Correlation functions of one-dimensional quantum systems

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A number of one-dimensional quantum systems like interacting fermions or spin chains can be described in terms of a generalized quantum sine-Gordon Hamiltonian. The transfer-matrix formulation of critical two-dimensional models also frequently leads to sine-Gordon Hamiltonians. We compute here the different correlation functions of such a model using a real-space renormalization technique. Our renormalization calculation gives us the exponents of the correlation functions as well as the corrections to the usual power-law behavior due to marginally irrelevant operators. Except on the critical line, where logarithmic corrections exist at all length scales, the correlation functions decay like power laws, with amplitude corrections from the marginal operators. The logarithmic corrections always appear at short length scale. The complete crossover between the two behaviors is given by our equations. The implication of our calculation for physical systems like the one-dimensional electron gas, quantum spin chains, and two-dimensional statistical systems is discussed.

I. INTRODUCTION

In one-dimensional quantum systems an ordered ground state with a broken continuous symmetry cannot exist, due to strong zero-point quantum fluctuations. Typical examples of this behavior are interacting fermions in one dimension[1,2] or Heisenberg quantum spin chains.[3] In a mean-field-like approach these systems would exhibit symmetry-breaking phase transitions at nonzero temperatures, however the order is destroyed by quantum fluctuations even at zero temperature. Nevertheless, the tendency of the system to order manifests itself in different correlations functions which in many cases exhibit power-law decay at long distances, frequently with nonuniversal exponents. Correspondingly, there are generalized susceptibilities which show power-law divergences as $T \rightarrow 0$.

Similarly, broken continuous symmetries are forbidden in two-dimensional systems at nonzero temperatures, in this case due to thermal fluctuations.[4] However, in some cases, below a certain temperature, quasi-long-range order, characterized again by power-law decay of correlation functions, exists. The most prominent example of this type of behavior is the two-dimensional XY model, and the associated two-component Coulomb plasma.[5,6]

In all the above examples, the power-law decay of correlation functions is related to the fact that the asymptotic long-distance properties of these models are described by a Gaussian model, or equivalently by a noninteracting massless boson theory. The Gaussian model also describes the critical properties of a number of two-dimensional models with a broken *discrete* symmetry like the eight-vertex and Ashkin-Teller models.[7–9] However, in all these cases there are corrections to this ideal behavior, coming from marginally irrelevant operators. As we shall show, these terms lead quite frequently to logarithmic corrections to the long-range behavior of correlation functions. These corrections can be important in the determination of the phase diagram of weakly coupled chains, e.g., quasi-one-dimensional conductors[10] or quasi-one-dimensional magnets. The corrections are also of significance if one wants to determine exponents numerically from finite-size calculations.[11–15]

It should be mentioned here that even though power-law correlations do occur frequently in low-dimensional systems, they are by no means universal. There are well-known examples where fluctuation effects are strong enough to lead to exponentially decaying correlations: the $O(n)$-vector model with $n>2$ in two dimensions,[16] for example, or quantum spin chains with *integer* spin.[17] On the other hand, power-law correlations do not universally imply an underlying noninteracting boson model: Wess-Zumino type models[18–20] (and some particular spin-chain models related to them[21]) do exhibit power-law correlations, however, the underlying field theory is not a noninteracting Bose field. Logarithmic corrections in this last case have been studied by Affleck *et al.*[15]

Corrections to the leading power-law behavior of *energies* in the presence of symmetry-breaking perturbations have been considered previously,[9,22,23] and logarithmic corrections have also been found in the spin-correlation function of the two-dimensional XY model.[6] In this paper we compute the different *correlation functions* of a generalized sine-Gordon model onto which the different physical systems related to the Gaussian model can be mapped. Our computation will give the long-range behavior of the correlation functions, including the different corrections, as well as the complete crossover between the short and large distance behavior. In all cases fluctuations play a major role and simple treatments such as straightforward perturbation expansions or RPA are insufficient. We therefore use a real space renormalization technique similar to that of Kosterlitz[6] or Nelson[24,25] to perform the calculation.

The plan of the paper is as follows. In Sec. II we introduce the model and the different correlation functions we will consider in the following. We give the renormalization equations for the coupling constants. In Sec. III the lowest-order renormalization equations are obtained for the most general case. Their behavior is studied in detail in some particular cases. In Sec. IV our results are applied to one-dimensional interacting fermions, quantum spin-chains, and two-dimensional critical systems. Finally in Sec. V we examine some consequences of our results.

II. GENERALIZED SINE-GORDON MODEL

We consider the generalized sine-Gordon Hamiltonian

$$H=H_0+\frac{2g_\phi}{(2\pi\alpha)^2}\int dx\,\cos\sqrt{8}\phi - \frac{2g_\theta}{(2\pi\alpha)^2}\int dx\,\cos\sqrt{8}\theta_\sigma\ , \tag{2.1}$$

where H_0 is defined by

$$H_0=\frac{1}{2\pi}\int dx\left[(uK)(\pi\Pi)^2+\left[\frac{u}{K}\right](\partial_x\phi)^2\right]\ , \tag{2.2}$$

where $\pi\Pi(x)=\partial_x\theta$ is the momentum density conjugate to ϕ:

$$[\phi(x),\Pi(x')]=i\delta(x-x')\ . \tag{2.3}$$

In order to regularize the theory we have to introduce a short distance cutoff α. We introduce the abbreviation

$$K=1+g_0/2\pi u\ . \tag{2.4}$$

For $g_\phi=g_\theta=0$ u is the velocity of the low-energy excitations of H, whereas K determines the exponents of different correlation functions, see Sec. III. H is invariant under the transformation $\phi\leftrightarrow\theta$ $g_\phi\leftrightarrow g_\theta$, $K\leftrightarrow 1/K$.

To describe the physical properties of our system we will consider the correlation functions

$$\begin{aligned}
R_0&=2\langle T_\tau \cos[\sqrt{2}\phi(r_1)]\cos[\sqrt{2}\phi(r_2)]\rangle\ ,\\
R_1&=2\langle T_\tau \cos[\sqrt{2}\theta(r_1)]\cos[\sqrt{2}\theta(r_2)]\rangle\ ,\\
R_2&=2\langle T_\tau \sin[\sqrt{2}\theta(r_1)]\sin[\sqrt{2}\theta(r_2)]\rangle\ ,\\
R_3&=2\langle T_\tau \sin[\sqrt{2}\phi(r_1)]\sin[\sqrt{2}\phi(r_2)]\rangle\ ,
\end{aligned} \tag{2.5}$$

where $r=(x,u\tau)$, and T_τ is the time-ordering operator. The generalization to other correlation functions will be given in Sec. III D.

From a perturbation expansion of the correlation functions we have previously derived the renormalization group equations for the change of the effective coupling constants under a change of the cutoff $\alpha\to e^l\alpha$:[10]

$$\begin{aligned}
\frac{dy_0}{dl}&=y_\theta^2(l)-y_\phi^2(l)\ ,\\
\frac{dy_\phi}{dl}&=-y_\phi(l)y_0(l)\ ,\\
\frac{dy_\theta}{dl}&=y_\theta(l)y_0(l)\ ,
\end{aligned} \tag{2.6}$$

where we have introduced y_ν for $g_\nu/\pi u$ and expanded to lowest order in y_0. From these equations one finds a critical plane $g_0-|g_\phi|+|g_\theta|=0$. This plane separates the two behaviors $g_0\to\pm\infty$. For $g_0\to-\infty$ the quantum fluctuation (kinetic energy) term in (2.2) is suppressed and $|g_\phi|$ increases under renormalization. Therefore one has long-range order of the ϕ field, and correlations of θ decay exponentially at large distance. The behavior for $g_0\to+\infty$ is obtained from the transformation $\phi\leftrightarrow\theta$. The average values of the ordered fields are given by

$$\begin{aligned}
&g_0\to-\infty\ \begin{cases} g_\phi>0 & \phi=\pi/\sqrt{8}\ ,\\ g_\phi<0 & \phi=0\ ,\end{cases}\\
&g_0\to+\infty\ \begin{cases} g_\theta<0 & \theta=\pi/\sqrt{8}\ ,\\ g_\theta>0 & \theta=0\ .\end{cases}
\end{aligned} \tag{2.7}$$

The case $g_\theta=0$ (or $g_\phi=0$) corresponds to the usual case of a Kosterlitz-Thouless[6,5] transition. There is (for $g_\theta=0$) a line of fixed points g_0^*, $g_\phi=0$ if $g_0>|g_\phi|$. If $g_0<|g_\phi|$ there is a renormalization to strong coupling with $g_0\to-\infty$, which gives the same ordered fields as before.

III. CORRELATION FUNCTIONS

A. Renormalization of the correlation functions

In the absence of g_ϕ and g_θ terms the functions R_i are easily computed:[2]

$$R_i(r_1-r_2)=\exp[-K_iU(r_1-r_2)]\ , \tag{3.1}$$

where

$$U(r-r')=\tfrac{1}{2}\ln\left[\frac{(x-x')^2+(u|\tau-\tau'|+\alpha)^2}{\alpha^2}\right]\ , \tag{3.2}$$

with $K_0=K_3=K$, $K_1=K_2=1/K$. If $r\gg\alpha$ then

$$U(r)\sim\ln(r/\alpha)\ . \tag{3.3}$$

We will use this approximate form of U in the following. Let us consider the functions $F_i(r_1-r_2)=R_i(r_1-r_2)\exp[K_iU(r_1-r_2)]$. In the absence of g_ϕ and g_θ, F_i reduces to the constant 1.

If we now include the interactions g_ϕ and g_θ and compute F perturbatively, the development is divergent for large r_1-r_2. To handle such a situation we use a method similar to that of Kosterlitz.[6] If $r_1-r_2\sim\alpha$ the development in powers of g_ϕ, g_θ is convergent and for small enough coupling constants $F_i\sim 1$. We will thus try to find a function I_i such that

$$\begin{aligned}
&F_i(r,\alpha e^l,g(l))=I_i(dl,g(l))\ ,\\
&F_i(r,\alpha e^{l+dl},g(l+dl))\ ,
\end{aligned} \tag{3.4}$$

where the different $g(l)$ are the solutions of (2.6), for an infinitesimal transformation $\alpha(l+dl)=\alpha(l)e^{dl}$. Here and in the following α always denotes the initial short-distance cutoff, whereas the rescaled cutoff is written as $\alpha(l)=\alpha e^l$. If we use

$$F_i(r,r,g(\ln(r/\alpha)))=O(1)\ , \tag{3.5}$$

provided that the couplings g are still small, and repeat (3.4) until the cutoff $\alpha(l)$ reaches r, we obtain

$$F_i(r,\alpha,g(\alpha))=\prod_{l=0}^{l=\ln(r/\alpha)} I_i(dl,g(l)) = \exp\left[\int_0^{\ln(r/\alpha)} \ln[I(dl,g(l))]\right] . \tag{3.6}$$

The algebra is left to the Appendix, and we obtain

$$\begin{aligned}
R_0(r)&= \exp\left[-K\ln(r/\alpha)+\int_0^{\ln(r/\alpha)} dl\{-y_\phi(l)+\tfrac{1}{2}[y_\phi^2(l)-y_\theta^2(l)]\ln[r/\alpha(l)]\}\right] ,\\
R_1(r)&= \exp\left[-K^{-1}\ln(r/\alpha)+\int_0^{\ln(r/\alpha)} dl\{+y_\theta(l)-\tfrac{1}{2}[y_\phi^2(l)-y_\theta^2(l)]\ln[r/\alpha(l)]\}\right] ,\\
R_2(r)&= \exp\left[-K^{-1}\ln(r/\alpha)+\int_0^{\ln(r/\alpha)} dl\{-y_\theta(l)-\tfrac{1}{2}[y_\phi^2(l)-y_\theta^2(l)]\ln[r/\alpha(l)]\}\right] ,\\
R_3(r)&= \exp\left[-K\ln(r/\alpha)+\int_0^{\ln(r/\alpha)} dl\{+y_\phi(l)+\tfrac{1}{2}[y_\phi^2(l)-y_\theta^2(l)]\ln[r/\alpha(l)]\}\right] .
\end{aligned} \tag{3.7}$$

From (2.6) we obtain

$$\tfrac{1}{2}\int_0^{\ln(r/\alpha)} dl(y_\theta^2-y_\phi^2)\ln\left[\frac{r}{\alpha(l)}\right] = -\tfrac{1}{2}y_0\ln(r/\alpha)+\tfrac{1}{2}\int_0^{\ln(r/\alpha)} y_0(l)dl\ . \tag{3.8}$$

As $K=1+y_0/2$ we have

$$\begin{aligned}
R_0&=\frac{\alpha}{r}L_1^{-1}L_2L_3^{-1}\ ,\\
R_1&=\frac{\alpha}{r}L_1L_2L_3\ ,\\
R_2&=\frac{\alpha}{r}L_1L_2^{-1}L_3^{-1}\ ,\\
R_3&=\frac{\alpha}{r}L_1^{-1}L_2^{-1}L_3\ ,
\end{aligned} \tag{3.9}$$

with

$$\begin{aligned}
L_1(r)&= \exp\left[\tfrac{1}{2}\int_0^{\ln(r/\alpha)} dly_0(l)\right] ,\\
L_2(r)&= \exp\left[\tfrac{1}{2}\int_0^{\ln(r/\alpha)} dl[y_\theta(l)-y_\phi(l)]\right] ,\\
L_3(r)&= \exp\left[\tfrac{1}{2}\int_0^{\ln(r/\alpha)} dl[y_\theta(l)+y_\phi(l)]\right] .
\end{aligned} \tag{3.10}$$

Equations (3.9) and (3.10) give the correlation functions to lowest nontrivial order in the coupling constants.

B. Solution for some particular cases

The general solution for the correlation function will be given in the next subsection, but it is interesting to study the equations for some simple cases. We treat here the limiting case $y_\theta=0$. We limit ourselves to $y_0\geq y_\phi$, i.e., the case where a fixed point at finite coupling exists. Other cases can be treated similarly provided that one limits oneself to sufficiently small distances, so that one remains in the range of validity of the renormalization equations. We will denote in the following the initial values of the coupling constants by y_ν and by y_ν^f the final ones, i.e., those for $l=\ln(r/\alpha)$.

Equations (2.6) are easily solved:

$$\begin{aligned}
y_0(l)&=\Delta/\{\tanh[\Delta l+\mathrm{artanh}(\Delta/y_0)]\}\ ,\\
y_\phi(l)&=\Delta/\{\sinh[\Delta l+\mathrm{artanh}(\Delta/y_0)]\}\ ,
\end{aligned} \tag{3.11}$$

where $\Delta=(y_0^2-y_\phi^2)^{1/2}$ is an invariant of the renormalization equations. We have from (3.10) $L_3=1/L_2$.

1. $y_0=y_\phi$

In this case $y_0(l)=y_0/(1+y_0 l)$ and

$$L_1=L_3=(y_0/y_0^f)^{1/2}\ . \tag{3.12}$$

If $r\to\infty$ then $y_0^f\sim 1/l$ and

$$\begin{aligned}
R_0(r)&=\frac{\alpha}{r}\ln^{-3/2}(r/\alpha)\ ,\\
R_1(r)&=R_2(r)=R_3(r)=\frac{\alpha}{r}\ln^{1/2}(r/\alpha)\ .
\end{aligned} \tag{3.13}$$

All the correlation functions decay with the same exponent, but logarithmic corrections enhance $R_{1,2,3}$ over R_0.

2. $\Delta\neq 0$

In this case we have

$$\begin{aligned}
L_1(r)&=[\cosh(\Delta l)+\frac{y_0}{\Delta}\sinh(\Delta l)]^{1/2}\ ,\\
L_3^2&=C_3\tanh\{[\Delta l+\mathrm{artanh}(\Delta/y_0)]/2\}\ ,
\end{aligned} \tag{3.14}$$

with $C_3=1/\tanh[\mathrm{artanh}(\Delta/y_0)/2]$. Then, from (3.9)

$$\begin{aligned}
R_0&=\frac{\alpha}{r}L_1^{-1}L_3^{-2}\ ,\\
R_3&=\frac{\alpha}{r}L_1^{-1}L_3^{2}\ ,\\
R_1&=R_2=\frac{\alpha}{r}L_1\ ,
\end{aligned} \tag{3.15}$$

As now we have $y_\phi\neq y_0$, R_3 and $R_{1,2}$ are no longer degenerate. There are two different qualitative behaviors depending on the values of $l=\ln(r/\alpha)$. If $\Delta l \ll 1$ one can expand the hyperbolic functions and it is easy to check that to lowest order in Δ the results (3.13) are recovered. Thus, for short enough length the system behaves as if the couplings were isotropic ($\Delta=0$). In particular all the correlation functions decrease with the *same* exponent, up to logarithmic corrections. At an "anisotropy" length given by $\ln(r/\alpha)=1/\Delta$ there is a crossover from a regime of isotropic correlation functions to a regime dominated by the anisotropies. By using $\Delta l \gg 1$ one can expand L_1 and L_2 and finds

$$\begin{aligned}
R_0(r)&=C_3^{-1}[(1+y_0/\Delta)/2]^{-1/2}\left[\frac{\alpha}{r}\right]^{1+\Delta/2}\left[1+2\left[\frac{y_0-\Delta}{y_0+\Delta}\right]^{1/2}\left[\frac{\alpha}{r}\right]^{\Delta}+O((\alpha/r)^{2\Delta})\right],\\
R_3(r)&=C_3[(1+y_0/\Delta)/2]^{-1/2}\left[\frac{\alpha}{r}\right]^{1+\Delta/2}\left[1-2\left[\frac{y_0-\Delta}{y_0+\Delta}\right]^{1/2}\left[\frac{\alpha}{r}\right]^{\Delta}+O((\alpha/r)^{2\Delta})\right],\\
R_1(r)&=R_2(r)=[(1+y_0/\Delta)/2]^{1/2}\left[\frac{\alpha}{r}\right]^{1-\Delta/2}[1+O((\alpha/r)^{2\Delta})]\,.
\end{aligned}\tag{3.16}$$

Thus the correlation functions decay with different exponents: $1+\Delta/2$ for R_0 and $R_3, 1-\Delta/2$ for R_1 and R_2. There are no logarithmic corrections but rather power-law corrections to the main divergence which become negligible when $r\to\infty$. The fact that logarithmic corrections have existed up to the "anisotropy" length manifest itself in the prefactors of the different correlation functions. If we go closer and closer to the isotropic limit the prefactors enhance $R_{1,2,3}$ over R_0.

C. General solution

We will now solve the complete Eqs. (3.7) and (3.10). We will restrict ourselves to $y_0>0$, $y_\theta>0$, $y_\phi>y_\theta$ and $y_0-y_\phi+y_\theta<0$. Other cases can be straightforwardly obtained by the same method. We introduce

$$\begin{aligned}
a^2&=(y_\theta+y_\phi)^2-y_0^2>0\,,\\
b^2&=(y_\theta-y_\phi)^2-y_0^2>0\,,\\
q&=\frac{(a^2-b^2)^{1/2}}{a}\,.
\end{aligned}\tag{3.17}$$

The solution of (2.6) is given by

$$y_0^f=b\,\mathrm{sc}[F(\arctan(y_0/b),q)-a\ln(r/\alpha)|q]\,,\tag{3.18}$$

where $\mathrm{sc}(x|q)$ is a Jacobi elliptic function. Moreover we have by straightforward integration of (3.10)

$$\begin{aligned}
L_1(r)&=\left[\frac{2y_\phi^2}{[(y_0^{f2}+a^2)(y_0^{f2}+b^2)]^{1/2}+y_0^{f2}-y_0^2+y_\phi^2+y_\theta^2}\right]^{1/4},\\
L_2(r)&=\left[\frac{y_0^f+(y_0^{f2}+a^2)^{1/2}}{y_0+y_\phi+y_\theta}\right]^{1/2},\\
L_3(r)&=\left[\frac{y_0+y_\phi-y_\theta}{y_0^f+(y_0^{f2}+b^2)^{1/2}}\right]^{1/2}.
\end{aligned}\tag{3.19}$$

If the system is not on the critical plane the renormalization Eqs. (2.6) leads to strong couplings. Therefore the solution (3.19) is only valid for length scales shorter than the correlation length of the ordered phase. For larger length scales one can use the renormalization equations to map the problem onto the exact solution of Luther and Emery for a particular value of the coupling constants.[26]

If the system is on the critical plane the couplings remain weak and the solution (3.9) is valid at arbitrary length scales. On the critical plane we have $y_0=y_\phi-y_\theta$. In terms of the variables y_0 and $y_\sigma=y_\phi+y_\theta$ the Eqs. (2.6) become

$$\begin{aligned}
\frac{dy_0}{dl}&=-y_0y_\sigma\,,\\
\frac{dy_\sigma}{dl}&=-y_0^2\,.
\end{aligned}\tag{3.20}$$

These are the standard Kosterlitz-Thouless equations.[6] The problem is then similar to that of the Sec. III B: there are power-law correlation functions, with corrections to the amplitude coming from the marginal operators. The only case where logarithmic corrections appear at arbitrary length scales is $y_0^2=y_\sigma^2$. This corresponds to $y_\theta=0$ and $y_0=y_\phi$ or $y_\phi=0$ and $y_0=-y_\theta$, i.e., a pure Kosterlitz-Thouless transition line.

D. Generalization to other correlations functions

The calculation of the previous section can easily be generalized to other correlation functions such as

$$R_{a,b}^{a',b'}(z,\tau)=\langle T_\tau[O_{a,b}^\dagger(z,\tau)O_{a',b'}(0,0)]\rangle\,,\tag{3.21}$$

where

$$O_{a,b}=e^{i(a\sqrt{2}\phi+b\sqrt{2}\theta)} . \tag{3.22}$$

In the absence of $y_{\phi}-$ and $y_{\theta}-$ interactions we have

$$R_{a,b}^{a',b'}=\delta_{a,a'}\delta_{b,b'}e^{-(a^2K+b^2/K)U(r)}e^{2iab\,\mathrm{Arg}(r)} , \tag{3.23}$$

where $\mathrm{Arg}(r)$ is the angle between r and the x axis.

If we now include the y_ϕ and y_θ interactions we can use the same method as above. If $a\neq 1 (b\neq 1)$ *there are no first order terms* such as [see (A1)]

$$\langle T_\tau e^{i\sqrt{2}\phi(r_1)}e^{i\sqrt{2}\phi(r_2)}e^{-i\sqrt{8}\phi(r_3)}\rangle \tag{3.24}$$

(or the equivalent term with $\phi\to\theta$). Of course such terms are also absent from the correlation functions if $y_\phi=0$ (resp. $y_\theta=0$). By using the same method as in the appendix we find, for $a\neq 1, b\neq 1$

$$R_{a,b}^{a,b}=\left[\frac{\alpha}{r}\right]^{a^2+b^2}e^{2iab\,\mathrm{Arg}(r)} \times \exp\left[-\frac{(a^2-b^2)}{2}\int_0^{\ln(r/\alpha)}y_0(l)dl\right] . \tag{3.25}$$

On the critical Kosterlitz-Thouless type lines the correlation functions $R_{a,b}^{a,b}$ have logarithmic correction factors $\ln^{(b^2-a^2)/2}(r/\alpha)$ (for $y_\theta=0$, $y_0=|y_\phi|$) or $\ln^{(a^2-b^2)/2}(r/\alpha)$ (for $y_\phi=0$, $y=-|y_\theta|$). For example, for $y_\theta=0$, $y_0=y_\phi$, $b=1$, $a=0$ this formula gives an exponent $\frac{1}{2}$ for the logarithmic correction, in agreement with (3.13) for $R_{1,2,3}$: for $y_\theta=0$ there is no first-order term for the θ correlation functions. On the other hand, as $y_\phi\neq 0$ formula (3.25) cannot give the exponent of the logarithmic corrections for $a=1$.

The case where $a=1$ or $b=1$ can be treated by a slight generalization of Sec. III A, the only change being the appearance of the angular part in functions containing both ϕ and θ. This angle-dependent part is not renormalized. As an example, we will give results for the case $a=1, b\neq 1$. It is more convenient to introduce linear combinations of the O operators. We will then consider

$$\begin{aligned} R_{1,b}^{(+)}&=\langle T_\tau \cos[\sqrt{2}\phi(z,\tau)]e^{-ib\sqrt{2}\theta(z,\tau)}\cos[\sqrt{2}\phi(0,0)]e^{ib\sqrt{2}\theta(0,0)}\rangle ,\\ R_{1,b}^{(-)}&=\langle T_\tau \sin[\sqrt{2}\phi(z,\tau)]e^{-ib\sqrt{2}\theta(z,\tau)}\sin[\sqrt{2}\phi(0,0)]e^{ib\sqrt{2}\theta(0,0,)}\rangle , \end{aligned} \tag{3.26}$$

with the result

$$\begin{aligned} R_{1,b}^{(+)}&=\left[\frac{\alpha}{r}\right]^{1+b^2}e^{2ib\,\mathrm{Arg}(r)}\exp\left[-\int_0^{\ln(r/\alpha)}dl\,y_\phi(l)dl-\frac{(1-b^2)}{2}\int_0^{\ln(r/\alpha)}y_0(l)dl\right] ,\\ R_{1,b}^{(-)}&=\left[\frac{\alpha}{r}\right]^{1+b^2}e^{2ib\,\mathrm{Arg}(r)}\exp\left[\int_0^{\ln(r/\alpha)}dl\,y_\phi(l)dl-\frac{(1-b^2)}{2}\int_0^{\ln(r/\alpha)}y_0(l)dl\right] . \end{aligned} \tag{3.27}$$

IV. PHYSICAL SYSTEMS

The generalized sine-Gordon Hamiltonian (2.1) can be related to a variety of one-dimensional quantum systems or two-dimensional classical systems.

A. Interacting fermions in one dimension

We use the standard "g-ology" description of one-dimensional interacting fermions.[1,2] As the important processes are those close to the Fermi surface, the spectrum is linearized around k_F and $-k_F$. The interactions between electrons are parametrized by constants. If the band is not half-filled (no umklapp process), one can show that all the processes at the Fermi level can be described by few constants.[10] The complete one-dimensional Hamiltonian then is

$$\begin{aligned} H=&\sum_{k,\sigma,r}v_F(rk-k_F)a_{rk\sigma}^\dagger a_{rk\sigma}\\ &+L^{-1}\sum_{k_1,k_2,p,\sigma,\sigma'}(g_{1\parallel}\delta_{\sigma,\sigma'}+g_{1\perp}\delta_{\sigma,-\sigma'})a_{+,k_1,\sigma}^\dagger a_{-,k_2,\sigma'}^\dagger a_{+,k_2+2k_F+p,\sigma'}a_{-,k_1-2k_F-p,\sigma}\\ &+L^{-1}\sum_{p,\sigma,\sigma'}(g_{2\parallel}\delta_{\sigma,\sigma'}+g_{2\perp}\delta_{\sigma,-\sigma'})\rho_{+,\sigma}(p)\rho_{-,\sigma'}(-p)+L^{-1}\sum_{k_1,k_2,p,\sigma}g_f a_{+,k_1,-\sigma}^\dagger a_{-,k_2,-\sigma}^\dagger a_{-,k_2+p,\sigma}a_{+,k_1-p,\sigma} , \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \rho_{r\sigma}(p)&=\sum_k a_{r,k+p,\sigma}^\dagger a_{r,k,\sigma} ,\\ \psi_{r\sigma}(x)&=L^{-1/2}\sum_k a_{rk\sigma}e^{ikx} , \end{aligned} \tag{4.2}$$

$r=\pm$ denotes right- and left-going fermions, $\sigma=\pm$ indicates spin up and down, and $a_{rk\sigma}(a_{rk\sigma}^\dagger)$ is the annihilation (creation) operator for a fermion in state (r,σ) with momentum k.

We now use the boson representation[27–30] of fermion operators, introduce charge (ρ) and spin (σ) density operators in the standard way,[2] and define the phase fields:

$$\phi_\nu(x),\theta_\nu(x) = \mp\frac{i\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2-ipx}[\nu_+(p)\pm\nu_-(p)] , \quad (4.3)$$

$\nu=\rho$ or σ .

In $A,B=$ . . . the upper sign refers to A, and α is a short-range cutoff parameter of the order of the lattice constant. In terms of boson operators the Hamiltonian is expressed by

$$H=H_\rho+H_\sigma+\frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx\,\cos(\sqrt{8}\phi_\sigma) - \frac{2g_f}{(2\pi\alpha)^2}\int dx\,\cos(\sqrt{8}\theta_\sigma) , \quad (4.4)$$

where H_ρ and H_σ are defined by

$$H_\nu=\frac{1}{2\pi}\int dx\left[(u_\nu K_\nu)(\pi\Pi_\nu)^2+\left[\frac{u_\nu}{K_\nu}\right](\partial_x\phi_\nu)^2\right] ,$$

$$u_\nu=\left[v_F^2-\frac{(g_\nu)^2}{4\pi^2}\right]^{1/2} , \quad K_\nu=\left[\frac{2\pi v_F+g_\nu}{2\pi v_F-g_\nu}\right]^{1/2} , \quad (4.5)$$

$$g_\rho,g_\sigma=g_{1\parallel}-g_{2\parallel}\mp g_{2_\perp} ,$$

and Π_ν is the momentum density conjugate to ϕ_ν. The charge part of the one-dimensional Hamiltonian corresponds to the free Hamiltonian (2.2), whereas the spin part is the complete Hamiltonian (2.1). The physically interesting correlation functions for the one-dimensional electron gas are the $2k_F$ charge density wave (CDW), spin density wave (SDW) or singlet (SS) and triplet (TS) superconducting Cooper pairing type. These fluctuations are described by the correlation functions[1,2]

$$R_i(x,\tau)=-\langle T_\tau O_i(x,\tau)O_i^\dagger(0,0)\rangle , \quad (4.6)$$

where τ is the Matsubara imaginary time, and

$$O_{\mathrm{CDW}}(x,\tau)=\frac{e^{2ik_Fx}}{\pi\alpha}\exp[-i\sqrt{2}\phi_\rho(x,\tau)]\cos[\sqrt{2}\phi_\sigma(x,\tau)] ,$$

$$O_{\mathrm{SDW}_x}(x,\tau)=\frac{e^{2ik_Fx}}{\pi\alpha}\exp[-i\sqrt{2}\phi_\rho(x,\tau)]\cos[\sqrt{2}\theta_\sigma(x,\tau)] ,$$

$$(4.7)$$

$$O_{\mathrm{SDW}_y}(x,\tau)=\frac{e^{2ik_Fx}}{\pi\alpha}\exp[-i\sqrt{2}\phi_\rho(x,\tau)]\sin[\sqrt{2}\theta_\sigma(x,\tau)] ,$$

$$O_{\mathrm{SDW}_z}(x,\tau)=\frac{e^{2ik_{Fx}}}{\pi\alpha}\exp[-i\sqrt{2}\phi_\rho(x,\tau)]\sin[\sqrt{2}\phi_\sigma(x,\tau)] .$$

O_{SS} and O_{TS} are obtained with $k_F=0$ and replacing $\phi_\rho\to\theta_\rho$. The spin part of the correlation functions corresponds to the four functions R_i of (2.5), whereas the charge part is easily obtained.[1,2]

In this model $y_\phi-y_0$ describes the spin anisotropy between the $x-y$ plane and the z axis, and y_θ the spin-anisotropy in the $x-y$ plane. For spin-independent interactions $y_\phi=y_0$ and $y_\theta=0$. For repulsive backward scattering ($y_\phi=y_0>0$), so that $y_\phi(l=\infty)=0$, and using (3.13) we find the asymptotic behavior

$$R_{\mathrm{SDW}_x}=R_{\mathrm{SDW}_y}=R_{\mathrm{SDW}_z}\approx\left[\frac{\alpha}{r}\right]^{1+K_\rho}\ln^{1/2}(r/\alpha) ,$$

$$R_{\mathrm{CDW}}\approx\left[\frac{\alpha}{r}\right]^{1+K_\rho}\ln^{-3/2}(r/\alpha) . \quad (4.8)$$

The three spin-correlation functions are identical, as expected by spin-rotation invariance, and decay with the *same* power-law behavior as the CDW function. It is only the logarithmic correction which, for $y_0>0$, favors spin- against charge-correlation functions. This can also be found, in momentum space, integrating the first-order renormalization equations for the correlation functions.[1] The pairing correlation functions are obtained from (4.8) replacing $K_\rho\to 1/K_\rho$. In a half-filled band umklapp scattering can lead to "frozen" charge fluctuations, i.e., the electrons are localized. The correlation functions in this situation are given by (4.8), with $K_\rho=0$. The correlation functions then have the same aymptotic behavior as that found for an isotropic $S=\frac{1}{2}$ antiferromagnet [Eqs. (4.12) and (4.14)]. This is of course expected on physical grounds: localized electrons interact via an exchange interaction, and therefore their properties are those of the spin-chain model.

The corrections found here can be important if one wants to determine which phase is realized in a physical system. This becomes especially important when interchain coupling it taken into account to describe quasi-one-dimensional systems.[10]

B. Quantum spin chains

A rather general Hamiltonian describing quantum spin chains is

$$H=-\sum_{i=1}^{N}[(S_i^xS_{i+1}^x+S_i^yS_{i+1}^y)+J_zS_i^zS_{i+1}^z-D(S_i^z)^2] , \quad (4.9)$$

where $\mathbf{S}_i^2=S(S+1)$, and the index i labels consecutive lattice sites. $D=0,J_z=1$ is the isotropic ferromagnet, $D=0,J_z=-1$ is the isotropic antiferromagnet (after a spin rotation by π around the z axis on every second lattice site), and in general there is both exchange and single-ion anisotropy, but the model is isotropic in the xy plane.

In a certain region in the parameter spaces D,J_z the model (4.9) has a massless excitation spectrum. In this region the long-range behavior of different correlation functions is believed to be governed by the Hamiltonian[3,31,8,9,32]

$$H=\frac{1}{2\pi}\int dx\left[(uK)(\pi\Pi)^2+\left[\frac{u}{K}\right](\partial_x\phi)^2+\frac{2g_\phi}{(2\pi\alpha)^2}\cos(\sqrt{2}m\phi)\right], \quad (4.10)$$

with u,K,g_ϕ functions of D,J_z, and $S\,(g_\phi\approx J_z)$, and $m=1\,(m=2)$ for S integer (S half-odd integer). The operators ϕ,Π are related to the operators used in Ref. 32 by $\phi=2\sqrt{S}\,\psi_1$, $\Pi=\chi_1/(2\sqrt{S}\,)$. In the massless region one has $g_\phi(l)\to 0$ for $l\to\infty$ under renormalization, whereas outside the massless region g_ϕ increases with increasing l. For half-odd integer S and $D=0$ the massless region is $-1\le J_z<1$,[33,17,13] whereas for integer S the lower limit of the massless region (for $D=0$) occurs at $J_{z,c}>-1$ (Refs. 17 and 32) [numerically one finds $J_{z,c}\approx -0.1$ for $S=1$ (Refs. 34–36)]. For $S>\frac{1}{2}$ and arbitrary D the boundaries of the massless region have to be determined numerically.[35,36]

Obviously, the Hamiltonian (4.10) is of the form (2.1), with $g_\theta=0$. Consequently, we can use the results of Sec. III B to obtain correlation functions. For half-odd integer S the spin operators (more precisely, the part of them giving the slowest decay of correlation functions) are

$$S^+(x)\approx e^{-i\sqrt{2}\theta(x)},\quad S^z(x)\approx e^{i\pi x}\cos[\sqrt{2}\phi(x)]\ . \quad (4.11)$$

On the critical line, i.e., the boundary to the antiferromagnetically ordered phase, we then have (after going back to lattice operators)

$$\langle S_x^+S_0^-\rangle=(-1)^x\langle S_x^zS_0^z\rangle\approx\frac{\alpha}{x}\ln^{1/2}(x/\alpha)\ , \quad (4.12)$$

whereas inside the massless phase there are corrections to the asymptotic power laws of the form (3.16). Another interesting operator is the dimerization of the nearest-neighbor exchange which intervenes for example in the spin-Peierls transition.[37] Its continuum representation is

$$(-1)^x(S_x^+S_{x+1}^-+S_x^-S_{x+1}^+)\approx\sin[\sqrt{2}\phi(x)]\ . \quad (4.13)$$

Again using the results of Sec. III B, one obtains for the corresponding correlation function

$$\langle(S_x^+S_{x+1}^-+S_x^-S_{x+1}^+)(S_0^+S_1^-+S_0^-S_1^+)\rangle\approx(-1)^x\frac{\alpha}{x}\ln^{-3/2}(x/\alpha)\ . \quad (4.14)$$

One should note that in the present antiferromagnetic case one has $g_\phi<0$, and consequently the roles of $\sin(\sqrt{2}\phi)$ and $\cos(\sqrt{2}\phi)$ are interchanged with respect to Sec. III B. We therefore find the same asymptotic behavior for the three components of the spin-spin correlation functions (4.12), as to be expected for an isotropic model. The different logarithmic factors in (4.12) and (4.14) can play an important role in discussing the competition between antiferromagnetic and spin-Peierls order in weakly coupled magnetic chains. These logarithmic corrections have not been taken into account in previous work on this problem.[38] Of course, the correction terms found here are closely related to those found in the analysis of finite-chain data.[11,15]

For *integer* S, $S^+(x)$ takes the same form as in (4.11), however, the argument of the cosine term in the Hamiltonian is changed. After a simple transformation of the fields $\phi\to 2\phi,\theta\to\theta/2$ and using the results of Sec. III D we find for the spin correlations on the boundary between the massless and the singlet phase

$$\langle S_x^+S_0^-\rangle\approx\left[\frac{\alpha}{x}\right]^{1/4}\ln^{1/8}(x/\alpha)\ . \quad (4.15)$$

The exponent of the logarithmic correction is the same as that found for the two-dimensional XY model,[6] to which the present model is related for $S=1$.[39] Contrary to the case of half-odd-integer spin, the S^z correlation function and the correlation function of the dimerization operator (4.14) always decay exponentially for integer S. The above results (4.12), (4.14), and (4.15) are straightforwardly generalized to time-dependent correlation functions: one simply replaces $x\to[x^2-(ut)^2]^{1/2}$.

Another model that can be transformed into (2.1) is the generalized anisotropic spin-chain model defined by

$$H=-\sum_i[(1+\gamma)S_i^xS_{i+1}^x+(1-\gamma)S_i^yS_{i+1}^y+\lambda_1S_i^zs_{i+1}^z+\lambda_2S_i^zS_{i+2}^z]\ , \quad (4.16)$$

where the S_i^α are spin-$\frac{1}{2}$ operators. After a Jordan-Wigner transformation[40] we find a Hamiltonian of the form (2.1) with the identifications (neglecting renormalization effects due to the continuum limit) $\gamma=g_\theta, g_\phi=\lambda_1+\lambda_2, g_0=\lambda_1$.

For $\lambda_2=0$ (4.16) is the XYZ model, solved exactly by Baxter.[41] Through spin rotations, all the different critical lines of the model can be transformed into (4.16) with $\gamma=0, |\lambda_1|\le 1$. One then has the model (4.9) with $S=\frac{1}{2}$, and the preceding results can be taken over. It is also of some interest to consider the correlation function of the operator for anisotropy in the xy plane. One has

$$S_x^xS_{x+1}^x-S_x^yS_{x+1}^y\approx\sin[\sqrt{8}\theta(x)]\ . \quad (4.17)$$

For the isotropic antiferromagnet ($\lambda_1=-1$) one has $K(l=\infty)=1$, and from (3.25) we find

$$\langle(S_x^xS_{x+1}^x-S_x^yS_{x+1}^y)(S_0^xS_1^x-S_0^yS_1^y)\rangle\approx\left[\frac{\alpha}{x}\right]^4\ln^2(x/\alpha)\ . \quad (4.18)$$

For $\gamma=0$ the model (4.16) has been studied numerically by Emery and Noguera.[14] They find a massless phase in a large part of parameter space, and determine correlation exponents, taking into account explicitly correction terms of the type discussed in the present paper.

C. Two-dimensional systems

Following the derivation of Ref. 42 one can see that the Hamiltonian (2.1) gives the partition function of a two-dimensional Coulomb gas of charges and magnetic monopoles. This model is related to the two-dimensional eight-vertex and Ashkin-Teller models and a generalized Villain model.[8,43,44] In the continuum limit the partition function of these different models reduces to those of a generalized Gaussian model, and can be shown to be equivalent to the trace over the imaginary time of the time evolution operator of the quantum Hamiltonian (2.2).[8,45].

In Ref. 8 the quantum-field operator

$$O'_{N,M}(r)=\exp[iN\phi'(x,\tau)+i2M\theta'(x,\tau)] , \tag{4.19}$$

is introduced, which for two-dimensional classical systems corresponds to a spin operator and a vortex excitation operator with vorticity M at site r. The correspondence between the notations of Ref. 8 and ours [see (3.22)] is $N=2a, M=b/2$ and $x^2=2/K$.

As an example, consider the XY model which corresponds to a Gaussian Hamiltonian with a vortex operator of vorticity 1.[6] In our model this leads to a term $\cos(\sqrt{8}\theta)$. The XY model is therefore described by the Hamiltonian (2.1) with $g_\phi=0$. The spin-spin correlation function is given by a spin operator with $N=1$, in terms of the ϕ' operators. In terms of the ϕ operators this corresponds to $a=\frac{1}{2}$, and therefore by using (3.25) we find an exponent $\frac{1}{8}$ for the logarithmic correction at the critical point, in agreement with the result of Kosterlitz.[6]

Many physical quantities of other two-dimensional models can be described in terms of a linear combinations of the $O_{a,b}$ operators,[8,9] and therefore in terms of the correlation functions of Sec. III D. For example, the energy operator in the Ashkin-Teller model is given by

$$O_{\rm AT}=\frac{1}{i}(O_{1,0}-O_{-1,0}) . \tag{4.20}$$

As in the preceding section, one has $g_\phi<0, g_\theta=0$, and therefore at the end of the critical line of the Ashkin-Teller model, or equivalently, for the four state Potts model, the correlation function of this operator behaves as

$$\langle O^\dagger_{\rm AT}(r)O_{\rm AT}(0)\rangle \sim \frac{\alpha}{r}\ln^{-3/2}(\alpha/r) . \tag{4.21}$$

A complete description of the physical quantities associated to these different operators for the various two dimensional models can be found in Ref. 8.

V. CONCLUSION

We have computed here the different correlation functions of a generalized sine-Gordon Hamiltonian, using a real-space renormalization technique. The long-range behavior of the different correlation functions is a power law with a nonuniversal exponent which depends on the parameters of the Hamiltonian. In addition to this power law and for certain values of the coupling constants logarithmic corrections can appear. Our computation allows us to obtain these logarithmic corrections which are of the form $\ln^\nu(r/\alpha)$ and to find the exponent ν. This exponent depends on the correlation function considered, but is independent of the coupling constants.

Away from the critical point the logarithmic corrections are replaced by corrections to the amplitudes of the correlation functions. The asymptotic behavior is simply a power law (up to power law corrections decaying at large r), but the *amplitude* of the correlation functions are different (even for correlation functions with the same exponent). In particular the amplitudes depend on the distance from the critical point and diverge for some correlation functions when the parameters approach their critical values. In fact this indicates that there is a crossover from a power-law regime with logarithmic corrections towards a pure power-law regime, but different amplitudes for different correlation functions. The full crossover as a function of the length scale is given by our calculation.

As discussed in the preceding section, our results apply to a large class of quantum one-dimensional or classical two-dimensional systems, to which the sine-Gordon Hamiltonian can be related. The corrections we find are important to take into account for the numerical or experimental determination of critical exponents. Moreover the amplitude corrections can also be important. In presence of a cutoff (thermal cutoff for example) the Fourier transform of the correlation functions will behave for small frequency and momentum as

$$R\sim\Delta^\nu(T/\epsilon)^\mu , \tag{5.1}$$

where Δ^ν comes from the amplitude corrections [see (3.16), μ from the power-law behavior of the correlation function and ϵ is the physical cutoff of the system (for example the bandwidth for an electron gas)]. Due to the amplitude corrections the *apparent* energy scale of the correlation functions $\epsilon'=\epsilon\Delta^{-\nu/\mu}$ can be considerably different from the real one. This is particularly important close to a point where logarithmic corrections appear, since in this case Δ can be very small [see for example (3.16)].

APPENDIX: A RENORMALIZATION OF CORRELATION FUNCTIONS

We derive renormalization group equations in a way similar to Refs. 6, 24, and 25. We compute the functions F_i in a development in powers of g_ϕ and g_θ. We find, for example for F_0

$$\begin{aligned}R_0(r_1-r_2)=&e^{-KU(r_1-r_2)}-\frac{2g_\phi}{(2\pi\alpha)^2}\int dx_3d\tau_3\langle T_\tau e^{i\sqrt{2}\phi(r_1)}e^{i\sqrt{2}\phi(r_2)}e^{-i\sqrt{8}\phi(r_3)}\rangle\\&+\frac{1}{2}\left[\frac{g\phi}{4\pi^2}\right]^2\sum_{\epsilon_i=\pm1}\int\frac{dx_3d\tau_3}{\alpha^2}\frac{dx_4d\tau_4}{\alpha^2}\langle T_\tau e^{i\sqrt{2}\phi(r_1)}e^{-i\sqrt{2}\phi(r_2)}e^{i\sqrt{8}\epsilon_3\phi(r_3)}e^{-i\sqrt{8}\epsilon_4\phi(r_4)}\rangle\\&+\frac{1}{2}\left[\frac{g\theta}{4\pi^2}\right]^2\sum_{\epsilon_i=\pm1}\int\frac{dx_3d\tau_3}{\alpha^2}\frac{dx_4d\tau_4}{\alpha^2}\langle T_\tau e^{i\sqrt{2}\phi(r_1)}e^{-i\sqrt{2}\phi(r_2)}e^{i\sqrt{8}\epsilon_3\theta_\sigma(r_3)}e^{-i\sqrt{8}\epsilon_4\theta_\sigma(r_4)}\rangle .\end{aligned} \tag{A1}$$

Computing the average values we find

$$F_0(r_1-r_2)=1-e^{2KU(r_1-r_2)}\frac{y\phi}{2\pi}\int_\alpha \frac{d^2r_3}{\alpha^2}e^{-2KU(r_1-r_3)}e^{-2KU(r_2-r_3)}$$
$$+\frac{y_\phi^2}{2}K^2U(r_1-r_2)\int_\alpha^{+\infty}\frac{dr}{\alpha}\left[\frac{r}{\alpha}\right]^{3-4K}-\frac{y_\theta^2}{2}U(r_1-r_2)\int_\alpha^{+\infty}\frac{dr}{\alpha}\left[\frac{r}{\alpha}\right]^{3-4/K}, \tag{A2}$$

where U has been introduced in (3.2). The two last terms in (A2) arise from the standard cumulant expansion[6,24,25] and give rise to the renormalization of the coupling constants. The term of first order in y_ϕ is important to take into account for the renormalization of the correlation functions. This term *only* appears in correlation functions of operators $\exp(ia\sqrt{2}\phi)$ with $a=1$ and does not contribute to the renormalization of the coupling constants. $\int_\alpha$ means that the domain of integration over r_3 excludes two circles of radius α around r_1 et r_2. If one changes α to $\alpha'=\alpha e^{dl}$, one finds

$$F_0(r_1-r_2)=1-y_\phi dl+\tfrac{1}{2}(K_y^2{}_\phi^2-y_\theta^2)U(r_1-r_2)dl$$
$$+y'_\phi\int_{\alpha'}\frac{dx_3d\tau_3}{\alpha'^2}e^{2KU'(r_1-r_2)}e^{-2KU'(r_1-r_3)}e^{-2KU'(r_2-r_3)}$$
$$+\frac{y'^2_\phi}{2}K^2U'(r_1-r_2)\int_{\alpha'}^{+\infty}\frac{dr}{\alpha'}\left[\frac{r}{\alpha'}\right]^{3-4K}-\frac{y'^2_\theta}{2}U'(r_1-r_2)\int_{\alpha'}^{+\infty}\frac{dr}{\alpha'}\left[\frac{r}{\alpha'}\right]^{3-4/K}, \tag{A3}$$

where

$$y'_\phi=y_\phi\left[\frac{\alpha'}{\alpha}\right]^{4-4K},\quad y'_\theta=y_\theta\left[\frac{\alpha'}{\alpha}\right]^{4-4/K}, \tag{A4}$$

and U' is the function U with the new value of α. The Eqs. (A4) are nothing but the renormalization Eqs. (2.6) for the coupling constants y_ϕ and y_θ. From (3.4) and (A3) we deduce

$$I(\alpha'/\alpha,y_\phi,y_\theta)$$
$$=\exp\left[-y_\phi+\frac{y_\phi^2}{2}\ln(r/\alpha)-\frac{y_\theta^2}{2}\ln(r/\alpha)\right]dl\ . \tag{A5}$$

Therefore we get

$$F_0(r)=\exp\int_0^{\ln(r/\alpha)}[-y_\phi(l)+\tfrac{1}{2}y_\phi^2(l)-\tfrac{1}{2}y_\theta^2(l)]dl\ , \tag{A6}$$

A similar derivation can be done for other correlation functions and we obtain (3.7).

[1]J. Sólyom, Adv. Phys. **28**, 209 (1979).

[2]V. J. Emery, in *Highly Conducting One-Dimensional Solids*, edited by J. T. Devreese *et al.* (Plenum, New York, 1979), p. 327.

[3]A. Luther and I. Peschel, Phys. Rev. B **12**, 3908 (1975).

[4]N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1967).

[5]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1973).

[6]J. M. Kosterlitz, J. Phys. C **7**, 1046 (1974).

[7]L. P. Kadanoff, Ann. Phys. (N.Y.) **120**, 39 (1979).

[8]M. P. M. den Nijs, Phys. Rev. B **23**, 6111 (1981).

[9]J. L. Black and V. J. Emery, Phys. Rev. B **23**, 429 (1981).

[10]T. Giamarchi and H. J. Schulz, J. Phys. (Paris) **49**, 819 (1988).

[11]G. Spronken, B. Fourcade, and Y. Lépine, Phys. Rev. B **33**, 1886 (1986).

[12]J. L. Cardy, J. Phys. A **19**, L1093 (1986).

[13]T. Ziman and H. J. Schulz, Phys. Rev. Lett. **59**, 140 (1987).

[14]V. J. Emery and C. Noguera, Phys. Rev. Lett. **60**, 631 (1988).

[15]I. Affleck, D. Gepner, H. J. Schulz, and T. Ziman (unpublished).

[16]E. Brézin and J. Zinn-Justin, Phys. Rev. B **14**, 3110 (1976).

[17]F. D. M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).

[18]E. Witten, Commun. Math. Phys. **92**, 455 (1984).

[19]V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B **247**, 83 (1984).

[20]I. Affleck, Nucl. Phys. B **265**, 409 (1986).

[21]I. Affleck, Phys. Rev. Lett. **56**, 746 (1986).

[22]M. Nauenberg and D. J. Scalapino, Phys. Rev. Lett. **44**, 837 (1980).

[23]L. P. Kadanoff, Phys. Rev. B **22**, 1405 (1980).

[24]J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, Phys. Rev. B **16**, 1217 (1977).

[25]D. R. Nelson, Phys. Rev. B **18**, 2318 (1978).

[26]A. Luther and V. J. Emery, Phys. Rev. Lett. **33**, 589 (1974).

[27]D. C. Mattis, J. Math. Phys. **15**, 609 (1974).

[28]A. Luther and I. Peschel, Phys. Rev. B **9**, 2911 (1974).

[29]F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).

[30]R. Heidenreich, R. Seiler, and D. A. Uhlenbrock, J. Stat. Phys. **22**, 27 (1980).

[31]F. D. M. Haldane, Phys. Rev. Lett. **45**, 1358 (1980).

[32]H. J. Schulz, Phys. Rev. B **34**, 6372 (1986).

[33]C. N. Yang and C. P. Yang, Phys. Rev. **150**, 327 (1966).

[34]R. Botet and R. Jullien, Phys. Rev. B **27**, 613 (1983).

[35]R. Botet, R. Jullien, and M. Kolb, Phys. Rev. B **28**, 3914 (1983).

[36]H. J. Schulz and T. A. L. Ziman, Phys. Rev B **33**, 6545 (1986).

[37]M. C. Cross and D. S. Fisher, Phys. Rev. B **19**, 402 (1979).

[38]S. Inagaki and H. Fukuyama, J. Phys. Soc. Jpn. **52**, 3620 (1983).

[39]M. P. M. den Nijs, Physica A **111**, 273 (1982).

[40]P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).
[41]R. J. Baxter, Ann. Phys. (N.Y.) **70**, 323 (1972).
[42]S. T. Chui and P. A. Lee, Phys. Rev. Lett. **35**, 315 (1975).
[43]T. T. Truong and K. D. Schotte, Phys. Rev. Lett. **47**, 285 (1981).
[44]L. P. Kadanoff, J. Phys. A **11**, 1399 (1978).
[45]D. J. Scalapino, M. Sears, and R. A. Ferrell, Phys. Rev. B **6**, 3409 (1972).