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Synchronization of perturbed sine-Gordon soliton oscillators

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We investigate the dynamics of systems composed of perturbed sine-Gordon soliton oscillators coupled through the boundary conditions. The perturbations are a dc forcing term and linear dissipation. The oscillations maintain their individual frequencies for low values of the coupling parameter. However, for particular values of this parameter, the reflections of the solitons at the boundary give rise to radiation exchange which allows spontaneous synchronization. We analyze our results in the light of recent experiments on long Josephson junction oscillators.

There is now growing interest in the study of collective synchronization in systems of nonlinear oscillators having different natural frequencies. An understanding of the dynamics of such systems could be relevant to collective phenomena in physics, chemistry, and biology.[1] In physics, collective synchronization effects are under investigation in the fields of statistical mechanics,[2] lasers,[3] long Josephson junction (LJJ) arrays,[4] and charge-density-wave systems.[1,5] Nonlinear oscillations that play a dominant role in the dynamics of the last two classes are ones based on the sine-Gordon equation:[5] perturbed sine-Gordon (PSG) equations model magnetic flux-quanta (fluxons) motion in long Josephson junctions and current oscillations in charge-density-wave compounds. In these systems temporal patterns are generated by the spatial degrees of freedom that give rise to solitons and, depending on boundary conditions and forcing terms, several kinds of oscillating modes are possible.[5] Since the space-time patterns are composed of solitons, and considering the broad scientific relevance of these nonlinear excitations, the collective synchronization of systems whose unit cell is regulated by a PSG equation is a very interesting problem.

The dynamics of PSG systems in the presence of ac and dc forcing terms has been the subject of several investigations for both periodic[6] and finite length[7,8] spatial boundary conditions. Among many interesting features observed are the possibility of phase locking the soliton oscillations to the external ac drive and the competing interactions and pattern conversion that take place in particular regions of parameter space. Previous studies have considered the case in which the external stimulation appears as an external forcing term in the PSG equation, either distributed over the whole spatial interval,[6-8] or at one end.[9] By contrast, in the present work we study situations where the ac excitation in one system is due to the oscillations taking place in another system and vice versa. Periodic interactions can be selected by a suitable choice of the perturbing terms (dc force and linear dissipation) and the boundary condition. The oscillations of each PSG system have a different natural frequency and the interaction is generated by a coupling at a common boundary. We analyze numerically the various phase locking regimes between the oscillators and the corresponding space-time patterns. This analysis can enable us to understand important properties of PSG systems, like superradiant behavior of LJJ's arrays,[4] so far investigated only indirectly by means of experimental techniques (measurements of power spectra and current-voltage characteristics).

Let us consider first the simplest case of two coupled PSG systems. We deal with the following perturbed sine-Gordon equation

$$\phi_{tt}-\phi_{xx}-\sin\phi=\rho+\alpha\phi_t\,, \tag{1}$$

over two finite spatial intervals (A and B). In Eq. (1) ρ represents a dc forcing term and $\alpha\phi_t$ a linear dissipation. At one end of the interval A we impose

$$\phi_x=0\,, \tag{2}$$

while at the other end Eq. (1) is coupled to one end of the interval B, though the equation

$$\phi_x=\zeta(\phi_{tt}^A-\phi_{tt}^B)\,, \tag{3}$$

where ζ is a coupling parameter. At the remaining end of the spatial interval B we also impose condition (2). The physical meaning of the boundary conditions (2) and (3) can be easily understood in terms of Josephson electrical quantities:[10] in this context we know that ϕ_x and ϕ_t are proportional to current and voltage, respectively. Therefore, Eq. (2) sets open circuit boundary conditions while, considering ζ as a normalized capacitance, Eq. (3) establishes that the current at the coupled end must be equal to the current flowing through the capacitor. Thus, our PSG systems interact at their common end through the time-dependent currents flowing in a coupling capacitor.

It is known[11,12] that a soliton solution of Eq. (1) scattering on boundaries such as the ones represented by Eq. (2) and Eq. (3) can be reflected as an antisoliton or annihilated depending upon the values of the parameters ρ, α, and ζ. Setting conditions which allow reflections of solitons at both the ends of the spatial interval, soliton-antisoliton oscillations with a frequency approximately equal to $u/2l$ are generated. Here $u\equiv[1+(4\alpha/\pi\rho)^{1/2}]^{1/2}$ is the soliton power balance velocity[11] and l is the normalized length of the spatial interval. We investigate the parameter range for which these oscillations can phase lock when the spatial intervals A and B are unequal. We concentrate on length variations because these enter linearly in the determination of the frequencies of the oscillators and also because making junction lengths equal within a specified uncertainty represents a crucial technological requirement in the fabrication of coherent LJJ's arrays.

Equations (1)–(3) were discretized and integrated by means of a third-order Runge-Kutta routine fixing a time step $\Delta t = \frac{1}{2}\Delta x = 0.025$. In all the simulations we set $\alpha = 0.1$. Then, to allow soliton reflections at the boundary regulated by Eq. (2), the values of ρ have to be greater than 0.1[11] This fact sets a lower limit for the power balance velocity of the solitons undergoing the reflection processes at about 0.62. Also, a detailed study of the boundary condition (3) (Ref. 12) has shown that, depending on the value of the parameter ζ, the solitons can be absorbed or reflected at the coupled end. In particular, if ζ is not much below the threshold $\zeta = 1-(1-u^2)^{1/2}$, we obtain soliton reflections at the coupled end and transmission of radiation between oscillators. We start the solitons with the above power balance velocity around the centers of the spatial intervals. The results prove to be independent of the particular points where the solitons were initialized provided that they were sufficiently far from the ends of the spatial intervals. The strategy of the simulations was the following: For a given value of oscillators lengths (and therefore frequencies) and for a given value of the coupling parameter ζ, we found the interval of ρ for which phase-locking phenomena occur. This procedure is equivalent to a search for phase locking in a dc series array of LJJ.[4] In this case the dc bias current feeding the junctions is varied in order to find a region of phase locking. Once we found the ρ locking ranges for a given ζ, we evaluated the ζ intervals over which phase locking can be maintained for fixed ρ and oscillators length.

The evolution of initial data in which the two solitons are started with the velocity $u = 0.686$ (corresponding to $\rho = 0.12$) for a coupling parameter $\zeta = 0.175$ is shown in Fig. 1. In this figure we have shown phase and phase derivative along the two spatial intervals at the time $t = 250$. The left interval has a length of 10.1 in normalized units while the right interval is 9.85 units long; the coupling point is near the center ($x = 10.1$) of the spatial interval in the figure. In Fig. 1(a) we see that the two solitons, while moving from right to left, occupy symmetrical positions. This is indeed the situation, for $\zeta = 0.175$, at all times beyond an initial transient. The solitons and antisolitons oscillate back and forth along the two lines maintaining their relative positions. A more quantitative way of looking at this phase locking effect is shown in Fig. 1(b). In this figure we show the time evolution of ϕ_t^S, the sum of the time derivatives of the phases at the left ends of the spatial intervals. We see that there is only one peak in the time sequence, which means that the solitons undergo simultaneous reflections. The peak of ϕ_t^S is about twice that of the two separate peaks observed when the two oscillators do not phase lock. From the ϕ_t^S waveform of Fig. 2(b) it is evident that there is only one frequency in the coupled PSG system. In Fig. 1(c) we show a typical ϕ_t^S output waveform obtained when the PSG systems are operated close to the phase-locking region ($\zeta = 0.24$, $\rho = 0.12$). In this case we see that subharmonic generation takes place; this behavior is qualitatively very similar

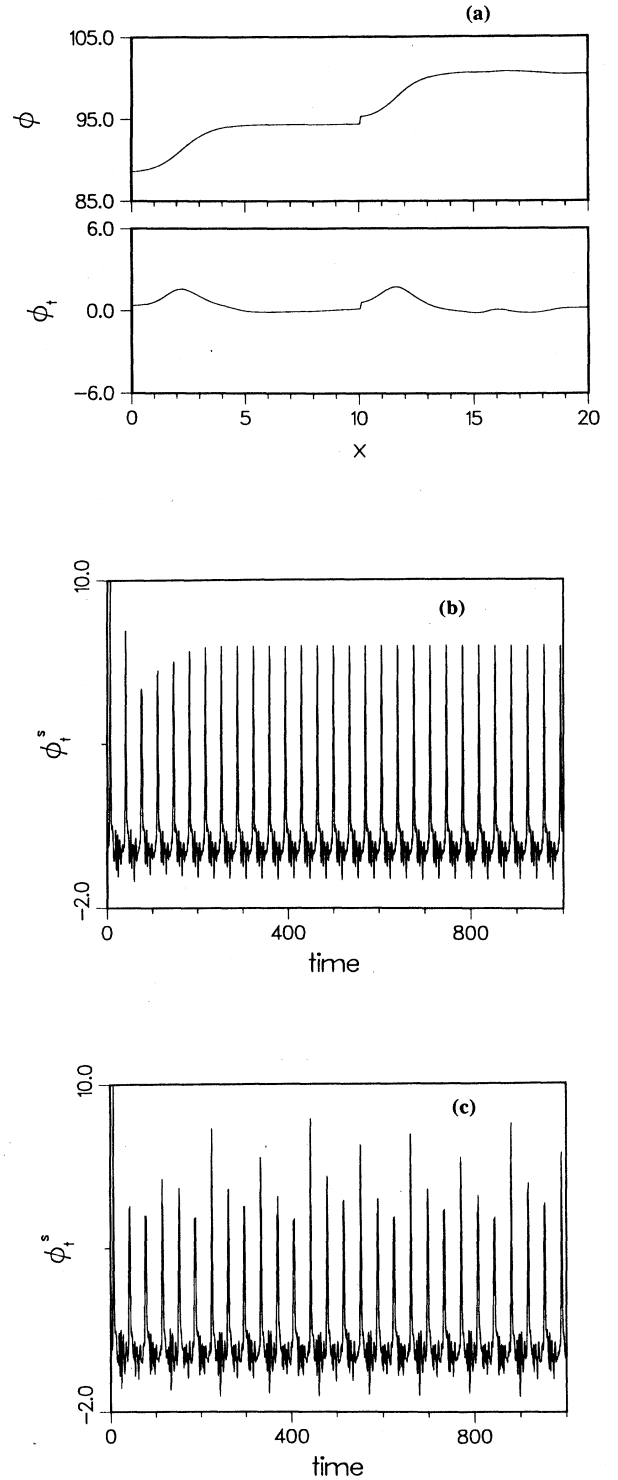

FIG. 1. Examples of different phase-locking regimes between two perturbed sine-Gordon oscillators: (a) phase and phase derivatives distributions along the coupled spatial intervals in a situation of IPL; (b) time evolution of the sum of the phase derivatives at the left ends of the two spatial intervals of (a); (c) subharmonic regime observed close to a parameter space region where IPL takes place. Dimensionless units are used.

to that observed for Eq. (1) on the finite spatial interval with a homogeneous external ac forcing term.[8]

An isochronous phase-locking (IPL) situation like the one in Figs. 1(a) and 1(b) is observed for $\zeta=0.175$ when ρ is varied in the interval [0.12,0.135]; we note that this corresponds to the interval [0.68,0.72] for the velocity u. This means that if the oscillators are locked for a given value of ζ they will remain locked for a frequency interval of about 5%. Fixing $\rho=0.12$, the IPL is also observed when ζ is varied in the interval [0.15,0.22]. We have investigated the dependence of the IPL upon the absolute lengths of the spatial intervals, choosing the two lengths 7.35 and 7.6 (i.e., different by about 3.4%). In this case the phase-locking interval in ρ is [0.145,0.15] for $\zeta=0.15$, while fixing $\rho=0.15$ the range of phase locking for ζ is [0.17,0.25].

Having obtained information about the phase locking of the two coupled oscillators, we investigated the possibility of locking a larger number of PSG oscillators having different internal frequencies. We did this for systems including three and four coupled PSG oscillators. The PSG systems were coupled in all cases by Eqs. (2) and (3) was imposed at the free ends of the two terminal intervals. For both the cases of three and four coupled PSG we observed features analogous to the ones of the two PSG systems for the same parameter ranges. In Fig. 2 we show a ϕ_t^S waveform for a system of four oscillators whose lengths differ within 2.5%. The IPL regime is evident and we can see that the peaks are about twice the ones of Fig. 1(b). For the simulations of Fig. 2 we set $\zeta=0.175$ and $\rho=0.12$.

If we think in terms of long Josephson junction oscillators, we note that ϕ_t^S corresponds to the time evolution of the sum of the ac Josephson voltages measured at the ends of the junctions. Since we have found that in conditions of IPL this voltage increases as the number of junctions, the power at the corresponding (single) frequency will increase as the square of the number of the junctions. We emphasize that this result has been found as a direct consequence of the internal dynamics of the PSG system after considering a realistic model for photon exchange

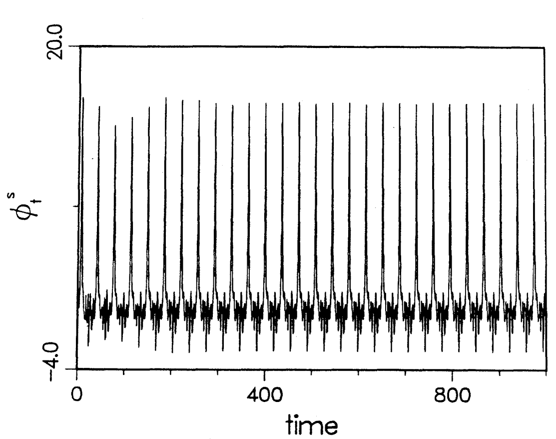

FIG. 2. The time evolution of the sum of the phase derivatives at the left ends of four coupled perturbed sine-Gordon oscillators. Note that the peaks are about twice as big as the ones in Fig. 1(b).

between the junctions. Because of the capacitive nature of the coupling equation (3), our simulated PSG system can model a situation in which the long Josephson junctions interact only through a mutual exchange of high frequency photons. Thus, it makes sense to compare our numerical data to the experiments reported in Ref. 4. These experiments have indeed demonstrated[5] that a considerable increase in the power of the emitted radiation can be obtained from dc-series arrays of long Josephson junctions. Considering that the junctions of Ref. 4 have a length of about 0.5 mm and a normalized length less than 5, a scatter of the junction physical lengths of about 3.4% (the numerically allowed coherence range for junctions whose normalized length is 7.5) implies that a 15 μm uncertainty in the fabrication process is allowable. This error is very plausible within the technology employed for the experiments[5] and so we would expect the observed synchronization phenomena on the basis of the simulations.

We further note that the results of the simulations are qualitatively very similar to the experimental ones. In Ref. 4 it was reported that the coherence could be observed only for particular bias current choices which is also what we found. However, as in previous experiments,[13,14] the measured intervals of coherence were very small (of the order of few MHz). Our results suggest that once that the junctions lock for a given ζ, it should be possible to observe coherent behavior between two long Josephson junctions over frequency ranges of the order of 1.5% of the natural frequency of emission (which is proportional to the velocity of the solitons) in the worst possible case. If a junction is operating at 10 GHz, this means that we should be able to observe intervals of coherent phase locking up to 150 MHz. Indeed, it has been shown that a long Josephson junction operated in the soliton-reflection mode can lock to an external source of electromagnetic radiation for frequency intervals as large as 80 MHz.[14] We speculate that the existence of large capacitive gaps between the junctions was the reason for the poor ranges of locking observed in the frequency-domain experiments. We do not exclude the possibility that if the value of the coupling parameter ζ is very small, the soliton oscillations could phase lock over very tiny ρ intervals. However, it may be difficult to identify these IPL regions in numerical simulations. Recent dc measurements[15] appear to confirm the poor coupling hypothesis because it has been shown that two long Josephson junctions phase lock very strongly with a suitably designed capacitive coupling.

The junctions length scatter allowing IPL is large enough that we could expect coherent behavior even from LJJ's arrays operating in millimeter wave regions. Active devices operating at these frequencies are presently a subject of great interest for the applied superconducting community.[16] A typical length of a LJJ operating at millimeter wave frequencies is about 60 μm.[15] Then, the 3.4% scatter on length determination allows a fabrication uncertainty of 2 μm. This is not an unreasonable margin considered the level of sophistication achieved in superconducting integrated technology.[16] Also, it should not be very difficult to design coupled LJJ devices[12] for which the value of the parameter ζ is of the order of magnitude giving rise to coherence in the numerical simulations.

In conclusion, we have reported on detailed space-time features accompanying coherent behavior of coupled perturbed sine-Gordon oscillators. Although the analysis performed is far from representing a fully comprehensive study, we believe that the phenomena presented should stimulate interest and further work in the general field of phase locking of nonlinear oscillators. Moreover, we have shown that our work should deserve attention from the applied Josephson community: an experimental confirmation of the results of our simulations would mean achievement of significant intervals of coherence for series arrays of long Josephson junction fluxon oscillators and, by consequence, a more stable increase of the emitted power.

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[1]See, e.g., S. H. Strogatz, C. M. Marcus, R. M. Westervelt, and R. Mirollo, Phys. Rev. Lett. **61**, 2380 (1988); Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984); A. T. Winfree, J. Theor. Biol. **16**, 15 (1967).

[2]H. Sompolinsky (unpublished).

[3]H. Haken, *Advanced Synergetics* (Springer, Berlin, 1983).

[4]R. Monaco, S. Pagano, and G. Costabile, Phys. Lett. A **131**, 122 (1988).

[5]A. R. Bishop, B. Horovitz, and P. S. Lomdahl, Phys. Rev. B **38**, 4853 (1988); *Spatio-Temporal Coherence and Chaos in Physical Systems,* edited by A. R. Bishop, G. Gruner, and B. Nicolaenko (North-Holland, Amsterdam, 1986).

[6]A. R. Bishop, K. Fesser, P. S. Lomdahl, W. C. Kerr, M. B. Williams, and S. E. Trullinger, Phys. Rev. Lett. **50**, 1095 (1983).

[7]M. Cirillo, J. Appl. Phys. **60**, 338 (1986).

[8]L. E. Guerrero and M. Octavio, Phys. Rev. A **37**, 3641 (1988).

[9]O. H. Olsen, and M. R. Samuelsen, Appl. Phys. Lett. **47**, 1007 (1985).

[10]A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).

[11]D. W. McLaughlin and A. C. Scott, Phys. Rev. A **18**, 1672 (1978).

[12]M. Cirillo, A. R. Bishop, P. S. Lomdahl, and S. Pace (unpublished).

[13]T. F. Finnegan and S. H. Wahlsten, Appl. Phys. Lett. **21**, 541 (1972).

[14]M. Cirillo, and F. L. Lloyd, J. Appl. Phys. **61**, 2581 (1986).

[15]M. Cirillo, I. Modena, P. Carelli, and V. Foglietti, Phys. Lett. **132A**, 20 (1988).

[16]S. Morohashi and S. Hasuo, J. Appl. Phys. **61**, 4835 (1987).