# Exact results for the scattering of electromagnetic waves with a nonlinear film

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Exact analytical results for the scattering of a TE electromagnetic wave with a nonlinear thin film is derived. We assume that the incident wave is monochromatic, and that the film is nonabsorbing and has a Kerr-like optical nonlinearity. Among other things, the reflection coefficient is calculated exactly as a function of the incident intensity. Optical bistable and multistable behaviors are found. The phenomena of induced resonances and induced transparency are also discussed.

## I. INTRODUCTION

There has been considerable interest in nonlinear (NL) surface and guided electromagnetic (em) waves in the last several years.[1] These waves either propagate at the interfaces or are guided by single or multilayered structures involving NL dielectric film characterized by a Kerr-like dielectric constant. In contrast, there is relatively little work on the scattering of electromagnetic waves with NL dielectric layers. Most of the studies on the scattering problem rely on the use of Kaplan's solution[2] which applies to a plane wave incident on a semi-infinite NL medium. However, a numerical study carried out by Tomlinson *et al.*[3] did not find any optical bistable behavior, in contrast to the theoretical prediction. This is because there is no memory mechanism so that information on the field configuration at the front of the beam can be stored and transmitted to the back of the beam. In this regard, a NL thin film is therefore interesting in that a portion of the incident wave is reflected from the surface at $z=d$ (see Fig. 1) and thus can strongly affect the wave that arrives at a later time, especially when the incident beam has a sufficient width, and the incident angle (measured with respect to the normal of the film) is not very close to $\pi/2$. A detailed understanding of the scattering, of course, can only be obtained from a time-dependent analysis of the problem, which unfortunately lies outside the scope of the present article.

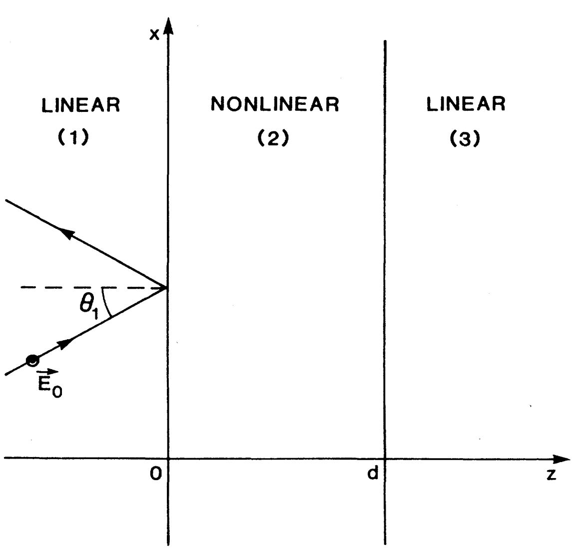


FIG. 1. The basic scattering configuration.

Besides its fundamental importance, work on the scattering of em waves with finite NL film will also be important for understanding a variety of phenomena such as the coupling efficiency of NL prism couplers.[4]

The problem of interest here is the scattering of a plane-TE-polarized em wave of a single frequency from a NL thin film. The nonlinearity is assumed to be Kerr-like. The electric field is separated into an intensity and a phase function. The Maxwell's equations are then solved exactly in terms of the Jacobian elliptic sine function. The integration constants are determined by matching the boundary conditions at the two interfaces. Results for the reflection coefficient are calculated as a function of the incident intensity, and optical bistable as well as multistable behaviors are found. Unlike the preliminary version of the present article,[5] full details of the calculation are given here, and the complete results for rather general parameter values are obtained for the case of a self-focusing material. In addition, the present paper also shows that by suitably adjusting the incident intensity, induced resonances as well as induced transparency of the film can occur.

An independent work on this problem has recently been published by Chen and Mills[6] The main results are quite the same. However, there are a few differences that are worth mentioning. (1) Our results are not restricted to normal incidence only. (2) In Ref. 6 a certain parameter has to be scanned through a fixed range to find particular values which yield a solution consistent with the boundary conditions. This in effect amounts to solving for the roots of a single but very complicated algebraic equation. This can be somewhat nontrivial because there can be multiple roots present.[7] However, in our work, by a suitable parametrization of the problem, there is no need to solve any equation numerically at all. Our analysis is therefore much simpler. (3) As a result, we are able to calculate the incident intensity at which induced resonance scattering occurs. The intensity at which the film becomes totally transparent is also calculated. (4) In Ref. 6 results were extended to bilayers and superlattices involving nonlinear dielectric layers[8] as well. Our results here apply only to a single nonlinear layer.

Marburger and Felber[9] did an analysis of a nonlinear Fabry-Perot interferometer nearly a decade ago and found multiple-valued transmission-versus-intensity characteristic. At first glance, there may seem to be similarities between their work and ours; however, there are major differences which we want to stress. (1) They consider circularly polarized waves while we consider TE-polarized waves. The difference in the polarization can be very significant, especially in a nonlinear analysis. (2) Our results apply to any angle of incidence while theirs apply only to normal incidence. (3) We also show that by a suitable choice of scaling of the various parameters, the basic results are, in general, independent of the incident angle. There are in total only four possible kinds of behaviors, depending on the ratios of the dielectric constant of the film and the media in front and at the back of it. (4) In a nonlinear Fabry-Perot interferometer, the front and the end of the cavity are covered by mirrors whose reflectivities are fixed. The situation in a nonlinear film is quite different in that the reflectivities at the two surfaces are not given but must be determined together with the reflectivity of the film itself in a totally self-consistent manner. This is comparatively much more difficult to handle.

## II. REDUCTION OF THE PROBLEM TO QUADRATURES

Let us consider here a single nonlinear film which occupies region (2) between $z=0$ and $d$ (see Fig. 1). It is assumed that the nonlinear material can be characterized by a dielectric function which depends on the instantaneous value of the local electric field intensity, i.e., $\epsilon_2=\epsilon_2(|\mathbf{E}(\mathbf{r},t)|^2)$. Regions (1) and (3), which correspond to $z>0$ and $z<0$, respectively, are taken to be linear, with dielectric constants $\epsilon_1$ and $\epsilon_3$.

The interest here is in monochromatic TE waves which are independent of $y$, and have the form of a plane wave along $x$. Thus one writes

$$\mathbf{E}(\mathbf{r},t)=\hat{\mathbf{y}}E_y(z)\exp[i(k_x x-\omega t)] , \tag{1}$$

$$\mathbf{H}(\mathbf{r},t)=[\hat{\mathbf{x}}H_x(z)+\hat{\mathbf{z}}H_z(z)]\exp[i(k_x x-\omega t)] , \tag{2}$$

where $E_y(z)$, $H_x(z)$, and $H_z(z)$ are in general complex-valued functions. On account of Maxwell's equations, one can eliminate $H_x(z)$ and $H_z(z)$ in favor of $E_y(z)$. The resulting equation for region (2) is[10]

$$\ddot{E}_y(\xi)=[\eta_x^2-\epsilon(|E_y(\xi)|^2)]E_y(\xi) , \tag{3}$$

where $\eta_x\equiv k_x c/\omega$, $\xi\equiv z\omega/c$ is the distance perpendicular to the film in dimensionless units, and the double overdot denotes differentiation with respect to $\xi$.

In order to solve Eq. (3), we express $E_y(\xi)$ in the form

$$E_y(\xi)=F(\xi)\exp\left[i\int_0^\xi d\xi\,\psi(\xi)+i\phi\right] , \tag{4}$$

where $F(\xi)$ and $\psi(\xi)$ are unknown real functions and $\phi$ is an unknown real constant. Clearly $F(\xi)$ represents the amplitude of the field, $\phi$ gives the overall phase at $z=0$, and the integral over $\psi$ is the accumulated phase within the film. Inserting Eq. (4) into Eq. (3) and equating the real and the imaginary parts results in the following two equations:

$$2\psi\dot{F}+\dot{\psi}F=0 , \tag{5}$$

$$\ddot{F}-[\eta_x^2+\psi^2-\epsilon(F^2)]F=0 . \tag{6}$$

Equation (5) can be integrated at once, the result being an exact expression relating the phase function $\psi(\xi)$ and the amplitude function $F(\xi)$:

$$\psi(\xi)\equiv\frac{l}{F^2(\xi)} , \tag{7}$$

where $l$ is an integration constant. An ordinary nonlinear differential equation for $F(\xi)$ is then obtained by substituting Eq. (7) into Eq. (6). The result is

$$\ddot{F}-\left[\eta_x^2+\frac{l^2}{F^4}-\epsilon(F^2)\right]F=0 . \tag{8}$$

It is convenient to decompose $\epsilon$ into a linear and a nonlinear part so that

$$\epsilon(F^2)=\epsilon_0+\epsilon_2(F^2) , \tag{9}$$

where, by definition, the nonlinear part has the property that $\epsilon(F^2)\rightarrow0$ as $F\rightarrow0$. For any given nonlinear dielectric function $\epsilon_2(F^2)$, Eq. (9) resembles a "classical equation of motion" and can be integrated to give the "energy integral":

$$\dot{F}^2/2+V(F)=e , \tag{10}$$

where

$$V(F)=\frac{KF^2}{2}+\frac{l^2}{2F^2}+\tfrac{1}{2}\int_0^{F^2}du\,\epsilon_2(u) , \tag{11}$$

and

$$K\equiv\epsilon_0-\eta_x^2 . \tag{12}$$

In Eq. (10), $e$ is an integration constant which plays the role of the total mechanical energy of the system.[11] Equation (10) can be integrated to give

$$\int_{F_0}^{F}dF\frac{1}{\sqrt{2[e-V(F)]}}=\pm(\xi-\xi_0) , \tag{13}$$

where $\xi_0$ is another integration constant which denotes the "time" at which the amplitude is given by $F_0=F(\xi_0)$. $F(\xi)$ can then be obtained by inverting this equation to give $F$ as a function of $\xi$. Substituting the amplitude function in Eq. (7), the phase factor can then be calculated via and additional integration. Thus the entire problem has been reduced to quadratures. For any given form of the nonlinear dielectric function $\epsilon_2$ the electric field can be determined completely by performing a total of three ordinary integrals.

Equation (3) can be derived from a Lagrangian of the form

$$L=\tfrac{1}{2}|\dot{E}_y|^2-\tfrac{1}{2}\int^{|E_y|^2} du\,\epsilon(u)+\tfrac{1}{2}\eta_x^2|E_y|^2 \ . \quad (14)$$

The fact that the field depends on a total of four real parameters, $\phi$, $l$, $e$, and $\xi_0$ (Ref. 12) can easily be understood. $\phi$ is needed because the field can only be determined up to a constant phase factor due to the gauge invariance of $L$ or of Eq. (3). The rotational invariance of $L$ implies that $l$ is the conserved angular momentum of the system. This is also related to the conservation of em flux along the $z$ direction, as can be seen by calculating the Poynting vector $\mathbf{S}$, which can be written, using Eqs. (1) and (2), as

$$\mathbf{S}(\xi)=\frac{\omega}{8\pi}\,\mathrm{Re}(\eta_x|E_y|^2,0,iE_y\dot{E}_y^*) \ . \quad (15)$$

With $E_y$ of the form given as in Eq. (4), Eq. (15) can be rewritten as

$$\mathbf{S}(\xi)=\frac{\omega}{8\pi}F^2(\xi)(\eta_x,0,\psi(\xi)) \ . \quad (16)$$

Therefore the vector $\boldsymbol{\eta}\equiv(\eta_x,0,\psi(\xi))$ is in the direction of energy flow. Equation (7), which states that $\psi(\xi)F^2(\xi)$ is a constant equal to $l$, is clearly a direct consequence of the conservation of flux along $\hat{\mathbf{z}}$.

Equation (10) for $e$ represents the conservation of the total mechanical energy of the system. Finally the existence of the constant $\xi_0$ simply reflects the time translational invariance of the Lagrangian. These four parameters will be fixed by the boundary condition which we will consider next.

## III. BOUNDARY CONDITIONS

A plane TE wave is assumed to incident from region 1 onto the nonlinear film at an angle $\theta_1$ (see Fig. 1). If we denote the complex reflection amplitude by $r$, then the combined incident and reflected wave in region 1 is given by

$$\mathbf{E}_1=\hat{\mathbf{y}}E_0[\exp(ik_{1z}z)+r\exp(-ik_{1z}z)]\exp(ik_{1x}x) \ , \quad (17)$$

where $k_{1x}\equiv k_1\sin\theta_1$, $k_{1z}\equiv k_1\cos\theta_1$, and $k_1\equiv\epsilon_1^{1/2}\omega/c$, and a time-harmonic factor has been suppressed. From Eq. (4), the electric field inside the nonlinear medium has the form

$$\mathbf{E}_2=\hat{\mathbf{y}}F(z)\exp\left[i\left[k_{2x}x+\int_0^{\xi}d\xi\,\psi(\xi)+\phi\right]\right] \ . \quad (18)$$

With $t$ as the complex transmission amplitude, the field to the right of the film can be expressed as

$$\mathbf{E}_3=\hat{\mathbf{y}}tE_0\exp\left[i\left[k_{3x}x+\eta_{3z}(\xi-\xi_d)+\int_0^{\xi_d}d\xi\,\psi(\xi)+\phi\right]\right] \ , \quad (19)$$

where $\eta_{i\lambda}$ is defined as $k_{i\lambda}c/\omega$, for $i=1,3$ and $\lambda=x,z$. From the requirement of the continuity of the tangential components of the fields across the interfaces at $z=0$ and $z=d$, the following condition are obtained:

$$k_{1x}=k_{2x}=k_{3x} \ , \quad (20)$$

$$E_0(r+1)=F(0)\exp(i\phi) \ , \quad (21)$$

$$\eta_{1z}E_0(r-1)=[i\dot{F}(0)-\psi(0)F(0)]\exp(i\phi) \ , \quad (22)$$

$$F(\xi_d)=tE_0 \ , \quad (23)$$

and

$$[\psi(\xi_d)F(\xi_d)-i\dot{F}(\xi_d)]=\eta_{3z}tE_0 \ . \quad (24)$$

First, it is clear that Eq. (17) is a direct consequence of the translational invariance along the $x$ direction. Next, if Eq. (16), for the field in region (3), is substituted in Eq. (15), one sees that $l$ is in fact proportional to the transmission coefficient:

$$l=\eta_{3z}|t|^2E_0^2 \ . \quad (25)$$

Next, we note that $t$ and the phase factors can be eliminated from Eqs. (20) and (21), with the following very interesting results:

$$\psi(\xi_d)=\eta_{3z} \quad (26)$$

and

$$\dot{F}(\xi_d)=0 \ . \quad (27)$$

Equation (26) means that the phase function $\psi(\xi)$ is continuous at $z=d$. This result comes from the fact that the energy flux along $\hat{\mathbf{z}}$ is a constant [i.e., Eq. (7) holds] and the electric field is continuous across $z=d$. Equation (27) implies that the electric field intensity must be an extremum at $z=d$. Since in general one cannot expect $\ddot{F}(\xi_d)$ to vanish, this means that the intensity must either be a maximum or a minimum at $z=d$. From the way the results in Eqs. (26) and (27) were derived, it is clear that this result is independent of the boundary conditions at the $z=0$ surface. Nor does it depend on the form of the nonlinearity inside the film. In fact, it is true even in the linear limit.

The physical origin of these effects is clearly due to the fact that the region with $z>d$ is a homogeneous linear medium, which extends all the way out to infinity. Consequently there is only an outgoing wave but no incoming wave at all. The result in Eq. (27) can also be derived very simply by noting that

$$\frac{d|\mathbf{E}|^2}{dz}=\mathbf{E}\cdot\frac{d\mathbf{E}^*}{dz}+\mathbf{E}^*\cdot\frac{d\mathbf{E}}{dz} \ , \quad (28)$$

and that $\mathbf{E}$ and $d\mathbf{E}/dz$ are continuous at $z=d$. Therefore, the quantity on the left-hand side of Eq. (28) must also be continuous at $z=d$. At $d^-$ this quantity is given by $2F(\xi_d)dF(\xi_d)/dz$, and at $d^+$, since the intensity in region (3) is constant, we have $d|t|^2/dz=0$. Therefore either $F(\xi_d)=0$ or $dF(\xi_d)/dz=0$. Moreover, Eqs. (7) and (26) imply that

$$l=\eta_{3z}F^2(\xi_d) \ , \quad (29)$$

and so $F^2(\xi_d)=l/\eta_{3z}$ must be nonzero. Thus again the result in Eq. (27) is obtained.

Next, in view of the difficulty associated with the "continuum problem" which has been discussed at great depth by Kaplan,[2] it is interesting to see if the four complex equations [(21)–(24)] obtained from the boundary conditions are sufficient to completely determine all the physical parameters of the theory. First, note that from the solution $F(\xi-\xi_0;e,l)$ the quantities $F(0)$, $\dot{F}(0)$, $F(\xi_d)$, $\dot{F}(\xi_d)$, and $\int_0^{\xi_d} d\xi\, \psi(\xi)$ in these equations can all be expressed as functions of $\xi_0$, $e$, and $l$. Thus the equations involve two complex unknowns, $r$ and $t$, and four real unknowns, $\xi_0$, $e$, $l$, and $\phi$, i.e., a total of eight real unknowns. Consequently, unlike the case of a semi-infinite NL medium, there are just the correct number of equations to determine all the parameters.

From Eq. (21), $r$ can be expressed as

$$r=\frac{F(0)}{E_0}\exp(i\phi)-1 . \tag{30}$$

When $r$ is substituted in Eq. (22) the result is

$$\{[\eta_{1z}+\psi(0)]F(0)-i\dot{F}(0)\}\exp(i\phi)=2E_0\eta_{1z} . \tag{31}$$

Eliminating $\phi$ from this equation yields the result

$$\dot{F}^2(0)+[\eta_{1z}+\psi(0)]^2F^2(0)=4E_0^2\eta_{1z}^2 . \tag{32}$$

The term with the derivative can be eliminated with the help of the energy integral [Eq. (10)], and together with Eq. (7) the following result is obtained:

$$2e+[(\eta_{1z}^2-K)]F^2(0)+2\eta_{1z}l-\int^{F^2(0)} du\, \epsilon_2(u) =4E_0^2\eta_{1z}^2 . \tag{33}$$

Making use of Eqs. (31) and (32), the phase angle $\phi$ can be expressed as

$$\cos\phi=\frac{\eta_{1z}F(0)+l/F(0)}{2E_0\eta_{1z}} , \tag{34}$$

$$\sin\phi=\frac{\dot{F}(0)}{2E_0\eta_{1z}} . \tag{35}$$

By combining Eqs. (30), (32), (34) and (35) the following result for the reflection coefficient can be obtained:

$$|r|^2=1-\frac{l}{\eta_{1z}E_0^2} . \tag{36}$$

This result, together with Eq. (25), yields

$$\eta_{1z}(1-|r|^2)=\eta_{3z}|t|^2 , \tag{37}$$

which is a direct consequence of the conservation of flux along the $z$ direction. The results derived here will be used later to determine the integral constants.

## IV. EXACT SOLUTIONS FOR A KERR-LIKE MEDIUM

Thus far the results apply for arbitary forms of the intensity-dependent dielectric constant. In the remaining part of the article, detailed results will be derived for the case of a Kerr-like medium where the dielectric constant can be written in the form

$$\epsilon_2=\alpha|\mathbf{E}(\mathbf{r},t)|^2 . \tag{38}$$

The analysis will be slightly different, depending on the sign of $\alpha$. Hereafter, only the case of a self-focusing medium, i.e., $\alpha>0$, will be considered.

To further simplify the notation, $K$ is rewritten as

$$K=\pm\gamma^2 , \tag{39}$$

where $\gamma\equiv|\epsilon_0-\eta_x^2|^{1/2}$, and the $+$ $(-)$ sign is used when $K$ is positive (negative). Note that when $K<0$ total internal reflection should occur in the linear limit at the boundary between (1) and (2). Also, a new length scale along $z$ is defined so that

$$\zeta\equiv\gamma\xi\equiv\gamma z\omega/c . \tag{40}$$

The integral in Eq. (11) can be easily performed, and Eq. (10) can then be rewritten in terms of a dimensionless intensity variable $W$, which is defined as

$$W(\xi)=\alpha F^2/\gamma^2 . \tag{41}$$

The result is

$$\tfrac{1}{2}\dot{W}^2+V(W)=0 , \tag{42}$$

where

$$V(W)\equiv W^3\pm 2W^2-4TW+2S^2 \tag{43}$$

and the constants $e$ and $l$ are redefined so that

$$T=\alpha e/\gamma^4 \tag{44}$$

and

$$S=\alpha l/\gamma^3 , \tag{45}$$

respectively. Since the "kinetic-energy" term in Eq. (42) cannot be negative, it is clear that a solution exists only in a region where $V(W)$ is not positive. From the form of $V$ and the fact that $W$ cannot be negative (since $\alpha$ is assumed positive here), the parameters $S$ and $T$ must be such that $V$ has a negative root (which always exists since $S^2>0$) and two positive roots. These roots will be denoted by $W_1$, $W_2$, and $W_3$, which are ordered so that

$$W_3\le 0\le W_2\le W_1 . \tag{46}$$

Of course these three roots can be found in terms of $S$ and $T$ by solving a cubic equation given by $V=0$; however, it turns out that there is actually no need to do so, as will be shown in the next section. It is clear that the physical region is given by $W_2\le W\le W_1$ and therefore $V$ will be expressed as

$$V(W)=-(W_1-W)(W-W_2)(W-W_3) . \tag{47}$$

The solution to Eq. (42) can be calculated exactly with the result

$$W(\xi)=W_1-(W_1-W_2)\,\mathrm{sn}^2\left[\left[\frac{W_1-W_3}{2}\right]^{1/2}(\xi-\xi_0)\right] , \tag{48}$$

where sn is the Jacobian elliptic sine function with modulus

$$k=\left[\frac{(W_1-W_2)}{(W_1-W_3)}\right]^{1/2} , \tag{49}$$

and $\zeta_0$ is an integration constant. With this result, the phase function $\psi$ can then be found from Eq. (7). A further integration gives the accumulated phase within the film as

$$\int_0^{\xi} d\xi\, \psi(\xi) = \frac{S}{W_1}\left[\frac{2}{W_1 - W_3}\right]^{1/2} \times \Pi\left[\left[\frac{W_1 - W_3}{2}\right]^{1/2}(\zeta - \zeta_0), \beta^2\right], \tag{50}$$

where

$$\beta \equiv \left[\frac{W_1 - W_2}{W_1}\right]^{1/2} \tag{51}$$

and $\Pi$ is the incomplete elliptic integral of the third kind with modulus $k$.

## V. INTENSITY-DEPENDENT REFLECTION COEFFICIENT

In this section, using the above exact solution, the reflection coefficient will be calculated exactly as a function of the incident intensity by imposing the boundary conditions. The boundary condition as given by Eq. (27) will first be imposed. It is easy to see that this condition implies that two different types of solution, (1) and (2), are in general possible. For solution (1), $\zeta_0$ is given by

$$\zeta_0 = \zeta_d , \tag{52}$$

where $\zeta_d \equiv \gamma\omega d/c$, with $d$ as the thickness of the film. At $z = d$ the intensity takes on a maximum value and is equal to $W_1$. On the other hand, for solution (2),

$$\zeta_0 = \zeta_d - \left[\frac{2}{W_1 - W_3}\right]^{1/2} K(k) , \tag{53}$$

where $K(k)$ is the complete the elliptic integral of the first kind, and the intensity at $z = d$ has a minimum value equal to $W_2$.

Next, with new parameters defined by

$$r_1 \equiv \eta_{1z}/\gamma , \tag{54}$$

$$r_3 \equiv \eta_{3z}/\gamma , \tag{55}$$

the boundary condition given in Eq. (29) can be rewritten as

$$r_3 = S/W(\zeta_d) . \tag{56}$$

The remaining boundary condition [Eq. (33)] can be expressed for a Kerr-like nonlinear medium as

$$2T + 2r_1 S + (r_1^2 \mp 1)W(0) - \tfrac{1}{2}W^2(0) = 4r_1^2 W_0 , \tag{57}$$

where $W_0$ is the dimensionless incident intensity defined by

$$W_0 = \alpha E_0^2/\gamma^2 . \tag{58}$$

At this point, note that $W(0)$ and $W(\zeta_d)$ are rather complicated functions of $W_1$, $W_2$, and $W_3$, which are themselves rather complicated functions of the parameters $S$ and $T$. For a given incident intensity $W_0$, one can find $S$ and $T$ by solving the two simultaneous transcendental equations. The reflection coefficient is then given from Eq. (36) by

$$|r|^2 = 1 - \frac{S}{r_1 W_0} . \tag{59}$$

It turns out that this somewhat complicated procedure can be avoided if, instead of $S$ and $T$, $W_1$, $W_2$, and $W_3$ are treated as parameters. As a result, there is an extra parameter which can be made use of to solve the equations parametrically.

To do so, one must be able to express $S$ and $T$ in terms of $W_1$, $W_2$, and $W_3$. This can be easily accomplished since the $W$'s are roots of the cubic equation $V(W)=0$ and therefore obey the following equations:

$$W_1 + W_2 + W_3 = \mp 2 , \tag{60}$$

$$W_1 W_2 + W_2 W_3 + W_3 W_1 = -4T , \tag{61}$$

and

$$W_1 W_2 W_3 = -2S^2 . \tag{62}$$

Now let us first consider solution (1) where we have

$$W(\zeta_d) = W_1 . \tag{63}$$

Equation (56) then gives $r_3 = S/W_1$, which gives $S$ as

$$S = r_3 W_1 . \tag{64}$$

It is clear from Eqs. (60), (61), (62), and (64) that there is actually only one free parameter. For solution (1) it turns out that it is convenient to choose $W_2$ as the free parameter. From these equations one finds that

$$W_1 = \frac{W_2(W_2 \pm 2)}{2r_3^2 - W_2} , \tag{65}$$

$$W_3 = \frac{-2r_3^2(W_2 \pm 2)}{2r_3^2 - W_2} . \tag{66}$$

Equation (61) then gives $T$ as a function of $W_2$ as

$$T = W_2(W_2 \pm 2)[W_2(2r_3^2 - W_2) - 4r_3^2(r_3^2 \pm 1)]/(2r_3^2 - W_2)^2 . \tag{67}$$

Therefore, for each value of $W_2$ one can calculate the incident intensity $W_0$ using Eqs. (57), (48), (52), (64), and (67). The reflection coefficient can also be calculated as a function of $W_2$ with the help of Eqs. (59) and (64). As a result, the behavior of the reflection coefficient as a function of the incident intensity can be readily found.

Next, it is important to note that not all values of the parameter $W_2$ are allowed, since the roots are supposed to be ordered as in Eq. (46). From this condition and Eqs. (65) and (66) it is easy to see that $W_2$ must be restricted so that, for $K > 0$,

$$(r_3^2 - 1) \le W_2 \le 2r_3^2 \quad \text{if } r_3 \ge 1 \tag{68}$$

and

$$0 \le W_2 \le 2r_3^2 \quad \text{if } r_3 \le 1 \ , \tag{69}$$

and, for $K < 0$,

$$(r_3^2+1) \le W_2 \le 2r_3^2 \quad \text{if } r_3 > 1 \tag{70}$$

and

$$2r_3^2 \le W_2 \le (r_3^2+1) \quad \text{if } r_3 < 1 \ . \tag{71}$$

On the other hand, for solution (2), it is more convenient to treat $W_1$ as the free parameter. The equations for $W(\zeta_d)$, $S$, and $T$ are the same as those for solution (1) except that $W_1$ and $W_2$ should be interchanged. The reflection coefficient can again be calculated as a function of the incident intensity parametrically. However, $W_1$ must be restricted so that, for $K > 0$,

$$0 \le W_1 \le (r_3^2-1) \quad \text{if } r_3 > 1 \ , \tag{72}$$

and if $r_3 > 1$ then this solution cannot be allowed. For $K < 0$, $W_1$ must be such that

$$2 \le W_1 \le (r_3^2+1) \quad \text{if } r_3 > 1 \tag{73}$$

and

$$(r_3^2+1) \le W_1 \le 2 \quad \text{if } r_3 > 1 \ . \tag{74}$$

The limit of infinite incident intensity corresponds to $W_2 = 2r_3^2$. For $K > 0$ the linear limit is represented by $W_2 = 0$ if $r_3 < 1$ and by $W_1 = 0$ if $r_3 > 1$, and for $K > 0$ it is $W_1 = 2$. As shown below, the results derived here reduce to those in the linear limit when $\alpha \to 0$. When $K > 0$ and $r_3 < 1$ only solution (1) is allowed, and only in that region does this solution have a linear analog. For $K > 0$ and $r_3 > 1$ as well as for $K < 0$ and $r_3 > 1$ or $r_3 < 1$, both solutions are allowed, but only solution (2) has a linear analog; solution (1) does not have a linear analog and so it exists only when the incident intensity is above some minimum threshold value.

## VI. LINEAR LIMIT

The purpose of this section is to show that the appropriate results are recovered in the linear limit when $\alpha \to 0$.

When $\alpha \to 0$ it is clear by examining Eq. (11) that, for $K > 0$, $W_1$ and $W_2$ must go to zero while $W_3$ must approach $-2$, and on the other hand, for $K < 0$, $W_2$ and $W_3$ must go to zero while $W_1$ must approach 2. The cases $K > 0$ and $K < 0$ have to be considered separately.

For $K > 0$ and $r_3 > 1$ only solution (1) is allowed. The linear limit corresponds to taking $W_2$ to zero. From Eqs. (65) and (66) one has

$$W_1 = W_2 / r_3^2, \quad W_3 = -2 \ . \tag{75}$$

This is just the right behavior one expects. It is also clear from Eq. (49) that in this limit, $k \to 0$ and

$$W(0) = W_2 \left[ \frac{1}{r_3^2} + \left( 1 - \frac{1}{r_3^2} \right) \sin^2 \zeta_d \right] , \tag{76}$$

$$S = \frac{W_2}{r_3} , \tag{77}$$

$$T = \frac{W_2}{2} \left[ 1 + \frac{1}{r_3^2} \right] . \tag{78}$$

$W_0$ can then be calculated from Eq. (57), and the reflection coefficient is found to be given by

$$|r|^2 = 1 - \frac{W_2}{r_1 r_3 W_0} = 1 - \frac{4 r_1 r_3}{(r_1 + r_3)^2 + (r_3^2 - 1)(r_1^2 - 1) \sin^2 \zeta_d} , \tag{79}$$

which is precisely the correct result in the linear limit when $K > 0$.

For $K > 0$ and $r_3 > 1$, one should consider solution (2) with $W_1 \to 0$. It is easy to see that the results are just the same as above except that $W_1$ and $W_2$ are interchanged. The reflection coefficient in the linear limit must be intensity independent and therefore must again be given by Eq. (79), which is just the result anticipated.

For $K < 0$, regardless of whether $r_3 > 1$ or $r_3 < 1$, the linear limit must correspond to taking solution (2) with $W_1 \to 2$. Solution (1), which has a maximum intensity at $z = d$ within the film, clearly has no linear analog and therefore need not be considered here.

To investigate this limit, let $W_1 = 2 + \delta$, where $\delta$ is an infinitesimal quantity which is positive if $r_3 > 1$ and is negative if $r_3 < 1$. One finds that

$$W_2 = \delta / (r_3^2 - 1) , \tag{80}$$

and

$$W_3 = -r_3^2 \delta / (r_3^2 - 1) \ . \tag{81}$$

With these results one finds from Eq. (49) that

$$k^2 = 1 - \frac{(r_3^2+1)\delta}{2(r_3^2-1)} \tag{82}$$

to first order in $\delta$. When $k \to 1$ the Jacobian elliptic functions become hyperbolic functions and Eqs. (48) and (57) yield the result

$$W(0) = \frac{\delta}{r_3^2 - 1} [1 + (r_3^2+1) \sinh^2 \zeta_d] \ . \tag{83}$$

In addition, in this limit one finds that

$$S = r_3 \delta / (r_3^2 - 1), \quad T = \delta / 2 \ . \tag{84}$$

Combining all these results, $W_0$ and $|r|^2$ can be calculated, and the following result is obtained:

$$|r|^2 = 1 - \frac{r_3 W_2}{r_1 W_0} = 1 - \frac{4 r_1 r_3}{(r_1 + r_3)^2 + (r_1^2 + 1)(r_3^2 + 1) \sinh^2 \zeta_d} , \tag{85}$$

which is just the expected result in the linear limit for $K < 0$.

## VII. INDUCED TRANSPARENCY

Let us investigate to see what happens when the incident intensity is at the threshold value where solution (2) becomes solution (1). If one sets $W_1 = W_2 = r_3^2 - 1$ for the case where $K > 0$ and $r_3 > 1$ (i.e., $\epsilon_3 > \epsilon_0$), one finds that the intensity within the film is uniform and is equal to $r_3^2 - 1$. The incident intensity for this to occur is given by

$$W_0 = (r_3^2 - 1)(r_1 + r_3)^2 / 4r_1^2 , \tag{86}$$

and the reflection coefficient has the same values as though the film is totally absent, i.e.,

$$|r|^2 = \frac{(r_1 - r_3)^2}{(r_1 + r_3)^2} . \tag{87}$$

This can be understood because at that incident intensity the effective dielectric constant of the entire film is given by

$$\epsilon_0 + \alpha E^2(0) = \epsilon_0 + (\epsilon_0 - \eta_x^2)(r_3^2 - 1) = \epsilon_3 , \tag{88}$$

i.e., the same as in region (3).

For $K < 0$, setting $W_1 = W_2 = r_3^2 + 1$ yields an intensity equal to $r_3^2 + 1$ uniformly within the film. The effective dielectric constant of the entire film is

$$\epsilon_0 + \alpha E^2(0) = \epsilon_0 + (\eta_x^2 - \epsilon_0)(r_3^2 + 1) = \epsilon_3 , \tag{89}$$

i.e., the same as that of region (3). Thus one expects and finds that $|r|^2$ is again given by Eq. (87). The critical incident intensity for this induced transparency to occur is

$$W_0 = (r_3^2 + 1)(r_1 + r_3)^2 / 4r_1^2 . \tag{90}$$

These results for included transparency can also be derived based on some rather simple considerations.

## VIII. INDUCED RESONANCE SCATTERING

In the linear case, resonant scattering occurs when the thickness of the film is precisely equal to an integral number of half-wavelengths. In the nonlinear case, since the effective dielectric constant and thus the effective wavelength are functions of the incident intensity, one expects that resonance scattering can occur for a film of a given thickness by adjusting the incident intensity. This section is devoted to an analysis of this interesting phenomenon of induced resonances.

First, one recalls that the function $\mathrm{sn}^2$ is periodic with a period of $2K(k)$, and therefore the distance between two adjacent maxima or minima of the intensity within the film is given by

$$\zeta_p = 2K(k) \left[ \frac{2}{W_1 - W_3} \right]^{1/2} . \tag{91}$$

One expects that resonances will occur for either type of solutions when the film thickness is such that

$$\zeta_d = 2mK(k) \left[ \frac{2}{W_1 - W_3} \right]^{1/2} , \quad m = 1, 2, \ldots . \tag{92}$$

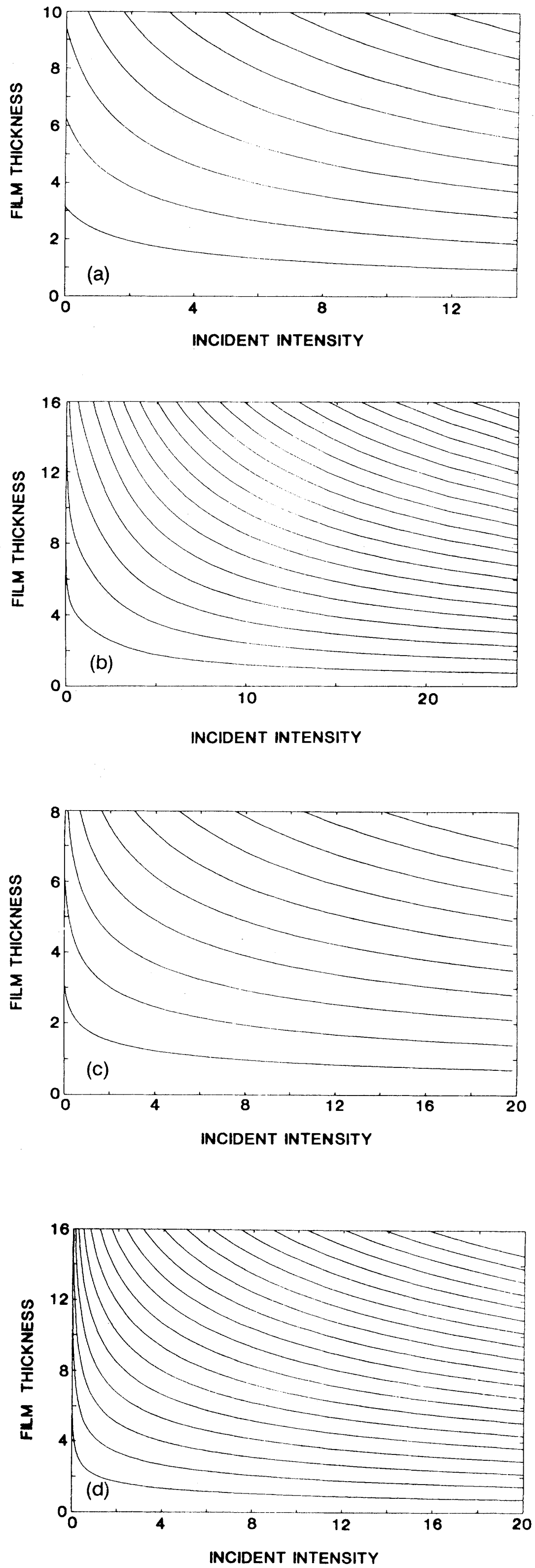


FIG. 2. For any film thickness the incident intensities at which resonances occur are shown for (a) $K > 0$, $r_3 = 0.5$; (b) $K > 0$, $r_3 = 2$; (c) $K < 0$, $r_3 = 0.5$; and (d) $K < 0$, $r_3 = 2$. The dimensionless incident intensity is defined in Eq. (58).

This claim is justified by the fact that when the above condition is satisfied, one can see from Eq. (48) that the intensities at $z=0$ and $d$ within the film have the same value. This in turn implies that $\dot{F}(0)=\dot{F}(\zeta_d)=0$. Then one can show from Eqs. (21) and (22) that $r$ is real and is given by

$$r=\frac{\eta_{1z}-\psi(0)}{\eta_{1z}+\psi(0)}, \tag{93}$$

which is exactly the same result at resonance in the linear case.

For a film of a given thickness, the incident intensity at which induced resonances occur can be calculated. For solution (1), $W(0)$ is equal to $W_1$, and one finds that the incident intensity is given by

$$W_0=\frac{(r_1+r_3)^2 W_2(W_2\pm 2)}{4r_1^2(2r_3^2-W_2)}. \tag{94}$$

From Eq. (92), $\zeta_d$ can be calculated as a function of $W_2$. Therefore one can easily plot $\zeta_d$ as a function of $W_0$ parametrically. This can be carried out similarly for solution (2). The results are shown in Fig. 2.

As is expected for $K>0$, there are no resonances in the linear limit ($W_0\rightarrow 0$) regardless of the thickness of the film. It can be seen from Eq. (49) that in the linear limit, $K$ approaches unity and $K(k)$ goes to infinity logarithmically. However, with a sufficiently strong incident intensity, the effective dielectric constant can become large enough so that the effective value of $K$ becomes positive and thus resonances can occur. Moreover, for a film of a given thickness, the number of resonances increases with increasing intensity. This can be understood because with increasing intensity the effective wavelength decreases and so more half-wavelengths can be fitted within the film.

Finally, it is possible to interpret the results of the reflection coefficient as a function of the incident intensity as shown in Fig. 3. The procedures for calculating these results have already been discussed at length in Sec. V. Here, for simplicity, the dielectric constants in regions (1) and (3) are taken to be identical, and the film thickness $\zeta_d$ is 3.

For $K>0$ and $r_3<1$, the dielectric constant of the film in the linear limit is higher than that of the outside. Therefore the reflection coefficient is not very large when $W_0\rightarrow 0$.[13] With higher incident intensity the effective dielectric constant of the films is even higher, and resonances can occur. The reflection coefficients at resonances are zero, and the incident intensities at which resonances occur correspond precisely with those given in Fig. 2(a). Between successive resonances the reflection coefficient rises to some maximum value. These maxima values increase with increasing intensity since the effective dielectric constant of the film also increases. It is clear from Fig. 3(a) that the reflection coefficient as a function of the incident intensity exhibits optical bistability.

For $K>0$ and $r_3>1$, the dielectric constant of the film is smaller than that of the outside region in the linear limit, but it is not so small that total internal reflection can

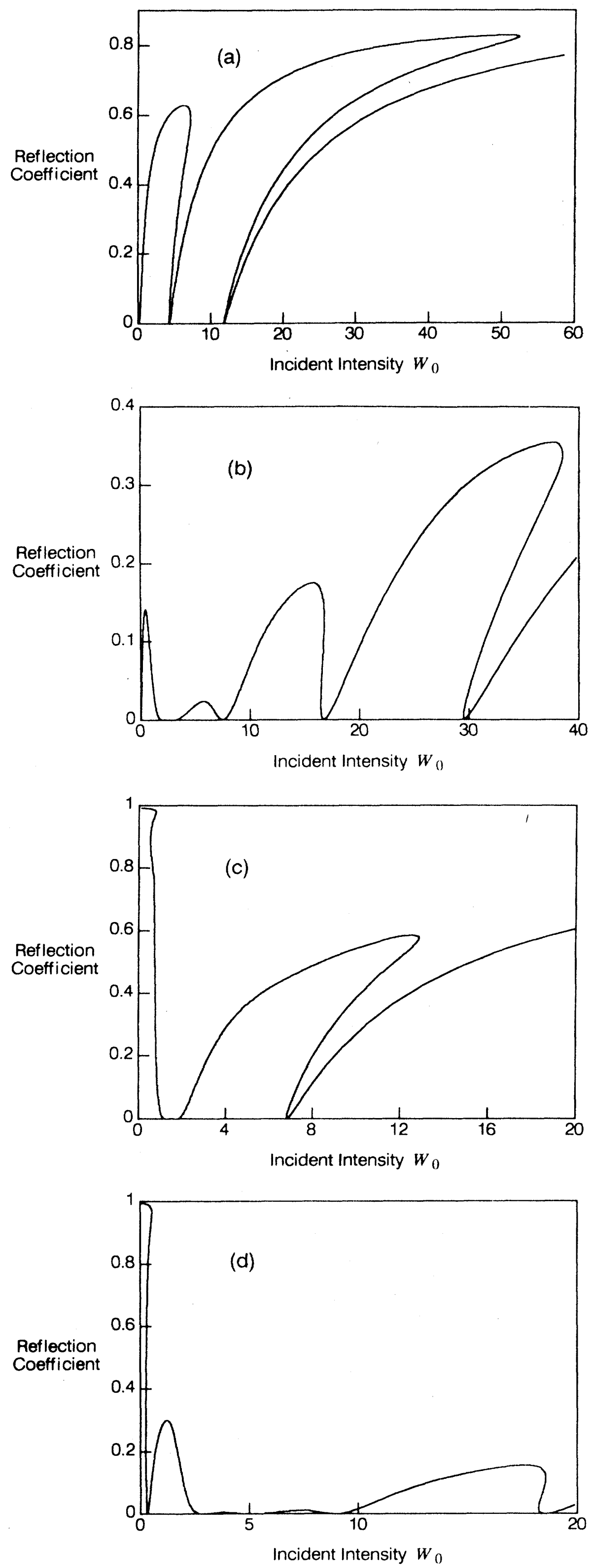


FIG. 3. Plots of the reflection coefficient as a function of the incident intensity in dimensionless unit for (a) $K>0$, $r_3=0.5$; (b) $K>0$, $r_3=2$; (c) $K<0$, $r_3=0.5$; and (d) $K<0$, $r_3=2$. The dielectric constants in regions (1) and (3) are taken to be identical, and the film thickness $\zeta_d$ is 3.

occur. The reflection coefficient is therefore not very large in the linear limit. As the incident intensity is increased, the effective dielectric constant of the film increases, and induced resonances occur. At $W_0=3$ induced transparency happens and the reflection coefficient is zero.

For $K<0$, regardless of whether $r_3>1$ or $r_3<1$, the incident angle $\theta_0$ is such that the scattering at the boundary of regions (1) and (2) is in the total internal reflection regime in the linear limit, thus the wave inside the film is evanescent and the reflection coefficient is close to unity. With increasing incident intensity, the effective value of $K$ increases and can become positive. The reflection coefficient therefore decreases with increasing intensity, and induced resonances as well as induced transparency of the film can occur. Optical bistability is possible not only at high intensity but also at relatively low intensity, as shown in Fig. 3.

## ACKNOWLEDGMENTS

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[10]Quantities which clearly refer to the nonlinear medium will not be labeled by any subscript.

[11]Actually $e$ does not correspond to the electromagnetic energy of the system.

[12]Note that $F(\xi_0)$ is presumed known. Alternately, we can fix the value of $\xi_0$, then $F(\xi_0)$ will be treated as an unknown parameter.

[13]It turns out that for the value of $\zeta_d$ $(=3)$ chosen here, the resonance condition is approximately satisfied in the linear limit, and therefore $|r|^2$ $(=0.011)$ is very small there.