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Critical behavior of the SU(3) ferromagnetic Heisenberg chain

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We have extracted the leading low-temperature contributions to the specific heat and the magnetic susceptibility from the thermodynamic Bethe-*Ansatz* equations of the SU(3) invariant ferromagnetic Heisenberg chain of spin 1. The critical exponents for the specific heat, the susceptibility, and the correlation length are $\alpha=-\frac{1}{2}$, $\gamma=2$, and $\nu=1$, respectively. The susceptibility and the correlation length exhibit logarithmic corrections, which are quenched by relatively small magnetic fields. In large fields, the energy required to flip a spin gives rise to an exponential activation.

The critical behavior of the one-dimensional isotropic spin-$\frac{1}{2}$ Heisenberg ferromagnet has been studied by a variety of methods, which yielded quite different critical exponents[1-8] of the susceptibility γ and of the specific heat α. This long-standing problem was resolved in part by recent numerical solutions of the coupled thermodynamic Bethe-*Ansatz* integral equations,[7,8] both yielding the exponent values $\alpha=-\frac{1}{2}$ and $\gamma=2$ as expected from spin waves. A gradual crossover into the critical region is observed with decreasing temperature, which explains the diversity of extrapolated exponents[1-6] from temperature ranges that were not low enough. Finite string-size scaling of the numerical data[7] suggests logarithmic corrections to the susceptibility. The above exponents, as well as logarithmic corrections to the susceptibility and correlation length, have later been analytically extracted from the thermodynamic Bethe-*Ansatz* equations[9] and extended to chains with arbitrary spin[10] and SU(2) invariance. Recently, Bonner and Müller[11,12] argued that not only is the hyperscaling hypothesis[13] not satisfied for the $S=\frac{1}{2}$ isotropic Heisenberg ferromagnet, but also the scaling assumption breaks down.

In this paper we report analytic results for the SU(3) Heisenberg ferromagnet by employing the same procedure as in Refs. 9 and 10. The Hamiltonian for the SU(3) Heisenberg chain[14] is

$$\mathcal{H}=J\sum_{i=1}^{N}[(\mathbf{S}_i\cdot\mathbf{S}_{i+1})+(\mathbf{S}_i\cdot\mathbf{S}_{i+1})^2]$$
$$=J\sum_{i=1}^{N}(P_{i,i+1}+1)\,, \tag{1}$$

where J is the ferromagnetic coupling ($J<0$), $\mathbf{S}$ are spin-1 operators, and $P_{i,i+1}$ permutes the spin components of the sites i and $i+1$. Model (1) has been diagonalized by Sutherland[14] by means of two nested Bethe-*Ansätze*.

Our main result is the critical behavior of the correlation length, the susceptibility, and the specific heat, which are given by

$$\xi\simeq(2|J|/T)[\mathcal{L}^{-1}+(\ln\mathcal{L})/\mathcal{L}^2+\cdots]\,, \tag{2}$$

$$\chi\simeq(1.7|J|/T^2)[\mathcal{L}^{-1}+(\ln\mathcal{L})/\mathcal{L}^2+\cdots]\,, \tag{3}$$

$$C\simeq1.2(T/|J|)^{1/2}+O(T/|J|)\,, \tag{4}$$

where $\mathcal{L}=\ln(|J|/T)$. The exponent $\alpha=-\frac{1}{2}$ has been obtained previously by numerically solving the Bethe-*Ansatz* equations.[15] A small but finite magnetic field smears the logarithmic corrections in (2) and (3) and a large magnetic field ($H\gg T$) leads to an exponential activation

$$F\simeq-2|J|-2H-(1/\pi)|J|^{-1/2}T^{3/2}$$
$$\times\exp(-3H/T)+\cdots\,. \tag{5}$$

For $J<0$ the ground state is ferromagnetically ordered. The excited states of the chain consist of magnons and bound states of magnons. The rapidities describing the ground and excited states are solutions of the discrete Bethe-*Ansatz* equations.[14] In the thermodynamic limit a bound state of n magnons is represented by a string of n complex rapidities. There are two sets of strings, since for $S=1$ there are two degrees of freedom per site. In thermal equilibrium the properties at finite T and H are given by the thermodynamic energies of these elementary excitations $\varepsilon_n^{(1)}$ and $\varepsilon_n^{(2)}$. It is usual to introduce the statistical weight of the excitations, $\eta_n^{(p)}=\exp(\varepsilon_n^{(p)}/T)$, $p=1,2$, which satisfy the following nonlinearly coupled integral equations[16]

$$\ln\eta_n^{(1)}=-2\pi(J/T)\delta_{n,1}G_2+G_2*\ln[(1+\eta_{n+1}^{(1)})(1+\eta_{n-1}^{(1)})/(1+\eta_n^{(2)})]+G_1*\ln[(1+\eta_{n+1}^{(2)})(1+\eta_{n-1}^{(2)})/(1+\eta_n^{(1)})]\,, \tag{6a}$$

$$\ln\eta_n^{(2)}=-2\pi(J/T)\delta_{n,1}G_1+G_2*\ln[(1+\eta_{n+1}^{(2)})(1+\eta_{n-1}^{(2)})/(1+\eta_n^{(1)})]+G_1*\ln[(1+\eta_{n+1}^{(1)})(1+\eta_{n-1}^{(1)})/(1+\eta_n^{(2)})]\,, \tag{6b}$$

for $n=1,2,\ldots$. Here the centered asterisk denotes a convolution,

$$G_p(\Lambda)=(1/\sqrt{3})[2\cosh(2\pi\Lambda/3)-(-1)^p]^{-1}\,, \tag{7}$$

and $\eta_n^{(p)}\equiv0$. These equations are completed by the asymptotic conditions

$$\lim_{n\to\infty}\{\ln[\eta_n^{(p)}(\Lambda)]\}/n=2H/T=2X_0\,, \tag{8}$$

and the free-energy per site is given by

$$F(T,H)=F(0,0)-T\int d\Lambda[G_2(\Lambda)\ln(1+\eta_1^{(1)})+G_1(\Lambda)\ln(1+\eta_1^{(2)})]\,, \qquad (9)$$

where $F(0,0)=J(2-\ln 3-\pi/3\sqrt{3})$.

The above equations have simple solutions in two limiting cases. (1) In the high-temperature limit (free spins), the driving terms in Eqs. (6) can be neglected. The solutions for the $\eta_n^{(p)}$ are then constants fixed by the field boundary condition (8) to be

$$\eta_n^{(1)}=\eta_n^{(2)}=\left[\frac{\sinh[(n+1)X_0]\sinh[(n+2)X_0]}{\sinh X_0\sinh(2X_0)}\right]-1\,, \qquad (10)$$

and the free-energy per site is then given by

$$F(T,H)=-T\ln[2\cosh(2X_0)+1]\,. \qquad (11)$$

(2) If $J<0$, all $\varepsilon_n^{(p)}(\Lambda)>0$ from Eqs. (6). At $T=0$ the driving terms dominate and $\eta_n^{(p)}=\infty$ for all n. The solutions obtained from (6) are

$$\varepsilon_n^{(p)}(\Lambda)=2nH+|J|\frac{n}{\Lambda^2+n^2/4}\delta_{p,1}\,. \qquad (12)$$

Then we have $S_z/N=1$ and $E/N=-2|J|-2H$, the ferromagnetic ground state.

In the critical region the solution is an interpolation between these free-spin and zero-temperature limits. Consider a low but finite temperature. We now argue that the functions $\varepsilon_n^{(1)}(\Lambda)$ show crossovers between expressions (10) and (12) as a function of Λ and n. For Λ or n sufficiently large, the driving term in (6) becomes negligible and the solution is (10). For small Λ and n, on the other hand, the driving terms dominate and $\varepsilon_n^{(1)}(\Lambda)$ is given by (12). For $\varepsilon_n^{(2)}(\Lambda)$, on the other hand, the large-n limit of (10) is identical to (12).

For small Λ and H we define a crossover index $n_c(T)$ by equating $\varepsilon_{n_c}^{(1)}$ from (10) with (12),

$$T\ln(n_c+\tfrac{3}{2})\simeq 2|J|/n_c\,, \qquad (13)$$

assuming that $n_c\gg 1$ at low T. For $n>n_c$ ($n<n_c$) the solution is then closer to the free-spin (strong-coupling) solution.

In thermal equilibrium, the correlation length, i.e., the average number of correlated spins, is approximately given by $\xi(T)\simeq n_c$. Note that bound states involving less (more) than n_c spins are unlikely (likely) to occur, so that n_c is the average number of correlated spins. If $T\gg|J|$, for all n and $n_c<1$, such that all spins are uncorrelated (free spins). On the other hand, as $T\to 0$ excitations are suppressed as n_c becomes very large and asymptotically ferromagnetic order is approached. Solving (13) iteratively for $T\ll|J|$ we obtain (2), corresponding to $\nu=1$, with logarithmic corrections.

Following Refs. 9 and 10 we now derive the low-temperature free energy by means of Eq. (9). For this purpose we determine $\ln[1+\eta_1^{(p)}(\Lambda)]$ using

$$\ln(1+\eta_1^{(p)})=2X_0+\sum_{m=1}^{\infty}([m+1]+[m-1])\ln(1+(\eta_m^{(p)})^{-1})-\sum_{m=1}^{\infty}[m]\ln(1+(\eta_m^{(q)})^{-1})-\delta_{p,1}(J/T)/(\Lambda^2+\tfrac{1}{4})\,, \qquad (14)$$

where $p,\ q=1,\ 2,\ q\neq p$, and $[m]$ is defined so that for any f

$$[m]f=1/\pi\int_{-\infty}^{\infty}d\Lambda'\frac{m/2}{(\Lambda-\Lambda')^2+(m/2)^2}f(\Lambda')\,. \qquad (15)$$

To a first approximation, $\eta_m^{(2)}$ is independent of Λ [Eqs. (10) and (12)] so that convolutions involving $\eta_m^{(2)}$ are easily integrated. The corresponding sums yield $2X_0-\ln[\sinh(3X_0)/\sinh X_0]$ if $p=1$ and minus twice this expression if $p=2$.

In the strong-coupling regime, $\eta_n^{(1)}\to\infty$ as $T\to 0$ so that the strong-coupling solution does not contribute to the integrals in (14). We assume $\eta_n^{(1)}(\Lambda)=\infty$ for $|\Lambda|<\Lambda_c(n)$ and $n<n_c$ and elsewhere given by the free-spin solution. This assumption of a sharp crossover does not affect the leading temperature dependence but only the amplitudes. $\Lambda_c(n)$ obtained via similar arguments as n_c is

$$\Lambda_c(n)\simeq\left[\frac{|J|}{T}\frac{n}{\ln[n(n+3)/2]}\right]^{1/2}. \qquad (16)$$

Following Refs. 9 and 10 we divide the contributions to $\ln(1+\eta_1^{(p)})$ arising from the remaining integrations in (14) into two parts: (a) $|\Lambda|>\Lambda_c(n)$, $n<n_c$ and (b) $n\geq n_c$.

(a) Expanding the free-spin solution for small fields X_0,

$$\ln[1+(\eta_m^{(1)})^{-1}]=\ln\left[1+\frac{2}{m(m+3)}\right]-\tfrac{2}{3}X_0^2+\cdots\,, \qquad (17)$$

and integrating (14) over the intervals $[\infty,\Lambda_c]$ and $[-\Lambda_c,-\infty]$, we obtain

$$\frac{1}{\pi}\sum_{m=1}^{n_c}\left[\ln\left[1+\frac{2}{m(m+3)}\right]-\tfrac{2}{3}X_0^2\right]\left[\tan^{-1}\left[\frac{(m+1)/2}{\Lambda_c-\Lambda}\right]+\tan^{-1}\left[\frac{(m-1)/2}{\Lambda_c-\Lambda}\right]+(\Lambda\leftrightarrow-\Lambda)\right]. \qquad (18)$$

Since $\Lambda_c\gg 1$ and the integration kernels in the free-energy expression (9) fall off exponentially fast so that only $|\Lambda|\ll\Lambda_c$ is relevant, we can expand Eq. (18) in powers of Λ/Λ_c. This Taylor expansion generates a power series in $T^{1/2}$ because $\Lambda_c\sim T^{-1/2}$, with the leading term given by $\Lambda=0$. The leading (as $T\to 0$) zero-field and small-field contributions are then

$$(T/|J|)^{1/2}\frac{2}{\pi}\sum_{m=1}^{n_c\to\infty}\ln\left[1+\frac{2}{m(m+3)}\right]\{m\ln[m(m+3)/2]\}^{1/2} \simeq 3.3(T/|J|)^{1/2}, \tag{19a}$$

$$-\frac{16}{3\pi}X_0^2\frac{0.57|J|}{T}\left[\frac{1}{\ln(|J|/T)}+\frac{\ln\ln(|J|/T)}{\ln^2(|J|/T)}+\cdots\right]. \tag{19b}$$

(b) Since the free-spin solution does not depend on Λ, the integrals in (14) are straightforward and we obtain

$$\sum_{m=n_c}^{\infty} 2\ln[1+(\eta_m^{(1)})^{-1}]\simeq\frac{4}{n_c}-4X_0+\tfrac{4}{3}n_cX_0^2+\cdots. \tag{20}$$

Expressions (19) and (20) are the contributions to $\ln(1+\eta_1^{(1)})$; those contributing to $\ln(1+\eta_1^{(2)})$ are very similar. Collecting all terms and inserting them into Eq. (9) we obtain the free energy per site in leading order as $T\to 0$:

$$F=-2|J|-1.65\frac{T^{3/2}}{|J|^{1/2}}-0.85|J|\frac{H^2}{T^2}\left[\frac{1}{\ln(|J|/T)}+\frac{\ln\ln(|J|/T)}{\ln^2(|J|/T)}+\cdots\right]. \tag{21}$$

The critical behavior is then similar to that of the SU(2) ferromagnetic Heisenberg chains.[9,10]

We have assumed in our calculation that the crossover from strong coupling to free spin is abrupt. Note that if we assume that the free-spin solution is valid for $|\Lambda|>a\Lambda_c$, $n<an_c$, and for all Λ if $n>an_c$, where $a\geq 1$, the temperature dependence of the free energy remains unchanged and only the amplitudes are rescaled.

We now discuss the critical behavior in an external field. Since the magnetic field favors magnetic ordering, the strong-coupling region will be enhanced by the field. Hence n_c is expected to grow with H, such that more excitations are suppressed. Equating $\varepsilon_{n_c}^{(1)}$ from (10) with (12) at finite H, we obtain

$$\frac{|J|}{n_c}=\frac{T}{4}\ln\left\{\left[1+e^{-2X_0}+e^{-4X_0}+\cdots+e^{-2n_cX_0}+\left[\frac{1-(-1)^{n_c}}{2}\right]e^{-2(n_c+1)X_0}\right]\right.$$
$$\left.\times\left[1+e^{-4X_0}+e^{-8X_0}+\cdots+e^{-2(n_c+1)X_0}+\left[\frac{1+(-1)^{n_c}}{2}\right]e^{-2n_cX_0}\right]-e^{-2n_cX_0}\right\}. \tag{22}$$

Hence, the magnetic field enhances n_c and ξ. If X_0 is small, we have $n_cX_0\sim H/T^2$, indicating that the exponent Δ (higher-field derivatives of the free energy) equals two (with logarithmic corrections). Note that n_cX_0 can be large at low T even if H is small and not all the exponential terms in (22) contribute. As a consequence, the logarithmic corrections in the correlation length (2) and the susceptibility (3) are quenched. The critical exponents in a weak field remain $\gamma=2$, $\alpha=-\frac{1}{2}$, and $\nu=1$.

For a strong field $H/T\gg 1$, considering only the first exponential term in (22) we get the crossover index

$$n_c\simeq 4(|J|/T)\exp(2H/T), \tag{23}$$

and the crossover rapidity

$$\Lambda_c\simeq(n|J|/T)^{1/2}\exp(H/T). \tag{24}$$

Both n_c and Λ_c are considerably enhanced by the field. To calculate the free energy we repeat the procedure used above with $\ln[1+(\eta_m^{(1)})^{-1}]\simeq e^{-2mX_0}$ as the free-spin solution. Since X_0 is large, only the $m=1$ term contributes to the leading order. Hence, part (b) can be neglected and part (a) yields

$$\frac{2}{\pi}(T/|J|)^{1/2}\exp(-3H/T). \tag{25}$$

Collecting all the terms we obtain (5) for the free energy to leading order. Note that the second term in (5) just gives the saturated magnetization and $2H$ is the Zeeman splitting. The last term in (5) can be rewritten as $(-8|J|/\pi)\xi^{-3/2}$, in terms of the correlation length ξ. Hence, in a strong magnetic field, ξ^{-1} replaces T in the zero-field free energy.[17] It should be mentioned that we implicitly assumed $H\ll|J|$ when dividing the integration regime into a weak- and a strong-coupling regime.

In summary, we obtained the critical behavior of the SU(3) invariant ferromagnetic Heisenberg chain of spin 1. The crossover index n_c represents the average number of correlated spins, i.e., the correlation length. We obtain $\nu=1$ with logarithmic corrections. From the zero-field free energy at low T we obtain that $\alpha=-\frac{1}{2}$ and $\gamma=2$ with logarithmic corrections.

The leading zero-field free energy, proportional to $T^{3/2}$, arises from the large $|\Lambda|$ values which correspond to small momenta. Hence the result is indeed due to long-wavelength magnons. The leading term of the susceptibility, on the other hand, cannot be explained within a simple magnon picture since strings of all orders contribute. The logarithmic corrections in χ arise from the temperature dependence of ξ. These logarithms are quenched by small magnetic fields.

In sufficiently large fields, $T \ll H \ll |J|$, we obtain that the temperature-dependent part of the free energy is proportional to $\xi^{-3/2}$, where ξ grows exponentially with $2H/T$.

Finally we discuss our results in the context of scaling and hyperscaling. The relations among the critical exponents[13] for a zero T_c are different from the usual ones where $T_c \neq 0$. Since in the ferromagnetic ground state all spins are aligned, the spin-spin correlation function is independent of the distance $\eta = 1$ and the magnetization is independent of the field $\delta = \infty$. The scaling and hyperscaling relations[13] yield then

$$\gamma = 1 + \nu, \quad \gamma = \Delta, \quad \text{and} \quad -\alpha_s = d\nu, \tag{26}$$

where $d = 1$ is the dimension. The first and second relations are satisfied, since $\chi \sim \xi/T$, including the logarithmic corrections, while the last one is not obeyed. The conclusion by Bonner and Müller[11,12] that the scaling and hyperscaling hypothesis break down is also valid for the SU(3) Heisenberg ferromagnet.

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