt{\pi}}\sum_{n\neq 0}\frac{a_n}{n}e^{-2\pi i n(x^+/L)} \ , \tag{A3}$$

with

$$t\pm x=x\pm, \quad [Q_R,\phi_{0R}]=-[Q_L,\phi_{0L}]=\frac{i}{2} \ , \qquad [\bar{a}_n,\bar{a}_m]=[a_n,a_m]=n\delta_{n+m,0} \ , \tag{A4}$$

the rest of the commutators vanishing.

From ϕ_R and ϕ_L we can form the linear combinations

$$\phi(x,t)=\phi_R(x-t)+\phi_L(x+t) \ , \qquad \tilde{\phi}(x,t)=\phi_R(x-t)-\phi_L(x+t) \ . \tag{A5}$$

$\tilde{\phi}$ is field *dual* to ϕ satisfying

$$\frac{\partial\tilde{\phi}}{\partial x}=-\frac{\partial\phi}{\partial t} \ . \tag{A6}$$

From (A5) and (A6) we see that the relation between ϕ_R and ϕ is nonlocal in terms of $\partial\phi/\partial t$. The fact that ϕ and $\tilde{\phi}$ are associated with spin waves and vortices can be inferred by writing the expansions (A2) on the plane by using the mapping Eq. (38) (ϕ is a scalar dimensionless field). We now give the bosonization rules with two different normalizations to make it easier for the reader to compare with the literature.

(a) *Normalization 1.* Here we follow Ref. 20:

$$\chi_R=\frac{1}{\sqrt{L}}\alpha_R:e^{i\sqrt{4\pi}\phi_R}: \ , \qquad \chi_L=\frac{1}{\sqrt{L}}\alpha_L:e^{-i\sqrt{4\pi}\phi_L}: \ , \tag{A7}$$

with α_R and α_L being Klein factors (cocycles) that ensure the anticommutation of χ_R and χ_L. The charge densities

$$:\chi_R^\dagger(x)\chi_R(x):=\chi_R^\dagger(x)\chi_R(y)-\frac{1}{2\pi i(x-y)}\bigg|_{x=y} =\alpha_R^\dagger\alpha_R\left[\frac{1}{\sqrt{\pi}}\partial_x\phi_R\right] \tag{A8}$$

(and are similar for χ_L in terms of ϕ_L). Since we want the currents not to depend on α_R we impose[20] $\alpha_R^\dagger\alpha_R=1=\alpha_L^\dagger\alpha_L$. The lowest representation for the α's is two dimensional given in terms of two Pauli matrices. We choose them real with

$$\alpha_{R,L}^\dagger=\alpha_{R,L}\alpha_R^2=\alpha_L^2=1$$

$$\{\alpha_R,\alpha_L\}=1 \ .$$

Then,

$$(\alpha_R\alpha_L)^2=-1 \ .$$

Since fermions bilinears of the form $\chi_R^\dagger\chi_L$ always have $\alpha_R\alpha_L$, we can diagonalize this operators by picking the eigenvalue $+i$ arbitrarily.

The normal-ordered Dirac Hamiltonian density is

$$:\mathcal{H}_R:=\chi_R^\dagger(x)[-i\gamma^5\overleftrightarrow{\partial}_y\chi_R(y)]-\frac{1}{2\pi}\frac{1}{(x-y)^2}\bigg|_{x=y} =\left[\frac{\partial\phi_R}{\partial x}\right]^2 \tag{A9}$$

and are similar for the left-handed fermions. Expectation values are given as

$$\langle 0|:e^{i\alpha\sqrt{4\pi}\phi_R(x^-)}::e^{i\beta\sqrt{4\pi}\phi_R(y^-)}:|0\rangle =\frac{\delta_{\alpha\beta}}{\left[-2i\sin\left[\frac{\pi}{L}(x^- -y^- +i\epsilon)\right]\right]^{\alpha\beta}} \ .$$

The "charge neutrality" rule $\alpha=\beta$ arises from the zero modes ϕ_{0R} in (A2) and the commutation relations (A4) (the vacuum $|0\rangle$ is annihilated by Q_R). On the plane

$$i(t\pm x)\to w-(\tau-ix), \quad \bar{w}=\tau+ix \ ,$$

$$Z=e^{2\pi w/L}, \quad \bar{Z}=e^{2\pi\bar{w}/L} \ ,$$

the above correlation function reads[12,13,18]

$$\left[\frac{L}{2\pi}\right]^{\alpha^2}\frac{\delta_{\alpha\beta}}{(Z-Z')^{\alpha\beta}} \ .$$

(b) *Normalization 2.* This is the normalization of Kadanoff and Brown[11]

$$:\mathcal{H}_R:=\frac{1}{2\pi}:\chi_R^\dagger(-i\gamma^5\overleftrightarrow{\partial}_x)\chi_R: \ ,$$

$$\{\chi_R^\dagger(x),\chi_R(y)\}=2\pi\delta(x-y) \ .$$

The bosonization is now in terms of a boson field normalized with

$$[\phi(x),\dot{\phi}(y)]=(\pi i\delta(x-y)) \ ,$$

$$\chi_R(x^-)=\left[\frac{2\pi}{L}\right]^{1/2}:e^{i2\phi_R(x^-)}: \ .$$

The ϕ_R used here is related to that in (A1) by a factor $\sqrt{\pi}$. Now in terms of ϕ_R and ϕ_L,

$$:\mathcal{H}:=:\mathcal{H}_R:+:\mathcal{H}_L:=\frac{1}{\pi}\left[\left[\frac{\partial\phi_R}{\partial x}\right]^2+\left[\frac{\partial\phi_L}{\partial x}\right]^2\right].$$

On the plane (see Ref. 17 for details)

$$\chi_R(Z)=:e^{2i\phi_R(Z)}:,$$

$$\langle 0|\chi_R^\dagger(Z)\chi_R(Z')|0\rangle=\frac{1}{(Z-Z')}.$$

APPENDIX B: FERMION BILINEARS

With the relations Eqs. (57a) and (57b) between the two Majorana fermions ψ_1 and ψ_2 and the Dirac fermion χ, we have the following:

$$:\chi_R^\dagger\chi_R:=\tfrac{1}{2}[\chi_R^\dagger(x),\chi_R(x)]=i\psi_{1R}\psi_{2R}$$

(and similarly for χ_L),

$$\chi_R^\dagger\chi_L=\tfrac{1}{2}[\psi_{1R}\psi_{1L}+\psi_{2R}\psi_{2L}+i(\psi_{1R}\psi_{2L}-\psi_{2R}\psi_{1L})],$$

$$\chi_R\chi_L=\tfrac{1}{2}[\psi_{1R}\psi_{1L}-\psi_{2R}\psi_{2L}+i(\psi_{1R}\psi_{2L}+\psi_{2R}\psi_{1L})].$$

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