Brief Reports

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Exact asymptotic behavior of the charge and spin susceptibilities in an interacting Fermi liquid

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We discuss some exact results concerning the large-wave-vector behavior of the charge and spin susceptibilities of an interacting Fermi liquid in its normal state. We have investigated three-dimensional as well as two-dimensional Coulombic systems for which new exact results have been established. Our microscopic analysis is easily generalized to other types of interactions. The relevance and the implications of our results are discussed.

The purpose of the present paper is to address the problem of the exact behavior at large wave vectors of both the charge and spin susceptibilities of an interacting Fermi liquid—a problem that has recently attracted some attention.[1–4] Our analysis will focus on three-dimensional (3D) as well as two-dimensional (2D) degenerate Fermi systems with Coulomb interactions but can be simply generalized to systems interacting via other types of potentials. The main motivation of the present study is provided by the need for a suitable approximation to the response functions, which is a necessary ingredient for the successful development of a qualitative and quantitative theory of various many-body phenomena.[5] A specific example is provided by our recent study of the effective mass and anomalous Landé g factor in inversion layers and in general in a 2D electron gas.[6–8]

Work on the behavior at large wave vectors of the susceptibilities of an interacting Fermi liquid has been carried out employing various techniques by several authors.[9–13] More recently Holas[1] has reviewed and discussed in detail the case of the charge susceptibility for the 3D case. Our aim is to recover in a simple and transparent way the well-known results for the charge and spin susceptibilities in 3D degenerate Coulombic systems, and to establish for the first time the corresponding results in the 2D case.

We begin by introducing a suitable definition for the response functions. By generalizing Niklasson's analysis,[13,3] $\chi_C(\mathbf{q},\omega)$ and $\chi_S(\mathbf{q},\omega)$, the charge and spin susceptibilities of an interacting Fermi system can, quite generally, be expressed as follows:

$$\chi_C(\mathbf{q},\omega)=\frac{\chi_0(\mathbf{q},\omega)}{1-v(\mathbf{q})[1-G_+(\mathbf{q},\omega)]\chi_0(\mathbf{q},\omega)} \tag{1}$$

and

$$\chi_S(\mathbf{q},\omega)=-\mu_B^2\frac{\chi_0(\mathbf{q},\omega)}{1+v(\mathbf{q})G_-(\mathbf{q},\omega)\chi_0(\mathbf{q},\omega)}, \tag{2}$$

where $v(\mathbf{q})$ is the Fourier transform of the appropriate interaction potential, and $\chi_0(\mathbf{q},\omega)$ differs from the familiar Lindhard susceptibility[14] in that the (plane-wave) occupation numbers $n(\mathbf{p})$ entering its expression are here taken to be the ones appropriate to the interacting system. Equations (1) and (2) do not have a physical content of their own and merely provide a definition for the many-body local fields $G_\pm(\mathbf{q},\omega)$. It is interesting to notice here that these quantities can be seen as the many-body analogues in the electron liquid of the familiar Clausius-Mossotti local fields of electromagnetism.[15] In general, however, the $G_\pm(\mathbf{q},\omega)$'s are (largely unknown) complex functions.

Although seemingly awkward and perhaps unsettling at a first superficial glance, Niklasson's definition of the $G_\pm(\mathbf{q},\omega)$'s provides a physically pleasing general definition for the class of vertex corrections first conceived by Hubbard.[16] In particular with this definition the $G_\pm(\mathbf{q},\omega)$'s do not display the (physically unappealing) $\mathbf{q}^2$ divergence for large $\mathbf{q}$ which must occur[1] if $\chi_0(\mathbf{q},\omega)$ in Eqs. (1) and (2) is taken to be the "plain" Lindhard response function.[14]

We consider first the large-wave-vector static limit (i.e., $\mathbf{q}\to\infty$ at $\omega=0$). In this case $\chi_0(\mathbf{q},\omega)$ and $G_\pm(\mathbf{q},\omega)$ are real and we can write down the following expansion for the function $\chi_0(\mathbf{q},0)$,

$$\chi_0(\mathbf{q},0)\approx-\frac{4m}{q^2}\left[n+\frac{4\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle}{q^4}+\frac{16\langle(\mathbf{k}\cdot\mathbf{q})^4\rangle}{q^8}+\cdots\right], \tag{3}$$

where n is the particle density. In Eq. (3) the notation $\langle f(\mathbf{k})\rangle$ indicates that an average over the (interacting) occupation numbers has been taken, i.e., $\langle f(\mathbf{k})\rangle=(1/V)\sum_{\mathbf{k}} n(\mathbf{k})f(\mathbf{k})$. The necessary other ingredients in the evaluation of the full response functions are the corresponding exact limiting values of the many-body fields $G_{\pm}(\mathbf{q},\omega)$. For the 3D case the appropriate values have been obtained by Niklasson[15] and by Zhu and Overhauser.[5] By making use of these results and of the expansion of Eq. (3) we readily obtain

$$\chi_{C,S}^{(3\mathrm{D})}(\mathbf{q}\to\infty,0)\approx-\frac{4m}{q^2}\left[n+\frac{4\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle}{q^4}+\frac{16\langle(\mathbf{k}\cdot\mathbf{q})^4\rangle}{q^8}-\frac{16\pi mn^2e^2}{3q^4}[1+(-1\pm3)g(0)]+\cdots\right], \tag{4}$$

where $g(0)$ is the value at the origin of the pair correlation function of the system. Here the upper sign corresponds to the charge, the lower to the spin response.

As far as the 2D Coulomb case is concerned the appropriate exact asymptotic values for the $G_{\pm}(\mathbf{q},\omega)$ have recently been obtained by Santoro and Giuliani.[4] In this case Eq. (3) still applies (n is in this case the areal density), and the results for the response functions are given by

$$\chi_{S,C}^{(2\mathrm{D})}(\mathbf{q}\to\infty,0)\approx-\frac{4m}{q^2}\left[n+\frac{4\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle}{q^4}\pm\frac{8\pi mn^2e^2g(0)}{q^3}+\cdots\right], \tag{5}$$

where in this case the upper sign corresponds to the spin, the lower to the charge response.[17,18]

Consider next the high-frequency regime in which the $\mathbf{q}\to\infty$ limit is taken in such a way that $\omega\gg q^4/mk_F^2$. Also in this limit $\chi_0(\mathbf{q},\omega)$ and $G_{\pm}(\mathbf{q},\omega)$ are real. The suitable expansion for the function $\chi_0(\mathbf{q},\omega)$ is given here by

$$\chi_0(\mathbf{q},\omega)\approx\frac{q^2}{m\omega^2}\left[n+\frac{3\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle}{m^2\omega^2}+\frac{n\mathbf{q}^4}{4m^2\omega^2}+\cdots\right], \tag{6}$$

As it turns out the exact values for the large-wave-vector limits of the $G_{\pm}(\mathbf{q},\omega)$ coincide with the ones of the static regime. As a consequence with the help of Eqs. (1), (2), and (6) we readily arrive at the sought results. In particular for the 3D case we have[10]

$$\chi_{C,S}^{(3\mathrm{D})}(\mathbf{q}\to\infty,\omega)\approx\frac{q^2}{m\omega^2}\left[n+\frac{3\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle}{m^2\omega^2}+\frac{n\mathbf{q}^4}{4m^2\omega^2}+\frac{4\pi n^2e^2}{3m\omega^2}[1+(-1\pm3)g(0)]+\cdots\right], \tag{7}$$

while for the 2D case we find

$$\chi_{C,S}^{(2\mathrm{D})}(\mathbf{q}\to\infty,\omega)\approx\frac{q^2}{m\omega^2}\left[n+\frac{3\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle}{m^2\omega^2}+\frac{n\mathbf{q}^4}{4m^2\omega^2}\pm\frac{2\pi n^2e^2g(0)q}{m\omega^2}+\cdots\right], \tag{8}$$

where the condition on the frequency is here $\omega^2\gg q^7/m^2k_F^3$.

A few remarks are in order here. The corrections involving $g(0)$ display a different wave-vector dependence in the 2D as compared to the 3D case. This feature simply stems from the different dispersion of the plasma modes in the two cases. While in 2D the first significant correction beyond the q^{-4} term is proportional to $g(0)$ and has always an opposite sign for the charge and spin response, in three dimensions the situation is more complicated. In fact in such a case the knowledge of both $g(0)$ and $\langle(\mathbf{k}\cdot\mathbf{q})^4\rangle$ (which must be finite) are necessary to determine the corresponding corrections. In all cases the knowledge of the average $\langle(\mathbf{k}\cdot\mathbf{q})^2\rangle$ is necessary. This quantity can either be simply evaluated with the use of the interacting occupation numbers $n(\mathbf{k})$,[19] or can alternatively be determined from the (exact) total kinetic energy of the system.

Finally we note that as far as the spin susceptibility in three dimensions is concerned, Eqs. (4) and (7), in both the static and the large frequency regimes the sign of the term involving $g(0)$ depends on the actual value of the latter, a sign change occurring for $g(0)=0.25$. From the approximately known values of $g(0)$ available in the literature it can be concluded that such a term is negative throughout the metallic range.

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