Perturbative analysis of long Josephson junctions having uniform bias and spatially varying penetration depth

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(Received 12 December 1988)

The effect that a uniform bias current has on the basic theoretical structure of a simple perturbation theory relating to single fluxon motion on long Josephson junctions is discussed. By including the bias directly into the general framework of the theory, it is possible to arrive at a fundamental limitation on the maximum value for the bias above which the validity of the usual power balance expression relating the loss factors, and the voltage to the applied current cannot be justified. As part of this effort, an interpretive generalization of the original theory is provided. This theory is then used to illustrate the way in which a step-wise variation in the London penetration depth affects the propagation of solitons on long junctions, and the influence that this has on the junction *I-V* curve. In the process of developing this theory and applying it to this problem, a formal connection was found between certain aspects of this approach and the technique outlined by McLaughlin and Scott. The results of the theoretical calculations are also shown to be in good agreement with experimental data.

I. INTRODUCTION

The propagation of fluxons on long junctions has been a subject of intense experimental and theoretical investigation for many years,[1-4] and with the advent of the new, high-T_c oxide superconductors, it is likely that research in this area will become even more attractive.[5] In the experimental arena, digital devices that are based on the concept of the fluxon as a fundamental unit of information have been successfully constructed and tested,[6,7] and analog devices like the flux flow oscillator and the vortex flow transistor are also being realized;[8,9] this is in addition to fundamental device studies relating to the basic physics of fluxon motion.[10-12] Theoretically, progress continues to be made toward understanding the nature of the interactions between fluxons and various perturbations, and the effect that this has on the output characteristics of the junction.[5,13,14]

The equation governing the propagation of electromagnetic waves of long Josephson junctions is given by[5]

$$\beta_L \phi_{xxt} + \phi_{xx} - \phi_{tt} = \sin\phi + \alpha\phi_t - \Gamma \ , \tag{1}$$

where the space and time variables have been normalized in units of the Josephson penetration depth λ_J and the Josephson frequency c/λ_J, α represents the quasiparticle leakage across the junction barrier, and β_L is due to quasiparticle losses in the junction electrodes. If the bias Γ is assumed to be uniform, Eq. (1) has as one of its solutions

$$\phi(x,t) = \sin^{-1}\Gamma \ . \tag{2}$$

Although this is a trivial result, it has recently been emphasized that proper consideration of (2) has a profound affect on fluxon propagation and their signature on the junction $I - \langle V \rangle$ curve.[10,15] In particular, it modifies the Lorentz contraction γ that a fluxon experiences:

$$\gamma \to \mu = (1-\Gamma^2)^{1/4}\gamma \ . \tag{3}$$

As a direct consequence of this, substituting μ for γ everywhere in the power balance relation

$$\Gamma = \frac{4\beta\gamma}{\pi}\left[\alpha + \frac{1}{3}\beta_L\gamma^2\right] \to \frac{4\beta\mu}{\pi}\left[\alpha + \frac{1}{3}\beta_L\mu^2\right] , \tag{4}$$

where $\beta = v/c$ is the steady-state velocity of the fluxon normalized to the speed of light on the junction, results in an expression that is significantly in better agreement with experimental observation.[10,15] From this discussion, it is obvious that it is important to properly include the effects of a uniform bias in any theory relating fluxon propagation to the junction $I - \langle V \rangle$ curve. Therefore, we consider here the incorporation of a uniform bias into a perturbation theory introduced by Rubinstein[16] and expanded upon by Fogel, Trullinger, Bishop, Krumhansl, and Currie.[17-20] This theory is then used to study not only the effect that a spatial inhomogeniety has on the dynamics of solitons traveling on long Josephson junctions, but also the result that this has on the junction *I-V* curve.

II. BASIC PERTURBATIVE APPROACH

The original formulation of the theory is based upon a direct linear expansion of the perturbed solution $\phi(x,t)$ about the single soliton solution:[16-20]

$$\phi(x,t) = \phi^0(x,t) + \psi(x,t) \ , \tag{5}$$

where $\phi^0(x,t)$ is the unperturbed soliton solution

$$\phi^0(x,t)=4\tan^{-1}\exp[\gamma(x-\beta t)] , \tag{6}$$

and $\psi(x,t)$ represents the influence of the perturbations on (6). This solution is then inserted into the perturbed sine-Gordon equation

$$\phi_{xx}-\phi_{tt}-\sin\phi=-f[x,t;\phi(x,t),\phi_x(x,t),\phi_t(x,t),\ldots] , \tag{7}$$

where $f(x,t;\ldots)$ represents the perturbing terms, and the resulting expression is linearized about the soliton solution (6). The result is

$$\psi_{xx}-\psi_{tt}-(1-2\,\text{sech}^2[\gamma(x-\beta t)])\psi=-f(x,t;\ldots) . \tag{8}$$

Transforming to the rest frame of the unperturbed soliton

$$x\rightarrow\xi=\gamma(x-\beta t) , \tag{9a}$$

$$t\rightarrow\tau=\gamma(t-\beta x) \tag{9b}$$

yields

$$\psi_{\xi\xi}-\psi_{\tau\tau}-(1-2\,\text{sech}^2\xi)\psi=-f(x,t;\ldots) . \tag{10}$$

The solution to (10) is found by noting that the Fourier time transform of the homogeneous equation related to (10),

$$\bar{\psi}_{\xi\xi}-(1-\omega^2-2\,\text{sech}^2\xi)\bar{\psi}=0 \tag{11}$$

is a linear eigenvalue problem whose eigenfunctions form a complete, orthonormal set of states. Specifically, Eq. (11) has one bound-state eigenfunction

$$\psi_b=\text{sech}\xi \tag{12a}$$

corresponding to the $\omega^2=0$ eigenstate, and the unbounded set of eigenfunctions

$$\psi_c=\omega^{-1}e^{ik\xi}(k+i\tanh\xi) \tag{12b}$$

corresponding to the continuous index k. The dispersion relation between k and ω is

$$\omega^2=k^2+1 ,$$

and the orthogonality and completeness relations are

$$\int_{-\infty}^{+\infty}\psi_b(\xi)\psi_b(\xi)d\xi=2 , \tag{13a}$$

$$\int_{-\infty}^{+\infty}\psi_c^*(\xi,k)\psi_c(\xi,k')d\xi=2\pi\delta(k-k') , \tag{13b}$$

$$\int_{-\infty}^{+\infty}\psi_c(\xi,k)\psi_b(\xi)d\xi=\int_{-\infty}^{+\infty}\psi_c^*(\xi,k)\psi_b(\xi)d\xi=0 , \tag{13c}$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\psi_c^*(\xi,k)\psi_c(\xi',k)dk+\tfrac{1}{2}\text{sech}\xi\,\text{sech}\xi'=\delta(\xi-\xi') . \tag{13d}$$

Therefore, any solution to (10) can be written as[17–20]

$$\psi(\xi,\tau)=A(\tau)\psi_b(\xi)+(2\pi)^{-1/2}\int_{-\infty}^{+\infty}b(\tau,k)\psi_c(\xi,k)dk , \tag{14}$$

where the expansion coefficients are given by the equations

$$A_{\tau\tau}(\tau)=\tfrac{1}{2}\int_{-\infty}^{+\infty}f(\xi,\tau,\ldots)\text{sech}\xi\,d\xi \tag{15a}$$

and

$$b_{\tau\tau}+\omega^2 b+\frac{(2\pi)^{-1/2}}{\omega}\int_{-\infty}^{+\infty}f(\xi,\tau,\ldots)\times e^{-ik\xi}(k-i\tanh\xi)dk , \tag{15b}$$

which are found by utilizing the orthogonality and completeness relations (13).

The physical interpretations of the eigenfunctions and expansion coefficients are straightforward.[16–20] The continuum eigenfunctions are related to the small amplitude oscillations excited by the perturbation. The presence of the soliton modifies the exponential form associated with simple plane-wave propagation through the prefactor $(k+i\tanh\xi)$ in (12b). The bound-state eigenfunctions are more closely tied to the soliton dynamics. To lowest order, $A(\tau)$ represents a time-dependent phase shift introduced into the soliton solution by the pertubation:

$$\phi^{\text{sol}}(\xi+\tfrac{1}{2}A(\tau))\sim\phi^0(\xi)+A(\tau)\text{sech}\xi . \tag{16}$$

Therefore, most of the attention is usually focused on finding the solution to Eq. (15a).

The approach outlined here has been used to study a variety of spatially-dependent perturbations,[17] weakly-interacting solitons,[16] and the behavior of fluxons subjected to a time-dependent driving term,[17] and the results were found to be in reasonable agreement with computer simulation. Kaup and Newell have pointed out that these same results are also obtained using more a general perturbative technique based upon the inverse scattering transform,[21] but in studies involving only the single soliton, we feel that the approach outlined here is simpler and more direct.

III. INTERPRETIVE GENERALIZATIONS AND A CONNECTION TO THE PERTURBATION THEORY OF MCLAUGHLIN-SCOTT

Prior to interacting with the perturbations, the soliton solution depends only upon the traveling coordinate ξ_0 [Eqs. (6) and (9)], where the subscript "0" is used to designate the initial equilibrium value for ξ. The result of the interaction is to add a temporal dependence to the argument of the soliton solution (16). Long after the interaction, the soliton relaxes back to its steady-state form (6), but will reflect influence of the perturbations as a change in the argument:

$$\xi_0\rightarrow\xi_f=\gamma_f(x-\beta_f t)+\delta\xi , \tag{17}$$

where $\delta\xi$ is the asymptotic phase shift introduced by the perturbation and the subscript "f" is used to denote final quantities. From Eq. (16), we see that asymptotic corrections to the phase, velocity, and Lorentz contraction factor can be generated by $A(\tau)$ provided it is of the form

$$\lim_{\tau\to\infty} A(\tau)\to a_1+a_2\tau = a_1+a_2\gamma_0(t-\beta_0 x) . \quad (18)$$

Inserting this into Eqs. (16) and (17) yields

$$\delta\xi=a_1/2 , \quad (19a)$$

$$\gamma_f\beta_f=\gamma_0\beta_0(1-a_2/2\beta_0) , \quad (19b)$$

and

$$\gamma_f=\gamma_0(1-a_2\beta_0/2) . \quad (19c)$$

To generalize this to times other than $\tau\to\infty$, we propose the simple ansatz

$$\xi_f+\delta\xi\equiv\xi_0+\tfrac{1}{2}A_0(\tau)+\tfrac{1}{2}A_\tau(\tau)\tau+\tfrac{1}{4}A_{\tau\tau}(\tau)\tau^2+\dots , \quad (20)$$

where $A_0(\tau)$, $A_\tau(\tau)$, and $A_{\tau\tau}(\tau)$ are associated with changes in the position, velocity, and acceleration in the unperturbed rest frame of the soliton, and ξ_f is defined as $\xi_f(\tau)=\gamma_f(\tau)[x-\beta_f(\tau)t]$. This leads to the following generalized definitions for the corrections to the overall phase shift, Lorentz contraction factor, and the soliton velocity:

$$\delta\xi=\tfrac{1}{2}A_0(\tau) , \quad (21a)$$

$$\gamma_f\equiv\gamma_0(1-\tfrac{1}{2}\beta_0 A_\tau) , \quad (21b)$$

and

$$\beta_f\equiv\beta_0\left[1-\frac{1}{2\beta_0}A_\tau\right]\Big/(1-\tfrac{1}{2}\beta_0 A_\tau) . \quad (21c)$$

Asymptotically, these results agree with Eqs. (19). However, they also describe the temporal change in these quantities over the entire time interval associated with the problem at hand, and therefore provide much more information than previously available. In addition, it is possible to estimate the acceleration of the soliton in the laboratory frame, which is a measure of the force exerted on the soliton by the perturbation. Differentiating (21c):

$$d\beta_f=-dA_\tau/[2\gamma_0^2(1-\tfrac{1}{2}\beta_0 A_\tau)^2 = -\tfrac{1}{2}(A_{\tau\tau}/\gamma_f^2)d\tau$$

and using $dt=\gamma_f d\tau$ and Eq. (15a), we find that this acceleration is given by

$$\frac{d\beta_f}{dt}=-\frac{1}{2\gamma_f^2}A_{\tau\tau} = -\frac{1}{4\gamma_f^2}\int_{-\infty}^{+\infty} f(\xi,\tau;\dots)\operatorname{sech}\xi\,dx . \quad (21d)$$

This expression is identical to the acceleration given by the McLaughlin-Scott theory, which illustrates a formal connection between the two theories.

There is some ambiguity in Eqs. (21b) and (21c), since they yield different estimates for the velocity of the soliton. From (21b) we have

$$\beta_f^2=\gamma_0^2\beta_0^2\left[1-\frac{A_\tau}{\beta_0}+\frac{1}{4}A_\tau^2\right]\Big/\gamma_f^2 , \quad (22a)$$

while (21c) can be rewritten as

$$\beta_f^2=\gamma_0^2\beta_0^2\left[1-\frac{A_\tau}{\beta_0}+\frac{1}{4}\left[\frac{A_\tau}{\beta_0}\right]^2\right]\Big/\gamma_f^2 . \quad (22b)$$

When $\beta_0\sim 1$, (22a) and (22b) are in agreement with one another. However, for small initial velocities, the highest-order correction to (22b) can become large. In fact, when $\beta_0\to 0$, Eq. (22b) predicts that the perturbation always leads to a nonzero value for β_f, as long as A_τ is nonzero. Obviously, this may not necessarily be true, especially if the soliton is already pinned to the site of an inhomogeneity. Therefore, (21b) will be used to self-consistently estimate the correction to the soliton velocity β_f and the Lorentz contraction factor γ_f, and will be shown to yield very accurate results when applied to the inhomogeneous penetration depth problem in Sec. V.

IV. INCORPORATION OF A UNIFORM BIAS

In order to properly consider the presence of a uniform bias the derivation leading to the basic equations (10), (15a), and (15b) must be modified in two ways. First, it can be shown[15] that the effect on the traveling-wave coordinate ξ [Eq. (9a)] of the unperturbed soliton solution (6) can be interpreted as a correction to the Lorentz contraction factor: $\gamma\to\mu$. We will assume that this extends to the time transformation (9b) as well. Second, the background field $\sin^{-1}\Gamma$ must be explicitly included in the basic expansion (5):

$$\phi(x,t)=\phi(x,t)+\sin^{-1}\Gamma+\psi(x,t) . \quad (23)$$

When (23) is inserted into the perturbed sine-Gordon equation (7) and the modified coordinate transformations are made, the result is

$$\psi_{\xi\xi}-\psi_{\tau\tau}-[1-2\operatorname{sech}^2\xi-2\Gamma(1-\Gamma^2)^{-1/2}\operatorname{sech}\xi\tanh\xi] = -(1-\Gamma^2)^{-1/2}g(\xi,\tau;\dots) , \quad (24)$$

where

$$g(\xi,\tau;\dots)=f(\xi,\tau;\dots)-\Gamma(1-2\operatorname{sech}^2\xi) .$$

The nonlinear mixing between the uniform bias field and the soliton solution has completely changed the basic eigenvalue problem by introducing an additional term to the potential. We have not been able to find an exact solution to the homogeneous equation associated with (24). However, by setting

$$2\Gamma/(1-\Gamma^2)^{-1/2}\ll 1 , \quad (25)$$

which corresponds to $\Gamma<0.45$, the correction to the potential can either be ignored or treated perturbatively within the framework of the existing theory. In that case the expansion coefficients $A(\tau)$ and $b(k,\tau)$ are still given by (15a) and (15b), except that $f(\xi,\tau,\dots)$ is now replaced by $g(\xi,\tau\ldots)$. It must be kept in mind, through, that the results are fundamentally limited to $\Gamma<0.45$.

V. EXAMPLE: EFFECT OF SUDDEN VARIATION IN PENETRATION DEPTH

In order to illustrate the way in which the interpretive generalizations (21a)–(21c) and the uniform bias modifications (23) and (24) are implemented, consider the specific problem

$$\partial_x(c^2\phi_x)-\phi_{tt}-\sin\phi=\alpha\phi_t-\beta_L\phi_{xxt}-\Gamma\ , \tag{26a}$$

where

$$c(x)=\begin{cases}1; & x<0,\\ c_0; & x>0\ .\end{cases} \tag{26b}$$

This type of spatial inhomogeneity occurs if there is an abrupt change in the London penetration depth in either of the junction electrodes. This problem has been examined by Sakai *et al.* when $\alpha=\beta_L=\Gamma=0$ using the McLaughlin-Scott theory, the results of which were found to be in excellent agreement with computer simulation.[14] Therefore the results of this calculation will be compared directly with McLaughlin-Scott in the limit where the losses and bias terms are zero, and in addition to this we will show the way in which this spatial inhomogeneity affects the junction $I-\langle V\rangle$ curve when these terms are reintroduced.

A. Loss and bias terms zero

When $\alpha=\beta_L=\Gamma=0$, Eq. (24) becomes

$$\partial_x(c^2\phi_x)-\phi_{tt}-\sin\phi=0\ . \tag{27}$$

The characteristic coordinates for this equation are found by solving

$$dx\pm c(x)dt=0\ .$$

For this reason, it is convenient to make the coordinate transformation

$$cdz=dx\ ,$$

and to replace x by z in Eqs. (6) and (9), reducing (27) to[14]

$$\begin{aligned}\phi_{zz}-\phi_{tt}-\sin\phi&=-\partial_z(\ln c)\phi_z\\ &=-(\ln c_0)\delta(z)\phi_z\ .\end{aligned} \tag{28}$$

In the rest frame of the unperturbed soliton, this is

$$\phi_{\xi\xi}-\phi_{\tau\tau}-\sin\phi=-(\ln c_0)\delta(\xi+\beta_0\tau)(\phi_\xi-\beta_0\phi_\tau)\ . \tag{29}$$

Inserting the expression for the soliton solution (6) and the expansions (5) and (14) gives

$$\begin{aligned}-A_{\tau\tau}\psi_b-(2\pi)^{-1/2}\int_{-\infty}^{+\infty}(b_{\tau\tau}+\omega b)\psi_c dk\\ =-(\ln c_0)\delta(\xi+\beta_0\tau)\\ \times\Big[2\,\mathrm{sech}\xi+A\psi_{b\tau}+(2\pi)^{-1/2}\int_{-\infty}^{+\infty}b(\tau,k)\psi_{c\xi}dk\\ -\beta_0A_\tau\psi_b-(2\pi)^{-1/2}\beta_0\int_{-\infty}^{+\infty}b_\tau(\tau,k)\psi_c(\xi,k)dk\Big].\end{aligned} \tag{30}$$

Multiplying through by ψ_b and using the orthogonality relations (13) yield

$$A_{\tau\tau}=(\ln c_0)(1-\tfrac{1}{2}\beta_0A_\tau)\mathrm{sech}^2\beta_0\tau \tag{31a}$$

and

$$\begin{aligned}b_{\tau\tau}-\beta_0\ln(c_0)b_\tau+\omega^2b=\frac{1}{\sqrt{2\pi\omega_0}}(\ln c_0)e^{ik\beta_0\tau}\mathrm{sech}\beta_0\tau\\ \times(k+i\tanh\beta_0\tau)\ .\end{aligned} \tag{31b}$$

In order to completely eliminate any coupling between A and b, the term proportional to ψ_ξ has been ignored. This should be a lower-order correction, since the ξ derivative of ψ_b is expected to dominate.

The solution to (31b) is readily found using Green's function techniques.[26] However, the focus of this paper will be on solving (31a), since $A(\tau)$ has a more direct influence upon the solitary wave form. Therefore, integrating (31a) directly yields

$$\begin{aligned}A_\tau&=\frac{2}{\beta_0}(1-\exp\{-[\tfrac{1}{2}(\ln c_0)(1+\tanh\beta_0\tau)]\})\\ &=\frac{2}{\beta_0}(1-e^{\sigma})\ ,\end{aligned} \tag{32}$$

where the initial condition $A_\tau(\tau\to-\infty)=0$ has been used. Inserting this into Eq. (21b) immediately yields the first-order correction to the Lorentz contraction factor (and hence the velocity):

$$\gamma_f=\gamma_0c_0^{\sigma}\ ,$$

or

$$2\ln(\gamma_0/\gamma_f)=(\ln c_0)(1+\tanh\beta_0\tau)\ . \tag{33}$$

Asymptotically,

$$\lim_{\tau\to\infty}\gamma_f=\gamma_0/c_0\ . \tag{34}$$

A comparison of the results of (34) with direct numerical calculations of β_f as a function of β_0 for various values of c_0 is shown in Fig. 1. This same agreement was also found by Sakai *et al.* upon numerical integration of the coupled partial differential equations for the soliton velocity and acceleration calculated using the McLaughlin-Scott perturbation theory.[14] However, it is also possible to extend the analysis of that paper and integrate those equations analytically. In order to provide a more detailed comparison between these two theories, we will digress to show this result.

The perturbation theory of McLaughlin and Scott provides two expressions relating the effects of the perturbations to modulations in the velocity and acceleration of the soliton.[1] The corrections to the velocity and acceleration due to the interaction between the soliton and a step-wise variation in c were shown (in the McLaughlin-Scott theory) to be given by[14]

$$\frac{dz}{dt}=\beta+\tfrac{1}{2}(\ln c_0)\beta\gamma z\,\mathrm{sech}^2\gamma z \tag{35a}$$

and

$$\frac{d\beta}{dt}=-\frac{1}{2\gamma}(\ln c_0)\mathrm{sech}^2\gamma z\ , \tag{35b}$$

where β and γ are assumed to be functions of z and t. These equations were then numerically integrated. However, it should be noted that these equations can be combined into the single equation

$$\frac{dU}{d\beta}=-\frac{2\beta\gamma^2}{\ln c_0}\cosh^2 U ,$$

where $U=\gamma z$, which is easily integrated analytically, yielding

$$2\ln(\gamma_0/\gamma_f)=\ln c_0(1+\tanh\gamma_f z) . \quad (36)$$

This is essentially equivalent to Eq. (33).

The reason for presenting these two theoretical approaches to solving this simple problem is to point out not only the similarity in the results, but the difference in the methodology. Both theories have their positive and negative points. We feel that the direct method we have developed here is much easier to use when studying problems involving the behavior of single solitons, because of the complicated way in which the space and time variables enter into the McLaughlin-Scott relations for the velocity and acceleration. In this case, it was possible to find a way to overcome this problem and arrive at an analytical solution, but this will not be true in general. On the other hand, the direct expansion method is restricted to the singe soliton problem only, while the McLaughlin-Scott approach is applicable to a much wider range of problems (breathers, multiple solitons, etc.). Finally, since the direct method has also been shown to yield the same results as a perturbative approach based upon the inverse scattering theory (again, single soliton only)[12] the connection to the McLaughlin-Scott theory illustrated here places all three of these theories on the same footing. Therefore, the choice between which theory to apply to the single soliton problem is essentially reduced to a matter of personal preference.

B. Addition of losses and uniform bias

When the loss and bias terms are added back into the problem, the application of the stretching transformation $cdz=dx$ yields the equation

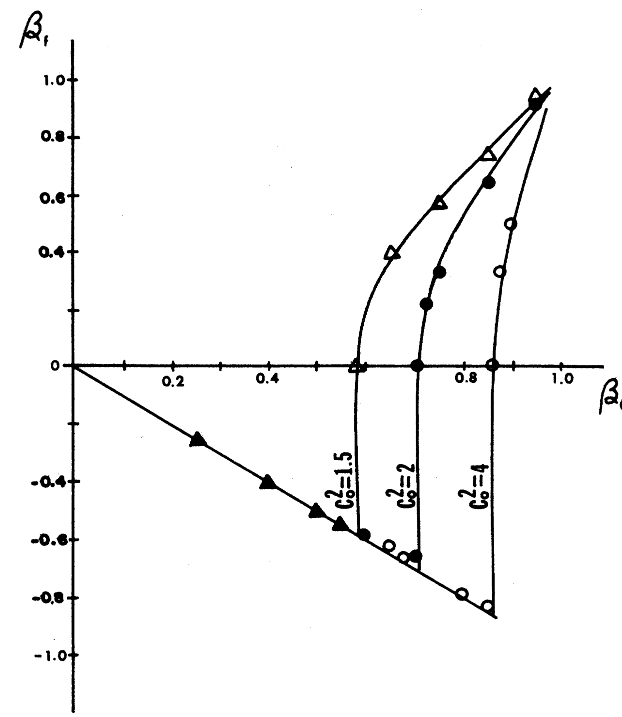

FIG. 1. Asymptotic values for β_f as a function of the initial soliton velocity β_0 and the size of the variation in c^2. The solid lines represent Eq. (34), while the points are the result of direct numerical simulation of (28). The closed triangles indicate that the same numerical result was obtained for all values of c^2 at that point.

$$\phi_{zz}-\phi_{tt}-\sin\phi=-\partial_z(\ln c)\phi_z+\alpha\phi_t-\frac{1}{c^2}\beta_L(\phi_{zzt}-\partial_z(\ln c)\phi_{zt})-\Gamma .$$

The loss term represented by β_L has now become coupled to the perturbation, which means that a wave encountering the abrupt variation in the penetration depth will dissipate some of its energy into the junction electrodes. Converting to the rest frame of the unperturbed soliton (with $\gamma\to\mu$) and inserting the perturbative expansion (23) yields

$$\psi_{\xi\xi}-\psi_{\tau\tau}-(1-2\,\text{sech}^2\xi)\psi\sim(1-\Gamma^2)^{-1/2}\Big[-\mu\partial_z(\ln c)\phi^0_\xi-\alpha\beta\mu\phi^0_\xi+\frac{\beta}{c^2}\mu^3\beta_L[\phi^0_{\xi\xi\xi}-\partial_z(\ln c)\partial^0_{zt}]-\Gamma(1-\cos\phi^0)+\Gamma\,\text{sech}\xi\tanh\xi\psi\Big] . \quad (37)$$

Note that the additional contribution to the potential has been included as part of the perturbation on the right-hand side of the equation. For simplicity, we have also assumed that the terms proportional to ψ and its derivative are small enough to be ignored in all of the perturbing terms on the right-hand side of (37). Finally, when ψ is expanded according to Eq. (14) and the orthogonality relations (13) are used, we find

$$A_{\tau\tau}\sim(\ln c_0)\text{sech}^2\beta\tau+(1-\Gamma^2)^{-1/2}\Big[2\alpha\beta\mu+\frac{\pi}{2}\Gamma+\beta\beta_L\frac{\mu^3}{3c_0^2}[(c_0^2+1)-(c_0^2-1)\tanh\beta\tau-\tfrac{1}{2}(4+3/\gamma^2)(c_0^2-1)\tanh\beta\tau\,\text{sech}^2\beta\tau]\Big] \quad (38)$$

for the expansion coefficient of the bound-state eigenfunction A. In arriving at this expression, we have used

$$c^2(z)=1+(c_0^2-1)\theta(z)$$
$$=1+(c_0^2-1)\theta(\xi+\beta\tau)\ , \quad (39a)$$

and the theory of distributions to generate the identity

$$\frac{1}{c^2}\frac{\partial(\ln c)}{\partial z}=\frac{1}{2c_0}2(c_0^2-1)\delta(z)$$
$$=\frac{1}{2c_0^2}\frac{\mu}{\gamma^2}(c_0^2-1)\delta(\xi+\beta\tau) \quad (39b)$$

prior to inverting the eigenvalue expansion to find $A_{\tau\tau}$.

Usually, the next step would be to integrate (38) to find A_τ. However, it turns out that direct application of the interpretative relations (21) to (38) is sufficient to obtain a power balance expression relating to the bias Γ and the loss factors to the soliton velocity β. As $\tau\to\pm\infty$, Eq. (38) becomes

$$\lim_{\tau\to\infty} A_{\tau\tau}\to(1-\Gamma^2)^{-1/2}\left\{2\alpha\beta\mu+\frac{\pi}{2}\Gamma+\beta\beta_L\frac{\mu^3}{3c_0^2}[(c_0^2+1)\mp(c_0^2-1)]\right\} . \quad (40)$$

Since $A_{\tau\tau}$ is proportional to the acceleration of the soliton, Eq. (40) indicates that there will be an asymptotic acceleration of the soliton unless all of the terms in the equation cancel one another. The alternative signs of the last term indicates that this balance between terms occurs at different steady-state velocities, depending upon the value of c. When $\tau\ll 0$, the soliton has not yet encountered the step, so that its power balance velocity is given by

$$\alpha\beta_<\mu_<+\frac{\pi}{4}\Gamma+\tfrac{1}{3}\beta_<\beta_L\mu_<^3\equiv 0\ . \quad (41a)$$

(The subscripts "<" are used to indicate that the soliton is traveling through the region $z<0$ prior to encountering the step.) Similarly, crossing over into the region $z>0$ where $c=c_0$ requires

$$\alpha\beta_>\mu_>+\frac{\pi}{4}\Gamma+\frac{1}{3c_0^2}\beta_>\beta_L\mu_>^3\equiv 0\ . \quad (41b)$$

The way in which these expressions is used to arrive at a relationship between the bias current Γ and the dc voltage depends upon the actual geometry of the device. If the step-wise variation occurs at a distance l from the end of a junction of total length L (that is, the spatial extent of the region having $c=1$ is l), then the voltage is given by

$$V=V_0\nu\frac{1}{(1+\beta_<(\nu-1)/\beta_>c_0)}\ , \quad (42)$$

where $\nu=L/l$ and V_0 is the voltage that would be generated on a homogeneous junction having $c=1$. Combining Eqs. (41) and (42) yields the desired result.

The variation in penetration depth produces two effects on the junction I-V curve. The first is a general shift in the voltage of the soliton resonant step, and the second is a change in its shape. Figure 2 illustrates the changes predicted by Eqs. (41) and (42) as a function of the size of the step-wise variation in c, while Fig. 3 compares the size of the voltage shift, as measured along the vertical portion of the resonant step, with the same quantity found by directly integrating Eq. (37) numerically.[22] At higher values of c we begin to see some deviation between the theoretical predictions and the simulation. We believe that this is because Eqs. (41) and (42) do not take into account any overall phase shifts that occur as a result of the interaction with the step, but only reflect steady-state conditions far from the perturbation. Near the step, localized accelerations will shift the position of the soliton from its equilibrium value, thereby increasing the amount of time it takes the soliton to traverse the junction. On the I-V curve, this results in a slight lowering of the voltage, as is shown by the numerically generated points in Fig. 3.

Figure 4 illustrates the change in the shape of the resonance as a function of the size of the change in c. Both curves were calculated using Eqs. (41) and (42) (when $c=1$, these equations reduce down to the usual power balance relation for a homogeneous junction[15]). The $c^2=4$ ($c=2$) resonance has been shifted in position to allow direct comparison with the $c^2=1$ curve. The overall change in the shape is not very large, even though the penetration depth has changed by a factor of 4. Clearly, the shift in the position of the resonance is the dominating feature.

We still have to examine the remaining terms in Eq. (38). The first term on the right-hand side of this equation is the same one that was studied in the lossless, undriven example. The force on the soliton represented by this term in localized near the perturbation, and was re-

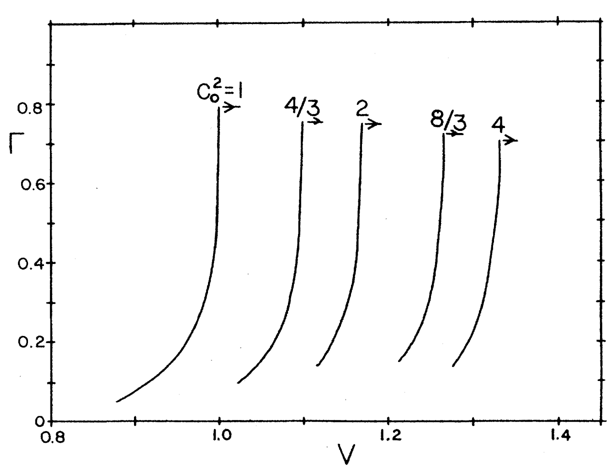

FIG. 2. Theoretical I-V curves calculated using Eqs. (41) and (42). The heights of these curves were found by numerically integrating Eq. (37). Values for α and β_L were 0.1 and 0.05, respectively.

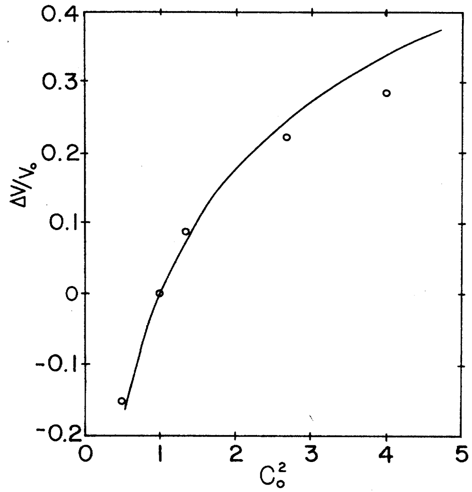

FIG. 3. The shift in voltage measured relative to $c^2=1$ $[V_0=V(c^2=1)]$ as a function of the size of the change in c^2. The solid line represents Eqs. (41) and (42), while the points were found by numerically integrating (37). The current at which the voltage was measured was $\Gamma=0.65$, which was well into the vertical region of the fluxon resonance. ($\alpha=0.1$ and $\beta_L=0.05$).

sponsible for modulating the soliton velocity in Eq. (33). The remaining term also represents a localized change in the soliton velocity, the origin of this force being the loss in the junction electrodes, which absorbs energy released by the soliton as a result of the interaction. Far from the perturbation, the localized corrections to the soliton velocity introduced by these terms do not affect the steady-state velocities given by Eq. (41). Therefore, in the presence of loss and bias, these terms will only contribute to the overall phase shift in the soliton position.

In analyzing the perturbed problem given by Eq. (38), there was also a piece representing the additional term in the potential. The reason for retaining this component was to justify extending the analysis to as large a value of Γ as possible, and to perhaps learn something about the instabilities that cause the junction to switch off the soliton resonance when $\Gamma<0$. However, this term gives no contribution to $A(\tau)$ when treated perturbatively, which suggests that it must be included as part of a more exact analysis of the homogeneous eigenvalue problem in Eq. (24). Therefore, it must be kept in mind that the power balance relations generated here are fundamentally limited to $\Gamma<0.45$, which suggests that perhaps the derivation of these expressions via the conservation of energy or momentum may also be limited.

To examine the effect that the step-wise variation in the penetration depth has on the height of the resonance, we have integrated the fully perturbed sine-Gordon equation numerically.[22–25] Figure 5 shows the value of Γ at which the junction switches off the single soliton resonance as a function of the loss parameter α ($\beta_L=0$) and the size of the variation in c^2. At low values of α, the presence of the step causes the junction to switch prematurely; the larger the step, the more severe the effect. Furthermore, it was found that the value obtained for the switching current was insensitive to the position of the step.[26]

In order to understand exactly what was happening, we generated field plots (not shown) illustrating dynamically the propagation of the soliton as it interacted with the step. These indicated that when the loss term α was small, the radiation generated by the collision could not be fully absorbed by the loss before the soliton interacted with the step again on its return trip. Therefore, energy was continually fed to the radiative modes faster than it could be dissipated. Ultimately, enough energy was final-

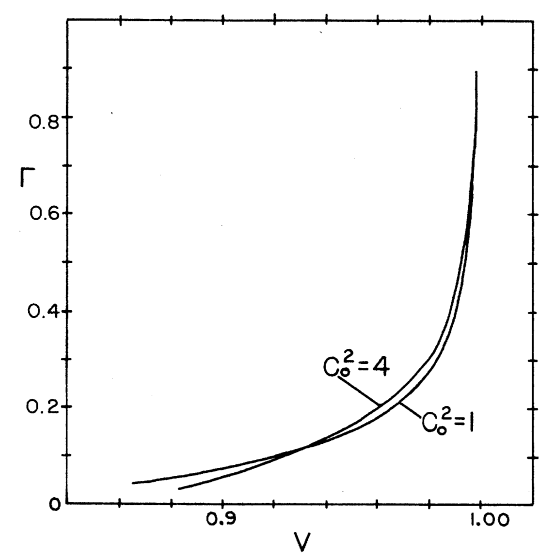

FIG. 4. A comparison of the shapes of the fluxon resonances found using Eqs. (41) and (42) for $c_0^2=1$ and $c_0^2=4$. ($\alpha=0.1$ and $\beta_L=0.05$).

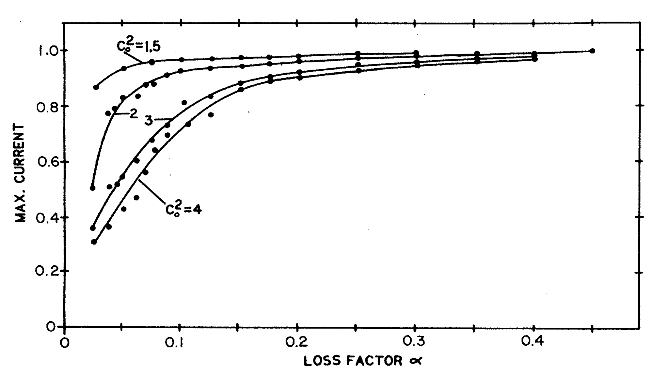

FIG. 5. The height of the fluxon resonance as a function of the size of the damping coefficient α and the size of the variation in c_0^2 These curves were generated numerically using an explicit finite difference scheme ($\beta_L=0$). In order to achieve consistent results, it was necessary to keep $\Delta x \leq 0.7\alpha$; the maximum value used for Δx (consistent with this condition) was 0.1. Δt was equal to $0.5\Delta x$ in all cases.

ly accumulated to create additional solitons, and the junction filled up with flux. One of the more interesting things that we observed was that the solitons were never generated at the point of the collision with the step. Instead, the energy was seen to accumulate at the edge of the junction opposite to the one that the soliton was heading toward. The reflection of the energy off of this edge then took the form of a pulse that grew into a soliton as it propagated away from that end of the junction. We expect that other spatial inhomogeneities will also produce this same effect, which serves to illustrate one way in which lack of control in a junction fabrication process can lead to a degradation in the *I-V* curve of the junctions.

Finally, we have been able to introduce the step-wise variations in the penetration depth on long NbN-Pn junctions, and have measured the change in the resonant voltage of the soliton as a function of the size of this variation. This was accomplished by making the NbN junction electrode much thinner than a penetration depth, and covering one-half of it with tin. The temperature of the junction was then varied, and the effect that this had on the penetration depth of the tin step was measured on the junction *I-V* curve. The results, presented in Fig. 6, are seen to be in agreement with the predictions of Eqs. (41) and (42).

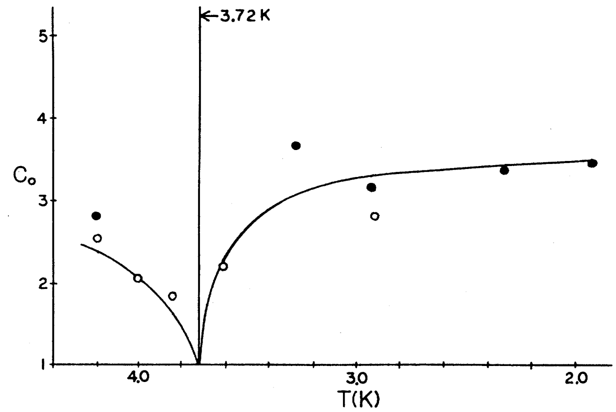

FIG. 6. The variation in the step height introduced by the tin layer as a function of the temperature. The solid curve was calculated directly from Eqs. (41) and (42), and the closed and open circles represent data taken from two separate junctions. The variation in c_0 with temperature above the T_c of the tin (3.72 K) is due to proximity effect between the NbN electrode and the tin. No such variation with temperature was observed on junctions having no tin layer.

VI. CONCLUSIONS

We have reformulated the direct perturbative approach introduced and developed by Rubinstein and others to allow for proper treatment of loss and bias term, and have demonstrated a more general connection between the perturbative solutions and the parameters governing the behavior of the solitary wave form (position, velocity, etc.). In the process of doing this, we have also shown a formal connection between this technique and the theory introduced by McLaughlin and Scott.

The theory was then used to study the effect that a sharp change in the penetration depth has on soliton dynamics, and the way in which this influences the junction *I-V* curve. Changes in the shape, position, and height of the fluxon resonances as functions of the size of the variation in the penetration depth were calculated theoretically and /or numerically, which illustrated both the generality and accuracy of the approach used here, and the extent to which spatial inhomogeneities affect the quality of the junction *I-V* curves. Experimental data was presented that supported these theoretical results. Finally, we have also found evidence to suggest that the usual power balance relation connecting the bias and loss terms to the D.C. average voltage may be fundamentally limited to bias values of $0.45I_c$.

ACKNOWLEDGMENTS

This work was partially supported by U.S. Air Force Office of Scientific Research Grant No. 86-0025.

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