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Resonant activation of a Brownian particle out of a potential well: Remarks on some recent theoretical and experimental investigations

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A unifying physical picture, the purpose of which is to account for both the experimental results of Devoret *et al.* [Phys. Rev. Lett. **53**, 1260 (1984)] and their more recent simulation data [Phys. Rev. B **36**, 58 (1987)] throughout a wide range of values of the friction γ is discussed. Our theoretical predictions are compared to the results of a digital simulation experiment, which we have carried out for this specific purpose.

The resonant activation of a current-biased Josephson tunnel junction has been recently studied by using the picture of a Brownian particle, with space coordinate x and velocity v, moving in a nonlinear potential $V(x)$ under the influence of both a coherent and oscillatory driving force and a standard fluctuation-dissipation process[1-3]

$$\begin{aligned}\dot{x}(t)&=v(t)\,,\\ \dot{v}(t)&=-\frac{\partial V(x)}{\partial x}-\gamma v(t)+A\cos(\omega_R t)+f(t)\,.\end{aligned}\tag{1}$$

$f(t)$ is assumed to be a white Gaussian noise defined by

$$\langle f(t)\rangle=0\,,\tag{2}$$

$$\langle f(0)f(t)\rangle=2\gamma k_B T\delta(t)\,.\tag{3}$$

By using the projection-operator method,[4] the system of Eq. (1) is described by a Fokker-Planck equation, where the effect of the coherent field is simulated by a v-diffusion term, with a diffusion coefficient dependent on x and v.

We know from the celebrated Kramers theory[5] that in the inertial case the process of escape from a potential well is given by (V_0 is the barrier intensity)

$$\Gamma(A=0)=\left[\frac{V_0}{k_B T}\right]\gamma e^{-V_0/k_B T}\,,\tag{4}$$

and is dominated by the energy diffusion around the bottom of the reactant well. This allows us to safely[6,7] make a mean-field approximation, which replaces the x- and v-dependent diffusion coefficient with its mean value. The resulting Fokker-Planck equation is shown[3] to lead immediately to the following effective temperature:

$$T_{\rm eff}=T+\frac{P(\omega_R)}{k_B\gamma}\,,\tag{5}$$

where $P(\omega)$ is the energy absorbed by the system per unit of time (when the steady state is reached) and is given by

$$P(\omega_R)=\frac{2A^2}{k_B T}\,\mathrm{Re}\int_0^\infty\langle v(0)v(t)\rangle_{\rm eq}e^{i\omega_R t}dt\,.\tag{6}$$

The nice aspect of this theory is that, according to the spirit of the Kubo linear-response theory,[8] $P(\omega_R)$ is evaluated in terms of a property of the system in the absence of the coherent excitation, i.e., $\langle v(0)v(t)\rangle_{\rm eq}$.

Thus, $\Gamma(A)$ is obtained from Eq. (4) by replacing T with $T_{\rm eff}$ of Eq. (5) and the enhancement of the rate process, defined by Devoret *et al.*[2] as

$$R\equiv\Gamma(A)/\Gamma(0)\,,\tag{7}$$

is written as follows:

$$R=\frac{k_B T}{k_B T+P(\omega_R)/\gamma}\times\exp\left[-\frac{V_0}{k_B T+P(\omega_R)/\gamma}\right]\exp\left[\frac{V_0}{k_B T}\right].\tag{8}$$

This leads us to the following expression for the observable $\ln R$:[2]

$$\ln R=-\ln\left[1+\frac{P(\omega_R)}{\gamma T}\right]+\frac{V_0}{k_B T}\,\frac{P(\omega_R)}{\gamma k_B T+P(\omega_R)}\,.\tag{9}$$

Under the assumption

$$P(\omega_R)/\gamma \ll k_B T , \tag{10}$$

the logarithm of R reads

$$\ln R \approx \left(\frac{V_0}{k_B T} - 1 \right) \frac{P(\omega_R)}{\gamma k_B T} . \tag{11}$$

From now on the problem reduces to that of evaluating the Laplace transform of $\langle v(0)v(t)\rangle_{eq}$ at the frequency ω_R. This is not an easy task and a detailed investigation has recently been totally devoted to discussing this intriguing problem.[9] In Ref. 9 this problem has been faced with different techniques. First of all, a very advanced recursion method[10] has been used. The convenience of each technique depends on the physical regime under investigation. It has been shown that when the damping γ is large enough, the major reason for the line broadening is the dissipation term $-\gamma v(t)$ itself appearing in Eq. (1). This is the counterpart of longitudinal relaxation processes in magnetic resonances, since the broadening is associated with an exchange of energy between system and "bath."[9] In the extremely underdamped limit, on the contrary, the main reason of broadening is the anharmonic nature of the potential V. A very small γ means that an almost vanishing stochastic force is sufficient to make the Brownian particles spread within the potential well. Due to the anharmonic nature of the potential V, these Brownian particles explore regions characterized by different local frequencies. When studying a process of escape from a potential well, a finite temperature means that the Brownian particle can "feel" the influence of local frequencies lower than that corresponding to the harmonic approximation at the bottom of the potential well.[11] When the system is excited by a coherent radiation field, the result is a line broadening of the absorption spectrum, which does not imply any significant exchange of energy between system and bath. This is reminiscent of transverse relaxation processes in the field of magnetic resonances.[9] When the former condition applies, the use of the recursion rules of Ref. 10 leads to reliable results.[3,12] On the contrary, when the latter condition applies, the recursion rule method should be supplemented by suitable asymptotic techniques, leading virtually to a resummation over an infinite continued fraction.[13] In this case it is convenient to follow a different procedure.

If the "longitudinal" relaxation process is to be totally neglected, the problem of the evaluation of $P(\omega_R)$ can be faced by adopting a technique which consists of taking advantage from the property of nonlinear oscillators, the oscillations of which are related to the energy E of an oscillatory trajectory in the well via a well precise relation, denoted by $\omega(E)$. We thus obtain[9]

$$\ln R = \frac{\pi V_0 A^2}{\gamma (k_B T)^4} \left[\left(\frac{dE}{d\omega} \right) \exp\left(-\frac{E(\omega)}{k_B T} \right) E(\omega) \right]_{\omega=\omega_R} . \tag{12}$$

The theory of Ref. 9 allows us to predict that the line shape of $\ln R$ on the high-frequency side is controlled by γ. The behavior on the low-frequency side is proved to exhibit universal behavior.[2] This is so because from Ref. 9 we see that in the *extremely underdamped limit*[9] the low-frequency side of the spectrum[11] consists of a series of "quantized" transition frequencies embodied within a broad spectrum and, on this side, the "longitudinal" relaxation process only contributes to making those transition overlap. Thus, as long as the "longitudinal" process is kept conveniently slower than the "transverse" one, this side of the spectrum must only depend on the potential shape and the temperature T. On the high-frequency side, on the other hand, according to Eq. (12) the spectrum should brusquely end at the frequency corresponding to the harmonic expansion at the bottom of the potential well. The presence of a finite "longitudinal" process establishes a further line broadening, proportional to γ, and this is the reason why this side of the spectrum is proven to sensibly depend on γ.[2]

For a quantitative illustration of our remarks above, taking also into account the effect of a finite "longitudinal" relaxation, we can generalize the result of Eq. (12) according to the lines of Ref. 12. This leads us to the following expression for $P(\omega_R)$:

$$P(\omega_R) = 2S^2 \frac{1}{1+P(\omega_R)/\gamma k_B T} \frac{\gamma}{2} \times \int_0^{V_0} \frac{E\sigma_{exc}(E)dE}{(\gamma/2)^2 + [\Delta\omega(E)]^2} , \tag{13}$$

where

$$\Delta\omega(E) \equiv \omega_R - \omega(E) , \tag{14}$$

$$S \equiv \frac{1}{2} \frac{A}{\sqrt{k_B T}} , \tag{15}$$

and

$$\sigma_{exc}(E) \propto \exp\left(\frac{E}{k_B T_{eff}} \right) . \tag{16}$$

$\omega(E)$ is precisely the inverse of $E(\omega)$ appearing in Eq. (12). According to the theory of Ref. 12, this expression also takes into account that, due to the absorption of ener-

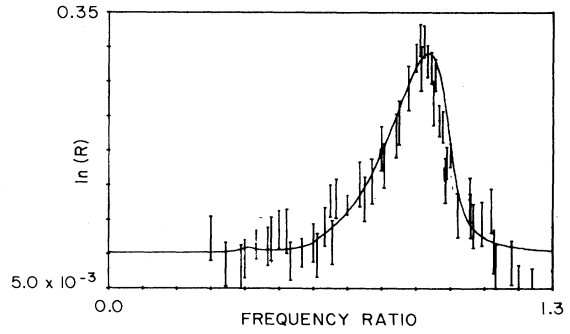

FIG. 1. $\ln R$ [Eq. (9)] vs ω_R/ω_B, where ω_B is the frequency at the bottom of the well. The solid line is the theory [Eq. (13)], the data with bars are the results of the computer simulation; the error bars are the one standard deviation confidence interval. The potential [Eq. (1)] is $V(x) = px + \cos(x)$. In the simulation $p = 0.73$, $k_B T = 0.1$, $A = 0.015$, and $\gamma = 0.05$ [see Eqs. (1) and (3)].

gy from the coherent field, the effective temperature of the system increases. The unperturbed energy distribution of the linear response theory is indeed replaced by the excited energy distribution of Eq. (16). This leads to the evaluation of T_{eff} via an iterative approach. The mere application of the linear-response theory would lead us to the same scaling properties as those pointed out by the authors of Ref. 2. This iterative approach, on the contrary, leads to a deviation from the predictions of the linear-response theory which can be significant around the top of the absorption curve.

To test this prediction we carried out a digital simulation experiment leading to results in qualitative agreement with those of Ref. 2. The algorithm used is described elsewhere,[14] and it is applied with a time step set to $\frac{1}{50}$ of the oscillation period at the bottom of the well and the residence time inside the well computed by averaging more than 2000 first passage times. On leaving the well, the particle was brought back to the bottom of the well with random velocity extracted from a Gaussian distribution with zero average and standard deviation $k_B T$ and random phase of $\cos(\omega_R t)$, extracted from a flat distribution between 0 and 2π. In Fig. 1 we compare our theoretical prediction to the results of our numerical simulation. We used an excitation field so weak that at the top of the absorption curve our iterative approach leads to a T_{eff} with a relative deviation from the prediction of the linear response of about 15%. Note that no fitting parameters have been used. The quantitative agreement is as satisfactory as that of other recent theoretical approaches.[15,16]

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