Langevin and Fokker-Planck equation with nonconventional boundary conditions for the description of domain-wall dynamics in ferromagnetic systems

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It is shown that domain-wall dynamics in ferromagnetic systems can be described in terms of a Langevin equation for the domain-wall velocity v. A detailed analysis is given of the complex structure of this equation around $v=0$, consequent to the peculiar role played by the domain-wall coercive field, which represents, on the one hand, a threshold for domain-wall movement and exhibits, on the other hand, stochastic fluctuations when the domain wall is in motion. By this analysis, a Fokker-Planck equation with specific boundary conditions is derived and solved in terms of a complete, orthogonal set of eigenfunctions. On this basis, the amplitude probability distribution and autocorrelation function of the v process are calculated and discussed, and their relationship with the Barkhausen effect observed in ferromagnetic materials is considered.

INTRODUCTION

Domain-wall (DW) dynamics in ferromagnetic systems exhibits intrinsic stochastic properties, consequent to the random nature of the perturbations encountered by a DW in its motion. The author has recently shown, through a critical reconsideration of literature results on this subject, that this stochastic behavior can be conveniently described by a Langevin equation.[1-3] This equation derives in a natural way from the well-known linear relationship, confirmed by many experiments,[4-6] between DW velocity v and magnetic field H in metallic systems,

$$k_1 v = H - H_c \equiv \Delta H \; , \tag{1}$$

where k_1 is a known coefficient describing eddy-current damping and H_c is the coercive field experienced by the DW. The internal field H, which includes the applied field H_a and a counterfield of magnetostatic origin H_m, can be expressed, when H_a increases at a constant rate, as

$$dH/dt = dH_a/dt - dH_m/dt = k_2(\langle v \rangle - v) \; , \tag{2}$$

where $\langle v \rangle$ is the DW average velocity and k_2 is another known constant.[1] On the other hand, the random DW interaction processes with the surrounding medium (lattice defects, other DW's) result in stochastic fluctuations of H_c which, as shown by several experiments,[7-9] are approximately describable in terms of a Wiener-Lévy stochastic process.[10] dH_c/dt is consequently a white Gaussian noise and the time derivative of Eq. (1),

$$k_1 dv/dt + k_2(v - \langle v \rangle) = -dH_c/dt \tag{3}$$

turns out to be precisely a Langevin equation for v.

The specific role played by the coercive field H_c, and in particular the fact that it represents a threshold for DW motion, takes however this description away from conventional Langevin models. In fact, the linear relation between v and ΔH expressed by Eq. (1), and consequently Eq. (3), hold only as long as $\Delta H > 0$. When $\Delta H < 0$, the coercive field interaction locks the DW in its position and $v=0$. In these conditions, H_m and H_c do not change in time and, in the place of Eq. (3), we have

$$d\Delta H/dt = dH_a/dt = k_2 \langle v \rangle, \quad \Delta H < 0 \; . \tag{4}$$

Equations (3) and (4) show that a drastic change in the behavior of the system is implied, passing from $\Delta H > 0$ to $\Delta H < 0$, by the presence of the coercive field threshold.

Despite these nontrivial complications, the problem is still amenable to an analytic treatment, when specific assumptions on the stochastic process dH_c/dt are made. In this connection, a case of relevant physical interest—discussed in Ref. 3—is obtained assuming that H_c is a Wiener-Lévy process with respect to the DW position, x. This assumption, which is supported by the results of specific measurements on a single moving DW,[8,9] is expressed by the equation

$$\langle dH_c \rangle = 0, \quad \langle |dH_c|^2 \rangle = 2A\,dx = 2Av\,dt \; , \tag{5}$$

where dH_c is the random increment of H_c corresponding to the small displacement dx, and A is an unknown constant. According to this model, the intensity of H_c time fluctuations is proportional, at each instant, to v. Owing to this fact, the point $\Delta H = 0$ (i.e., $v = 0$) behaves as an impenetrable barrier for the process, which remains confined in the region $\Delta H > 0$ ($v > 0$) and is here fully characterized by Eqs. (3) and (5), Eq. (4) playing no significant role. The Fokker-Planck equation associated with these equations can be analytically solved and a complete statistical characterization of the process is obtained.[3]

The aim of the present paper is to show that there is a further case of interest where the problem can be rigorously treated, namely, when H_c fluctuations depend on the DW dynamic state not through the instantaneous DW velocity v, but only through its mean value $\langle v \rangle$,

$$\langle dH_c \rangle = 0, \quad \langle |dH_c|^2 \rangle = 2A\langle v \rangle dt \ , \tag{6}$$

so that H_c is a Wiener-Lévy process directly with respect to time. In Ref. 2, it is suggested that this model may be useful to characterize the effect of DW internal degrees of freedom, not directly describable in terms of the DW position x. Nonetheless, the behavior of real systems is expected to lie somewhere midway between the two complementary descriptions expressed by Eqs. (5) and (6), and the very fact of obtaining analytic solutions for both of them provides a proper, general basis for a deeper comprehension of DW dynamics in ferromagnetic systems. In particular, as will be further discussed later, these descriptions are expected to clarify the dynamic connection between DW motion and the Barkhausen effect[11,12] in ferromagnetic materials.

The model discussed in this paper is worthy of some attention also from a different viewpoint, as an interesting example of stochastic process which can be rigorously given an analytic description, despite the fact that nonconventional boundary conditions and singular terms in the solutions are obtained as a consequence of the peculiar behavior of the process around $\Delta H = 0$. We will show that this fact has its root in the existence of a variational principle for our problem, which is but a natural generalization of the principle governing the solutions of the Sturm-Liouville problem in the theory of ordinary differential equations.[13]

THE MODEL

Let us rewrite Eqs. (3), (4), and (6) in terms of the dimensionless variables

$$h = \Delta H / k_1 \langle v \rangle, \quad h_c = -H_c / k_1 \langle v \rangle \tag{7}$$

and of the time constant, τ, and the dimensionless parameter, a, defined as

$$\tau = k_1 / k_2, \quad a = k_1 \sqrt{\langle v \rangle / A\tau} \ . \tag{8}$$

We obtain

$$dh/dt - 1/\tau = 0 \quad \text{when } h < 0 \ , \tag{9}$$

$$dh/dt + (h-1)/\tau = dh_c/dt \quad \text{when } h > 0 \ , \tag{10}$$

$$\langle dh_c \rangle = 0, \quad \langle |dh_c|^2 \rangle = (2/a^2) dt/\tau \ . \tag{11}$$

We recall that, in this description, $v = \langle v \rangle h \Theta(h)$ [$\Theta(h)$ is the Heaviside step function] so that h is a normalized DW velocity. $h(t)$ is a stationary Markov process and, as such, its statistical properties are fully controlled by the conditional probability density $P(h,t|h_0)$, where $h = h_0$ represents the initial condition at $t = 0$. Our basic objective is to derive a Fokker-Planck equation for P, and, to this end, we have first to clarify the behavior of the process around $h = 0$. This is conveniently done by discretizing our problem in time, i.e., considering $h(t)$ at times separated by short steps Δt, with $\epsilon = \Delta t / \tau \ll 1$, and by investigating the dependence on ϵ of the probability flow across $h = 0$ implied by Eqs. (9)–(11).

Let us thus consider the transition density $M_\epsilon(h|h')$ from h' to h in a step ϵ. Standard considerations on the properties of Markov processes[10] show that Eqs. (9)–(11) are equivalent to the limit for $\epsilon \to 0$ of a Markov chain characterized by

$$h' < 0: \quad M_\epsilon(h|h') = \delta(h - h' - \epsilon) \ , \tag{12a}$$

$$h' > 0: \quad M_\epsilon(h|h') = (a/\sqrt{4\pi\epsilon}) \exp(-a^2 \Delta h_c^2 / 4\epsilon) \ , \tag{12b}$$

$$\Delta h_c = h - h' - \epsilon(1 - h') \ . \tag{12c}$$

Using Eqs. (12) in the Chapman-Kolmogorov equation for $P(h,t|h_0)$, we can write (the dependence on h_0 is understood)

$$P(h,t+\Delta t) - P(h,t) \simeq \epsilon\tau \partial P/\partial t = P(h-\epsilon,t)\Theta(\epsilon - h) - P(h,t) + (a/\sqrt{4\pi\epsilon}) \int_0^\infty dh' \exp(-a^2 \Delta h_c^2/4\epsilon) P(h',t) \ , \tag{13}$$

with Δh_c expressed by Eq. (12c). The basic implications of this equation on the structure of P around $h = 0$ are schematically shown in Fig. 1, and can be pointed out by simple order-of-magnitude considerations. Let us assume, as a starting point, that P is different from zero only for $h > 0$, and let C be the limit value of P for $h \to 0^+$. Equation (13) shows that this condition is strongly nonstationary. The integral in Eq. (13) implies in fact that, after a time step Δt, P acquires a value $\sim C$ (where $\sim$ means "of the order of") inside $h < 0$, up to $h \sim -\sqrt{\epsilon}/a$. This corresponds to a probability flow, from $h > 0$ into $h < 0$, $\sim C(\sqrt{\epsilon}/a)/\Delta t \sim C/a\sqrt{\epsilon}$, which therefore would become divergent in the limit $\epsilon \to 0$. This cannot be the case and an opposite flow of the same order of magnitude must be present. In fact, the previous considerations show that P must be assumed to be different from zero also in a region $\sim \sqrt{\epsilon}/a$ inside $h < 0$. If C' is the limit value of P in this region for $h \to 0^-$, the terms before the integral in Eq. (13) imply the existence of a probability flow, from $h < 0$ into $h > 0$, $\sim C'\epsilon/\Delta t \sim C'$, which can balance out the previous flow if $C' \sim C/a\sqrt{\epsilon}$. In the region $h < 0$, therefore, P attains values $\sim C/a\sqrt{\epsilon}$ in a h interval $\sim \sqrt{\epsilon}/a$ and will consequently be describable, in the limit $\epsilon \to 0$, in terms of a Dirac singularity $\sim (C/a^2)\delta(h)$. Our basic objective is to determine the exact strength of this singularity. The detailed calculations by which this information is obtained are reported in Appendix A. They show that the singular contribution to P is precisely $(C/a^2)\delta(h)$. Recalling that C is the limit value of P for $h \to 0^+$ we can summarize the result of our analysis by saying that the conditional probability density $P(h,t|h_0)$ of the process described by Eqs. (9)–(11) is defined for $h \geq 0$, and, in this region, is expressed as

$$P(h,t|h_0) = p(h,t|h_0)[1 + \delta(h)/a^2] \ , \tag{14}$$

where p is regular for $h \to 0^+$. Notice that the Dirac singularity, although located at the boundary of the h re-

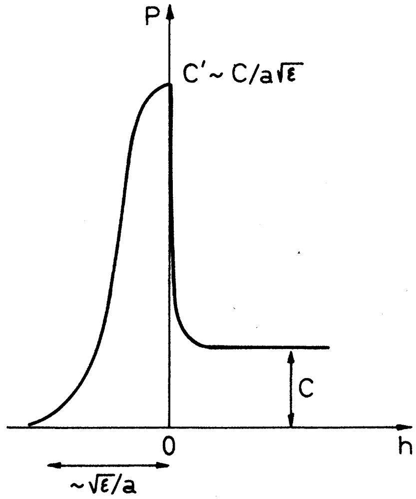

FIG. 1. Schematic representation of the behavior of $P(h,t)$ vs h around $h=0$ implied by Eq. (13).

gion of interest, will fully contribute to any integral over h which starts from $h=0$. The function $p(h,t|h_0)$ is governed by the Fokker-Planck equation associated with Eqs. (10) and (11) in the region $h>0$,

$$\tau\partial p/\partial t-\partial(h-1)p/\partial h-(1/a^2)\partial^2 p/\partial h^2=0\ . \tag{15}$$

The complications discussed in Appendix A, being limited to an interval $\sim\sqrt{\epsilon}/a$ around $h=0$, do not affect this equation. A specific boundary condition for p is however implied by Eq. (14) and by the fact that P must be normalized. Actually, taking the time derivative of the equation

$$\begin{aligned}1&=\int_0^\infty dh\, P(h,t|h_0)\\&=\int_0^\infty dh\, p(h,t|h_0)+(1/a^2)p(0,t|h_0)\end{aligned} \tag{16}$$

and making use of Eq. (15), we obtain

$$(\tau\partial p/\partial t+a^2 p-\partial p/\partial h)_{h=0}=0\ . \tag{17}$$

Our problem is therefore reduced to finding a solution of Eq. (15) which decays exponentially to zero for $h\rightarrow+\infty$ and satisfies the boundary condition (17) at $h=0$ and the initial condition $p(h,0|h_0)=\delta(h-h_0)$ at $t=0$.

SOLUTION OF THE FOKKER-PLANCK EQUATION

The problem is conveniently discussed by introducing the new variable

$$z=a(h-1)\ . \tag{18}$$

The probability density expressed by Eq. (14), when normalized with respect to z, becomes [the dependence on z_0 (i.e., h_0) is again understood]

$$P(z,t)=p(z,t)[1+\delta(z+a)/a],\quad z\ge -a\ , \tag{19}$$

and Eqs. (15) and (17) take the form

$$\tau\partial p/\partial t-\partial zp/\partial z-\partial^2 p/\partial z^2=0\ , \tag{20}$$

$$(\tau\partial p/\partial t+a^2 p-a\partial p/\partial z)_{z=-a}=0\ . \tag{21}$$

The solution of Eq. (20) is well known in the theory of Markov processes when p is defined over the entire z axis. In this case, Eq. (20) admits a complete set of eigensolutions decaying exponentially to zero for $z\rightarrow\pm\infty$, of the form

$$p_n(z,t)=\exp(-z^2/2)H_n(z/\sqrt{2})\exp(-nt/\tau)\ , \tag{22}$$

where $H_n(z/\sqrt{2})$ is the Hermite polynomial of order n. The set of eigenfunctions $H_n(z/\sqrt{2})$, $n=0,1,\ldots,$ can be obtained by a variational principle, involving the action integral

$$\mathcal{S}=\int_{-\infty}^{\infty}dz(df/dz)^2\exp(-z^2/2) \tag{23}$$

together with the normalization condition

$$(f,f)\equiv\int_{-\infty}^{\infty}dz\, f^2(z)\exp(-z^2/2)=1\ . \tag{24}$$

We will show that the effect of the boundary condition (21) amounts to a simple generalization of these results.

In analogy with Eq. (22), let us look for an eigensolution of Eq. (20) of the form

$$p_n(z,t)=\exp(-z^2/2)f_n(z)\exp(-\lambda_n t/\tau),\quad \lambda_n\ge 0\ , \tag{25}$$

where the eigenvalue λ_n is not necessarily an integer. By inserting Eq. (25) in Eq. (20), we obtain the ordinary differential equation

$$d^2 f_n/dz^2-z\, df_n/dz+\lambda_n f_n=0\ , \tag{26}$$

which would coincide with the equation for Hermite polynomials if λ_n were an integer. The solutions of Eq. (26) for a generic λ can be expressed in terms of parabolic cylinder (or Weber) functions $D_\lambda(z)$.[13] In fact, two linearly independent solutions of Eq. (26) are given by $\exp(z^2/4)D_{\lambda_n}(z)$ and $\exp(z^2/4)D_{\lambda_n}(-z)$. Considering the asymptotic behavior of $D_\lambda(z)$ for $z\rightarrow\pm\infty$, it turns out that only the former solution guarantees an exponential decay of p_n for $z\rightarrow+\infty$, and we conclude that

$$f_n(z)=\exp(z^2/4)D_{\lambda_n}(z)\ . \tag{27}$$

The set of permitted eigenvalues λ_n is obtained by imposing that p_n must satisfy the boundary condition (21). In terms of f_n, we obtain

$$(a\, df_n/dz+\lambda_n f_n)_{z=-a}=0\ . \tag{28}$$

By inserting Eq. (27) in Eq. (28) and making use of the recurrence relations for parabolic cylinder functions we arrive at the equation

$$\lambda_n(\lambda_n-1)D_{\lambda_n-2}(-a)=0\ , \tag{29}$$

which determines the set of eigenvalues of the problem as a function of a. Notice that $\lambda_0=0$ and $\lambda_1=1$ are always solutions of Eq. (29), and, according to Eq. (27), the corresponding eigenfunctions are simply $f_0(z)=1$ and $f_1(z)=z$.

It is a remarkable result that the set of eigensolutions expressed by Eqs. (27) and (29) provides an orthogonal basis for our problem. Orthogonality holds with respect to the scalar product

$$(f_n,f_m)\equiv\int_{-a}^{\infty}dz\,f_n(z)f_m(z)\exp(-z^2/2)\times[1+\delta(z+a)/a]\ , \qquad (30)$$

where the last factor in the integrand is exactly the same as in Eq. (19). The proof of this statement is given in Appendix B. On the other hand, completeness derives from the existence of a variational principle for our problem. In fact, as shown in Appendix C, the eigenfunctions $f_n(z)$ can be obtained from the action integral

$$\mathcal{S}=\int_{-a}^{\infty}dz\,(df/dz)^2\exp(-z^2/2)\ , \qquad (31)$$

together with the normalization condition

$$(f,f)\equiv\int_{-a}^{\infty}dz\,f^2(z)\exp(-z^2/2)[1+\delta(z+a)/a]=1 \qquad (32)$$

and, as such, provide a complete set of eigenfunctions. The comparison of Eqs. (31) and (32) with Eqs. (23) an (24) points out the simple generalization by which the boundary condition at $z=-a$ expressed by Eq. (28) is added to the Fokker-Planck Equation (20) to obtain our model. In terms of the basis of eigenfunctions $f_n(z)$, the conditional probability density $P(z,t|z_0)$ of interest becomes

$$P(z,t|z_0)=\exp(-z^2/2)[1+\delta(z+a)/a]\times\sum_{n=0}^{\infty}C_n(z_0)f_n(z)\exp(-\lambda_n t/\tau)\ . \qquad (33)$$

The coefficients $C_n(z_0)$ are obtained by requiring that $P(z,0|z_0)=\delta(z-z_0)$. Exploiting the orthogonality properties of f_n with respect to the scalar product (30), we obtain

$$C_n(z_0)=f_n(z_0)/(f_n,f_n)\ . \qquad (34)$$

Equations (33) and (34) solve completely our problem.

DISCUSSION

Equation (33) determines all the statistical properties of $z(t)=a[h(t)-1]$, where $h(t)=v(t)/\langle v\rangle$. The stationary amplitude probability density of the process corresponds to the eigenvalue $\lambda_0=0$ and is thus given, in terms of the variable h, by the expression (see Fig. 2)

$$P_0(h)=K\exp[-a^2(h-1)^2/2][1+\delta(h)/a^2],\quad h\geq 0 \qquad (35)$$

where K is a proper normalization constant and the fact that $f_0(z)=1$ has been taken into account. A straightforward integration shows that the mean value of $(h-1)$ calculated through Eq. (35) is zero, which implies that the mean DW velocity is just $\langle v\rangle$, consistently with the initial assumptions of our model. The Dirac singularity in Eq. (35) describes the finite probability of finding the DW at rest because of the pinning action of the coercive field. Its presence implies that $v(t)$ exhibits an intermittent behavior in time, with burstlike events separated by finite time intervals where $v(t)=0$. This behavior is at the origin of the Barkhausen (B) effect commonly observed in ferromagnetic materials.[11,12] The signal detected in B effect experiments measures in fact the rate of change of magnetic flux and is thus basically proportional to the DW velocity v. The mentioned burstlike events (often termed B clusters) are actually observed in experiments, and have been considered the essential aspect of B noise by many authors.[11,12,14,15] It is worth remarking that in the model where H_c fluctuations are described by Eq. (5), a power-law divergence of $P_0(h)$ for $h\rightarrow 0$, instead of a Dirac singularity at $h=0$, is obtained, which implies the existence of scaling properties in the time and amplitude distribution of B clusters.[3]

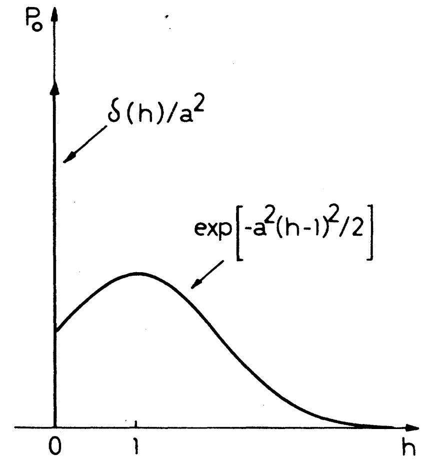

FIG. 2. Stationary amplitude probability density $P_0(h)$ vs normalized DW velocity $h=v/\langle v\rangle$, as predicted by Eq. (35). The Dirac singularity describes the finite probability of finding the DW at rest.

The intermittent behavior of $v(t)$ tends to disappear at high DW velocities since, according to Eq. (8), $a\rightarrow\infty$ when $\langle v\rangle\rightarrow\infty$. In this limit v shows only small fluctuations around $\langle v\rangle$, the value $v=0$ is never reached and a simple Gaussian process is obtained.

The autocorrelation function $R(\Delta t)$ of $h(t)$ can be calculated as

$$R(\Delta t)=\int\int dh_1dh_2(h_1-1)(h_2-1)P_0(h_1)P(h_2,\Delta t|h_1)\ , \qquad (36)$$

where P and P_0 are given by Eqs. (33)–(35). This integral is straightforwardly calculated by noticing that $(h-1)\propto f_1(z)$ and exploiting the orthogonality properties of the eigenfunctions $f_n(z)$. We obtain

$$R(\Delta t)=\langle(h-1)^2\rangle\exp(-\Delta t/\tau)\ , \qquad (37)$$

which implies that h fluctuations are characterized by a Lorentzian spectrum of time constant τ [see Eq. (8)]. The quantity $\langle(h-1)^2\rangle$ measures the intensity of h fluctuations around its average $\langle h\rangle=1$, and can be calculated through Eq. (35). We obtain

$$\langle (h-1)^2\rangle = \frac{\exp(-a^2/2)+(\pi/2a^2)^{1/2}+(\sqrt{2}/a)\gamma(\frac{3}{2};a^2/2)}{\exp(-a^2/2)+a\sqrt{\pi/2}[1+\Phi(a/\sqrt{2})]} , \tag{38}$$

where $\Phi(x)$ is the probability integral defined by Eq. (A4) and

$$\gamma(\alpha;x)=\int_0^x du\, u^{\alpha-1}\exp(-u) \tag{39}$$

is the incomplete Γ function. A simple behavior is obtained in the two limits $a \ll 1$ and $a \gg 1$ (see Fig. 3),

$$\langle (h-1)^2\rangle \simeq (\pi/2a^2)^{1/2} \text{ when } a \ll 1 \text{ (low } \langle v\rangle) , \tag{40}$$

$$\langle (h-1)^2\rangle \simeq 1/a^2 \text{ when } a \gg 1 \text{ (high } \langle v\rangle) . \tag{41}$$

According to our previous considerations, the results expressed by Eqs. (37)–(41) should be applicable to the interpretation of B effect fluctuations. It is not the aim of the present paper to give any accurate comparison between the present model and B noise experimental results, but some interesting conclusions can be very simply obtained from the fact, already discussed in previous papers,[1] that the time constant τ is proportional to the differential premeability μ of the portion of the hysteresis loop where DW motion is considered. This implies, on the one hand, that B noise cutoff frequency is expected to be proportional to $1/\mu$ [Eq. (37)]. On the other hand, B noise intensity should be, depending on the average DW velocity $\langle v\rangle$, proportional to $\mu^{1/2}$ [low $\langle v\rangle$, Eqs. (8) and (40)] or to μ [high $\langle v\rangle$, Eqs. (8) and (41)] and the spectrum maximum, which is proportional to $\tau\langle (h-1)^2\rangle$, should correspondingly be proportional to $\mu^{3/2}$ or to μ^2. These predictions are in fact in good agreement with literature results,[16,17] although the lack of accurate information on $\langle v\rangle$ introduces inevitably some arbitrarity in the comparison between theory and experiments.

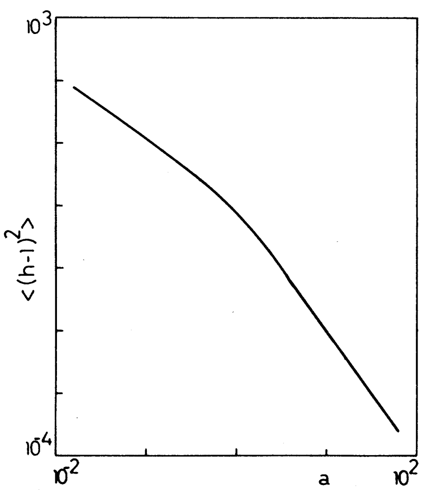

FIG. 3. Intensity of normalized DW velocity fluctuations vs dimensionless parameter a defined by Eq. (8), as predicted by Eq. (38). The asymptotic behavior for $a \ll 1$ and $a \gg 1$ is given by Eqs. (40) and (41).

CONCLUSION

The common aim of this paper and of Ref. 3 is to develop a proper theoretical frame for the description of DW motion in ferromagnetic systems. The main difficulty opposing this task is represented by the peculiar role played by the DW coercive field H_c, which represents on the one hand a threshold for DW movement and exhibits, on the other hand, stochastic fluctuations when the DW is in motion. This leads to a Langevin equation for the DW velocity v having a complex structure around $v=0$. The corresponding Fokker-Planck equation can, however, be rigorously solved in terms of a complete set of orthogonal eigenfunctions. In particular, in the case of the model discussed in this paper, the properties of these eigenfunctions and the variational principle associated with them point out that our Langevin approach to DW motion is quite a simple generalization of the standard Langevin model considered in the theory of Markov processes. We believe that important information on DW dynamics will be obtained by comparing the predictions of the present model and of that of Ref. 3 with the results of Barkhausen effect experiments. This comparison will be the subject of specific forthcoming papers.

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APPENDIX A

Following the order-of-magnitude analysis given in the text, we shall look for a solution of Eq. (13) which, in an interval $\sim\sqrt{\epsilon}/a$ around $h=0$, has the structure expressed by the equation

$$P(h,t)\simeq Q(h,t)\Theta(\epsilon-h)+[C(t)+\Delta(h,t)]\Theta(h-\epsilon) ,$$
$$|h| \lesssim \sqrt{\epsilon}/a , \tag{A1}$$

where $\Delta(h,t)$ is a correction term $\sim C$. According to the discussion given in the text, $Q \sim C/a\sqrt{\epsilon}$ so that, when we consider Eq. (13) for $h<0$, $\epsilon\tau\partial P/\partial t \sim C\sqrt{\epsilon}/a$. All terms of Eq. (13) having a weaker dependence on ϵ must therefore balance out. When Eq. (A1) is substituted in Eq. (13), we find in fact several terms $\sim C$, whose sum must be equal to zero. This gives the equation [dependencies on t are understood, C' represents $Q(h=0)$]

$$-\epsilon\partial Q/\partial h+F(h)+G(h)=0, \quad h<\epsilon , \tag{A2}$$

where

$$F(h)=\epsilon C'(a/\sqrt{4\pi\epsilon})\exp(-a^2h^2/4\epsilon)+(C/2)$$
$$\times[1+\Phi(ah/\sqrt{4\epsilon})] , \tag{A3}$$

$$\Phi(x)=(2/\sqrt{\pi})\int_0^x du\, \exp(-u^2) , \tag{A4}$$

and

$$G(h)=(a/\sqrt{4\pi\epsilon})\int_{\epsilon}^{\infty}dh'\exp[-a^2(h-h')^2/4\epsilon]\Delta(h') . \tag{A5}$$

As previously mentioned, we are basically interested in the integral ΔQ of Q, which, in the limit $\epsilon\to 0$, gives the strength of the Dirac singularity at $h=0$. ΔQ can be calculated from Eq. (A2), exploiting the fact that

$$\Delta Q=-\int_{-\infty}^{\epsilon}dh\,(h-\epsilon)\partial Q/\partial h\simeq-\int_{-\infty}^{\epsilon}dh\,h\partial Q/\partial h . \tag{A6}$$

Multiplying Eq. (A2) by h and working out some known integrals, we obtain

$$\Delta Q=C'\sqrt{\epsilon}/a\sqrt{\pi}+C/2a^2-(1/\epsilon)\int_{-\infty}^{\epsilon}dh\,hG(h) . \tag{A7}$$

The last term of Eq. (A7) can be evaluated by considering the behavior of Eq. (13) in the region $h>\epsilon$. Here $P\sim C$, so that $\epsilon\tau\partial P/\partial t\sim\epsilon C$, but several terms $\sim C$ are again obtained when Eq. (A1) is substituted in Eq. (13). Equating their sum to zero, we obtain

$$-C-\Delta(h)+F(h)+G(h)=0, \quad h>\epsilon . \tag{A8}$$

Let us multiply Eq. (A8) by h and integrate from ϵ to $+\infty$. It can be checked from Eq. (A5) that

$$\int_{\epsilon}^{\infty}dh\,h\Delta(h)=\int_{-\infty}^{\infty}dh\,hG(h) . \tag{A9}$$

Exploiting this relation and working out the integral containing $F(h)$ we obtain

$$\int_{-\infty}^{\epsilon}dh\,hG(h)=\epsilon(C'\sqrt{\epsilon}/a\sqrt{\pi}-C/2a^2) . \tag{A10}$$

From Eqs. (A7) and (A10), we therefore obtain

$$\Delta Q=C/a^2 . \tag{A11}$$

This result implies that, in the limit $\epsilon\to 0$, the behavior of $P(h,t)$ around $h=0$ can be described by the singular term $(C/a^2)\delta(h)$, where C is the limit for $h\to 0^+$ of $P(h,t)$ in the region $h>0$.

APPENDIX B

Let us consider the scalar product given by Eq. (30), and let us multiply it by λ_n. We obtain

$$\lambda_n(f_n,f_m)=\lambda_n\int_{-a}^{\infty}dz\,f_n(z)f_m(z)\exp(-z^2/2)+[(\lambda_n/a)f_n(z)f_m(z)\exp(-z^2/2)]_{z=-a} . \tag{B1}$$

Equation (26) implies that

$$\lambda_n\exp(-z^2/2)f_n=-d[\exp(-z^2/2)df_n/dz]/dz , \tag{B2}$$

while Eq. (28) shows that

$$[(\lambda_n/a)f_n(z)]_{z=-a}=-(df_n/dz)_{z=-a} . \tag{B3}$$

Inserting Eqs. (B2) and (B3) in Eq. (B1) and integrating by parts, we obtain

$$\lambda_n(f_n,f_m)=\int_{-a}^{\infty}dz(df_n/dz)(df_m/dz)\exp(-z^2/2) . \tag{B4}$$

The right-hand side of Eq. (B4) no longer contains λ_n and is symmetric in n and m. This implies that the same result is also obtained when we multiply by λ_m instead of λ_n,

$$\lambda_n(f_n,f_m)=\lambda_m(f_n,f_m) , \tag{B5}$$

which shows that $(f_n,f_m)=0$ whenever $\lambda_n\neq\lambda_m$.

APPENDIX C

Let us consider the variation

$$\delta[\mathcal{S}-\lambda(f,f)]=0 , \tag{C1}$$

where $\mathcal{S}$ and (f,f) are given by Eqs. (31) and (32), and λ is a Lagrange multiplier. By following the usual procedure with an integration by parts, and taking into account that δf is by no means constrained to be equal to zero at $z=-a$, we obtain

$$-\int_{-a}^{\infty}dz\,\delta f\{d[\exp(-z^2/2)(df/dz)]/dz+\lambda\exp(-z^2/2)f\}-\{\delta f[df/dz+(\lambda/a)f]\exp(-z^2/2)\}_{z=-a}=0 . \tag{C2}$$

By imposing that both the integral and the contribution at $z=-a$ must be zero for an arbitrary variation δf, we obtain a differential equation for $f(z)$ equivalent to Eq. (26) and a boundary condition at $z=-a$ coincident with Eq. (28).

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