Critical behavior of $n \to 0$ gauge-invariant theory: Self-avoiding random surfaces

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We study the critical behavior, in a mean-field approximation, of a grand-canonical ensemble of nonplanar self-avoiding random surfaces (SAS's) which arises as an $n \to 0$ limit of a gauge-invariant theory. We show explicitly that our model is equivalent to the usual ϕ^4 theory near the critical point, with the upper critical dimension equal to 4. The fractal dimension of SAS is 2.

I. INTRODUCTION

An understanding of the behavior of interacting random surfaces is an important current problem both in particle physics and in condensed matter physics,[1] and has recently seen considerable activity.[2–16] Self-avoiding surfaces (SAS's) are a natural extension[1] of self-avoiding walks (SAW's) and are closely related to string theories of lattice gauge systems.[17–19] In particular, random surfaces have been found useful in the random surface representation of lattice gauge theories.[3] The self-avoiding constraint forbids a given bond to be shared by more than two plaquettes in SAS's (Ref. 3) and a given site by more than two bonds in SAW's.[20] The initial attempt at an $n=0$ description by Maritan and Omero[2] for SAS's was shown to be incorrect by Durhuss *et al.*[3] It was noticed by Durhuss *et al.*[3] that the weight for surfaces of fixed boundary could become unbounded if there were *no* restrictions on the nature of the allowed surfaces. For example, the Euler characteristic must be restricted for the weights to make sense as $n \to 0$. However, such a restriction could make the significance of these surfaces for gauge theories highly questionable. Nevertheless, one still hopes that some form of $n=0$ description for SAS's might exist, in view of the similarities that exist between gauge theories and spin systems.[21,22] This will be highly desirable for obtaining a field-theoretic description of SAS's which is lacking at present. Such a description is necessary to understand the critical behavior in physical systems involving statistical fluctuations of surfaces. Our understanding of the critical phenomena in surfaces is far from complete. There are competing conjectures regarding the upper critical dimension D_c^H, above which self-avoidance becomes irrelevant.[3–17] Various values of D_c^H have been identified, depending on the characteristics of the model. Different models of SAS's exhibit different universality classes and our understanding of the field is at present incomplete. Therefore, it is very important to produce a model of SAS's which can be investigated analytically.

Our aim in the present work is to extend the $n \to 0$ analogy from SAW's (Ref. 23) to SAS's. We find that the natural extension requires considering nonplanar surfaces. This natural model of *nonplanar* SAS's is very general and turns out to be *equivalent* to the $n \to 0$ limit of a spin model with Z_2 gauge invariance involving n-component spins $\mathbf{S}$ ($\mathbf{S}^2=n$) located on the links of a d-dimensional hypercubic lattice Z^d. This model, which turns out to be an extension of the model of Ref. 3, reduces to several models, each with distinct characteristics. We focus our attention on one of the models and solve it in a mean-field approximation for arbitrary n. For $n=0$, our calculation yields some novel features. Our model possesses a first-order transition at zero field and a critical point at nonzero field. Near the critical point, our model is equivalent to the usual ϕ^4 field theory. We conjecture that the upper critical dimension D_c^H is 4, and the lower critical dimension D_c^L is reasonably expected to be 1, just like the SAW problem.[24] This violates the conjectural relation $D_c^H=2D_c^L$.[17] The fractal dimension D_f is 2. Our values of D_c^H and D_f are consistent with the results of Durhuus *et al.*[3] Since nonplanar diagrams arise in gauge theories, domain walls, etc., our results should prove very useful in these fields.

The layout of the paper is as follows. We introduce our surface model in Sec. II. This is accomplished by considering a gauge-invariant theory in the limit $n=0$. Here, n stands for the number of components of the spins which are located at the links of the lattice. As said previously, each spin is constrained to have a fixed length: $\mathbf{S}^2=n$. This mapping enables us to relate a grand-canonical ensemble of nonplanar surfaces with the spin model as $n \to 0$. We consider various limits of the spin model. Each such limit describes a certain physically interesting model of SAS's. In Sec. III, we present a mean-field analysis. Within this framework, we find a line of first-order phase transitions terminating in a critical point C. The critical behavior about C is studied in Sec. IV. We demonstrate that the critical behavior is equivalent to that of a ϕ^4 field theory with usual Landau-like exponents. In particular, we identity the upper critical dimension D_c^H as 4 and the fractal dimension D_f of a surface as 2. Section V includes a discussion of other interesting properties. In particular, we suggest that the lower critical dimension D_c^L can be reasonably taken as 1, and we estimate the number of Hamilton surfaces. A brief summary of our results is also given.

II. SURFACE MODEL

Let us consider a finite lattice with N sites. Every plaquette on the lattice has four links associated with it and,

correspondingly, four n-component spins **s**, **s**, **s**$''$ and **s**$'''$. However, only first m components of these spins interact with each other. Let us also include a gauge-fixing external magnetic field

$$\mathbf{H}=H(1,1,\ldots,1,0,\ldots,0)$$

with m' nonzero and equal components. The system is described by the following anisotropic Hamiltonian:

$$\mathcal{H}_n = K\sum_{[P]} O_P + H\sum_{[B]} O_B \ , \tag{1}$$

with

$$O_P = \sum_{\alpha=1}^{m} S^{(\alpha)}S'^{(\alpha)}S''^{(\alpha)}S'''^{(\alpha)}$$

and

$$O_B = \sum_{\alpha=1}^{m'} S^{(\alpha)} \ ,$$

K and H both non-negative. We will also restrict m and m' such that $m \geq m'$. The first sum is over each plaquette P and the second one over each bond B. In the absence of **H**, we have local *gauge invariance* under

$$Z_2 : S^{(\alpha)} \to -S^{(\alpha)} \ , \tag{2}$$

which is broken by **H**. A similar anisotropic interaction has been used to describe self-avoiding rings and walks.[23]

The partition function Z_n is defined by

$$Z_n = \prod_{[B]} \mathrm{Tr}_B \exp\mathcal{H}_n \ ,$$

where

$$\mathrm{Tr}_B(\times) = \int d\Omega_n(\times)/\Omega_n \ ,$$

and where Ω_n is the solid angle. It is well known[20] that as $n \to 0$, a spin component can occur, at most, twice:

$$\mathrm{Tr}(1) = \mathrm{Tr}[S^{(\alpha)}]^2 = 1 \ ,$$

$$\mathrm{Tr}[S^{(\alpha)}]^{2p} = \mathrm{Tr}[S^{(\alpha)}]^{2p-1} = 0, \quad p=2,\ldots,$$

as $n \to 0$. This ensures that a bond can, at most, be shared by two plaquettes. Following Ref. 23, we find that the only diagrams that survive are the ones shown in Fig. 1.

(a) There are closed surfaces, Fig. 1(a). Each closed surface of area $|S|$ contributes a weight of $mK^{|S|}$. The factor m originates because the "color" index α runs from 1 to m in each closed surface. Each plaquette gives a factor of K.

(b) Each elementary plaquette, Fig. 1(b), can appear twice due to $K^2O_P^2/2$ in the expansion of $\exp(KO_P)$, see (1), and contributes a weight of $mK^2/2$, and not mK^2 as in (a). This is a closed surface with only two plaquettes.

(c) There are open surfaces, Fig. 1(c), with boundary γ, determined by bonds, each belonging to exactly one occupied plaquette. Such a diagram must have a factor of $H^{|\gamma|}$ from the boundary because, each spin on the boundary can be made to appear twice with a term $HS^{(\alpha)}$ from (1), as shown by crosses in Fig. 1(c). Here, $|\gamma|$ denotes the length, i.e., the number of bonds belonging to γ. Since $\alpha=1,\ldots,m'$, each surface can have only m' different colors. The total weight is $m'H^{|\gamma|}K^{|S|}$.

(d) Each bond, which is unoccupied by any plaquette, can appear in two different states: (i) The corresponding spin does not appear in the diagram, Fig. 1(d), and (ii) The spin appears via $H^2[S^{(\alpha)}]^2/2$, $\alpha=1,\ldots,m'$, Fig. 1(e). The total weight for these two possibilities[20,23] is $z_0 = 1+m'H^2/2$.

Now, it is not hard to see that as $n \to 0$, $Z_n \to Z_0$ given by

$$Z_0 = \sum_{|S|,|\gamma|} m^C m'^{C'} \left(\tfrac{1}{2}\right)^{C_0} H^{|\gamma|}K^{|S|} z_0^{N_B - N_0} \ , \tag{3}$$

where C is the number of connected closed SAS's, C' the number of connected open surfaces (i.e., with nonempty boundary), C_0 the number of connected double plaquettes; $N_B = Nd$ is the total number of bonds and $N_0 = 2|S| + |\gamma|/2$ is the number of bonds occupied by plaquettes. Two SAS's are disconnected if they do not share a bond; sharing a corner is not sufficient for their connectness. Moreover, our SAS's are not necessarily planar and have very *rich* topology: They may possess *handles* and/or *holes*.

We can define a related partition function $\hat{\mathbf{Z}}$ for SAS's by

$$\hat{\mathbf{Z}}(\kappa,\eta) = Z_0/z_0^{N_B} = \sum_{|S|,\gamma|} m^C m'^{C'} \left(\tfrac{1}{2}\right)^{C_0} \eta^{|\gamma|}\kappa^{|S|} \ , \tag{4}$$

where $\eta^2 = H^2/z_0$ and $\kappa = K/z_0^2$. As $\eta \to 0 (H \to 0)$, $\kappa \to K$, and we have only closed SAS's in (4):

$$\hat{\mathbf{Z}}_{\text{closed}}(\kappa,0) = \sum_{|S|,|\gamma|=0} m^C \left(\tfrac{1}{2}\right)^{C_0} K^{|S|} \ . \tag{5}$$

By letting $m = n \to 0$, we find that all closed surfaces disappear in (4) and we have only open surfaces:

$$\hat{\mathbf{Z}}_{\text{open}}(\kappa,\eta;m=0) = \sum_{|S|,|\gamma|\neq 0} m'^{C'} \eta^{|\gamma|}\kappa^{|S|} \ . \tag{6}$$

If one is interested in studying the behavior of only one SAS, it is achieved by setting $m = m' = n$ and evaluating $(\partial Z_n/\partial n)_{n=0}$ at fixed K and H (note that $Z_0 = 1$):

$$(\partial Z_n/\partial n)_{n=0} = (\partial \ln Z_n/\partial n)_{n=0} = N_B H^2/2 + N_P K^2/2 + \sum_{|S|>2,|\gamma|} K^{|S|}H^{|\gamma|} \ . \tag{7}$$

Here N_P denotes the number of plaquettes in the lattice. Each term in the sum in (7) denotes the statistical weight of a *single* surface, whether closed or open. This limit, in which one probes the behavior of a single surface, is also

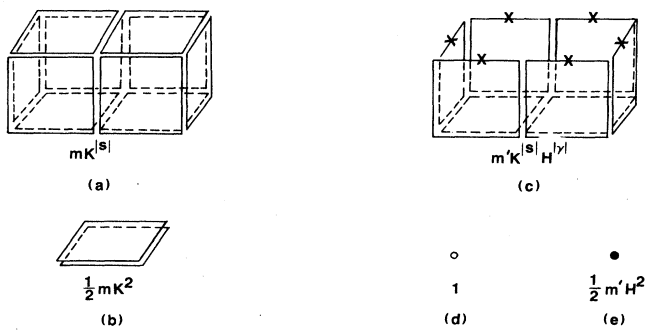

FIG. 1. Various diagrams surviving the $n \to 0$ limit.

known as the dilute limit. As $H \to 0$, one obtains only closed SAS's in (7).

In many cases, one is interested in considering a Wilson loop. In this case, one must consider an open surface whose boundary is defined by the *single* contour Γ of the Wilson loop. Let

$$W_\Gamma = \prod_i S_i^{(1)}, \quad S_i \epsilon \Gamma , \tag{8}$$

denote the product of $S_i^{(1)}$ over the contour Γ of the Wilson loop. The quantity of interest is the average value of the operator W_Γ. To obtain this, we set $m = n$ and $H = 0$ in (1). By setting $m = n \to 0$, all closed surfaces disappear. Therefore, $Z_o = \hat{Z} = 1$. Now consider the average value

$$\langle W_\Gamma \rangle_n = \prod_B \mathrm{Tr}_B (e^{\mathcal{H}_n} W_\Gamma) / \prod_B \mathrm{Tr}_B e^{\mathcal{H}_n} .$$

As $n \to 0$, we find that only diagrams that survive are those whose boundary is the contour Γ:

$$\langle W_\Gamma \rangle_o = \sum_{\substack{S \\ \partial S = \Gamma}} K^{|S|} . \tag{9}$$

Each term in the sum (9) denotes the weight of a *single* surface S with boundary $\partial S = \Gamma$. It is, in principle, possible to evaluate $\langle W_\Gamma \rangle_o$.

It should be remarked that the weights in (4) are determined by $|S|$ and $|\gamma|$ both. In this respect, our model differs from previous models.[3-16] At $H = 0$ and $m = m' = 1$, our weights are identical to those used by Durhuus *et al.*,[3] except that we have a grand-canonical ensemble. Durhuus *et al.* consider a *single* surface in their discussion. Majority of the work, except Ref. 3, deals only with *planar* surfaces. As we have indicated, we put no restriction of planarity on our surfaces. It is conceivable that due to nonplanarity, our surfaces belong to a new university class of surfaces. As we will see, however, our results about D_c^H and D_f are in agreement with that of Durhuus *et al.*[3] who also consider nonplanar surfaces in general.

It is easy to generalize our technique to describe self-avoidance among D-dimensional manifolds ($D = 0$ for SAW's, $D = 1$ for SAS's, etc.). We will not do this here.

Let ϕ_P and ϕ_γ denote the density of plaquettes and of the boundary in the thermodynamic limit $N_B \to \infty$. This limit will be assumed in the following. It should be remarked that our mapping between the spin system and self-avoiding surfaces is established for finite N_B. In other words, $n = 0$ limit is taken before $N_B \to \infty$. However, we will assume in the following that the two limits ($n \to 0$ and $N_B \to \infty$) can be interchanged. Such an interchange is believed to be valid in the disordered phase of the spin system but may be invalid in its ordered phase. We will come back to this point in the last section. We observe from (4) that $\phi_P = \kappa \partial \hat{\omega} / \partial \kappa$, $\phi_\gamma = \eta \partial \hat{\omega} / \partial \eta$ where $\hat{\omega} = \ln \hat{Z} / N_B$. As $\eta \to 0$, i.e., $H \to 0$ we find that $\phi_\gamma = MH \to 0$, where M is the magnetization of the system. Let us consider $H \to 0$. At small K, we expect to have very few closed SAS's each with a finite area so that ϕ_P is zero in the thermodynamic limit. As K increases, we expect to observe a phase transition at some $K = K_0$. For $K > K_0$, we will have $\phi_P > 0$, whereas $\phi_P = 0$ below K_0 with a singularity at $K = K_0$. This behavior is similar to the behavior observed in SAW model.[23,24] As we will see below, the transition is, in fact, a first-order one in the mean-field approximation and is expected to become a continuous one in lower dimensions.

As $K \to \infty$ ($H \to 0$), we expect the SAS to cover maximum number of plaquettes. This corresponds to covering *every* bond of the lattice. We will call such a SAS a Hamilton surface in analogy with Hamilton walks that cover each site.[25]

III. MEAN-FIELD SOLUTION

We will now solve (1) in a mean-field approximation.[26] For the sake of convenience, we will consider $m = m' = 1$. In the presence of a field, the magnetization $\mathbf{M} = \langle \mathbf{S} \rangle$ points along $\alpha = 1$ direction. Let us write $\mathbf{S} = \mathbf{M} + \boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ is the fluctuation with zero mean: $\langle \boldsymbol{\sigma} \rangle = 0$. Setting

$$S^{(\alpha)} = M^{(\alpha)} + \sigma^{(\alpha)}$$

in (1) and expanding, we find that

$$\mathcal{H}_n = KN_P M^4 + Kn_P M^3 \sum_{[B]} \sigma^{(1)} + \cdots + H \sum_{[B]} S^{(1)} , \tag{10}$$

where N_P is the total number of plaquettes and $n_P = 2(d-1)$ is the number of distinct plaquettes at each bond. The missing terms in (10) contain higher powers of σ's and describe the effects of correlations among the fluctuating field σ. However, in the mean-field approximation,[26] these fluctuation correlations are neglected. Now, reexpressing $\sigma^{(\alpha)} = S^{(\alpha)} - M^{(\alpha)}$, we find that the mean-field approximation for $\mathcal{H}_n$ is given by

$$\mathcal{H}_{no} = -(\tfrac{3}{4}) KN_B M^4 + (KM^3 + H) \sum_{[B]} S^{(1)} , \tag{11}$$

where Kn_P has been replaced by K and where we have used the fact that $N_B n_P = 4N_P$. The corresponding free-energy $\omega_n = \ln Z_n / N_B$ per bond is given by

$$\omega_n = -(\tfrac{3}{4}) KM^4 + \ln z_n , \tag{12}$$

where $z_n(x) = \mathrm{Tr} \exp[x s^{(1)}]$ is the single-spin partition function in a field $x = KM^3 + H$. This single-spin partition function is given by

$$z_n(x) = \Gamma(n/2)(x\sqrt{n}\,/2)^{1-n/2} I_{n/2-1}(\sqrt{n}\,x) ,$$

where I_ν is the modified Bessel function of order ν (see Ref. 26 for details).

It is evident that our mean-field calculation does not destroy the n dependence in the free energy. In other words, our free energy ω_n depends not only on the average $M = \langle S^{(1)} \rangle$, but also on the transverse components through z_n.

As $n \to 0$, $z_0(x) = 1 + x^2/2$, and we have

$$\omega_0 = -(\tfrac{3}{4}) KM^4 + \ln[1 + \tfrac{1}{2}(KM^3 + H)^2] . \tag{13}$$

The equilibrium value of M is obtained by maximizing ω_0 with respect to M. Note that our definition of ω differs by a minus sign from the useful definition of the free energy. Therefore, we must maximize ω_0 and not minimize it. This yields the following equation of state:

$$H = f(x) = x - K[x/(1+x^2/2)]^3 , \tag{14}$$

or

$$HM = g(M) = 1 - KM^4 - (1-2M^2)^{1/2} . \tag{15}$$

We have shown $f(x)$ in Fig. 2 for various values of K. For very small H and small K, we have a solution of M which is proportional to H along AB. However, for K between K_0 and K_c, to be determined herein, there is a first-order transition at some $H = H^*(K)$ from $x = x_B$ to $x = x_C$, so that the free energies are equal: $\omega_o(x_B) = \omega_o(x_C)$. At $H = 0$, the first-order transition occurs from $M = 0$ to $M = M_0 \simeq 0.697$ at $K = K_0 \simeq 4.927$ ($x = x_0 \simeq 1.668$), and is similar to the transition obtained for $n = 1$ by Balian *et al.*[18] This first-order transition line AC defined by $H^*(K)$ is shown in Fig. 3 and terminates at a *critical* point C with $H = H_c(K_c)$, $K = K_c$ at which we have an inflection point in $f(x)$ with zero slope ($\partial H/\partial M = 1/\chi = 0$): $f'(x_c) = f''(x_c) = 0$. This corresponds to $g'(M_c) = H_c$, $g''(M_c) = 0$. The point C is a critical point since the susceptibility $\chi = \partial M/\partial H$ *diverges* at this point. Numerically, we find that $x_c \simeq 0.662$, $K_c \simeq 2.153$, $H_c \simeq 0.317$, and $M_c \simeq 0.543$. The magnetization M is continuous in K at $H = H_c$. The same is true of the plaquette energy $\epsilon_p = \partial\omega_0/\partial K$.

As $K \to \infty$, we have from Ref. 14, $x^4 \sim 8K$, i.e., $M^4 \sim 2/K \to 0$. This should not be surprising for $n = 0$, as it is shown that M is indeed zero at $K = \infty$.[26] It is easily seen that the spontaneous magnetization decreases continuously from $M = M_0$ at $K = K_0$ to $M = 0$ at $K = \infty$. This is, again, a common feature for $n = 0$.[26]

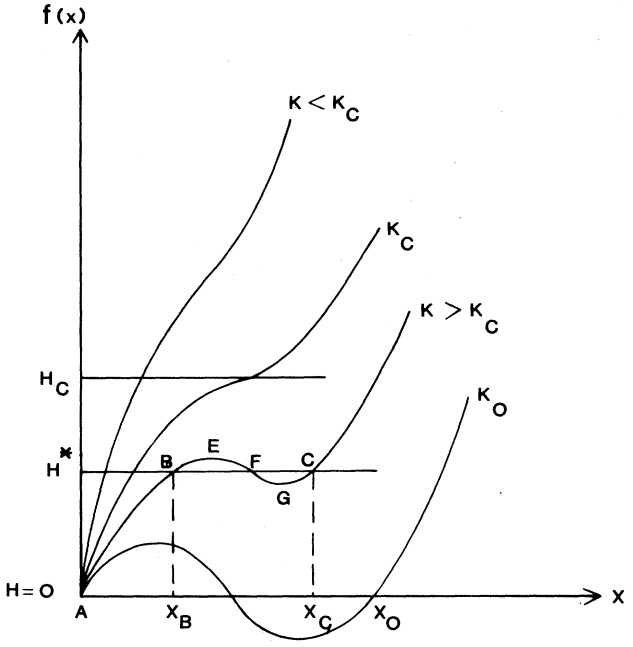

FIG. 2. The graphical solution of (14) for various K and H.

IV. CRITICAL BEHAVIOR

The divergence of χ as we approach the critical point implies that the correlation length ξ also diverges as we approach the critical point. Therefore, there is hope that the study of our model near the critical point will enable us to obtain a *continuum theory* of SAS's whose scaling properties, i.e., long-distance properties, are identical.

With this hope, let us now expand our theory near the critical point. We find the equation of state (15) to be more suitable for this purpose. We also find it convenient to use new variables τ, h, and ϕ in the following. They are defined as follows:

$$K = K_c(1+\tau), H = H_c(1+h), \text{ and } M = M_c(1+\phi) . \tag{16}$$

We expand $g(K,M)$ around the critical point:

$$g(K,M) = 1-(1-2M^2)^{1/2} - K_c M^4 - K_c M^4 \tau = g(K_c,M) - K_c M^4 \tau \simeq g(K_c,M_c) + g'(K_c,M_c)(M_c\phi) + g''(K_c,M_c)(M_c\phi)^2/2 + g'''(K_c,M_c)(M_c\phi)^3/6 - K_c M^4 \tau .$$

Using $g(K_c,M_c) = H_c M_c$ and $g'(M_c) = H_c$, $g''(M_c) = 0$, we find that

$$HM = H_c M_c(1+h)(1+\phi) \simeq H_c M_c(1+\phi) + (g'''/6)(M_c\phi)^3 - K_c M_c^4(1+\phi)^4\tau . \tag{17}$$

Simple algebra now shows that the preceding equation reduces to

$$h = A\phi^3 - B\tau(1+3\phi) , \tag{18}$$

where

$$A = 2M_c^2(5M_c^2-1)/[3H_c(1-2M_c^2)^{5/2}] > 0 ,$$
$$B = K_c M_c^3/H_c > 0 . \tag{19}$$

Defining a new *ordering* field

$$h' = h + B\tau , \tag{20}$$

we can rewrite the preceding equation of state as

$$h' = A\phi^3 - 3B\tau\phi . \tag{21}$$

The ordering field h' has the following significance: $h' = 0$ is tangential to the transition line AC near C (Fig. 3). Therefore, h' is asymptotically orthogonal to the transition line AC near C. The ordering field h' remains zero as we move along AC. In this sense, h' acts like the symmetry breaking in ordinary magnetic systems.

It is now clear that the equation of state[21] is identical to the equation of state in the usual ϕ^4-field theory. This

field theory is defined by the action

$$S=\tfrac{1}{2}(\partial\phi)^2+\frac{m_0^2}{2};\phi^2+\frac{\lambda_0}{4}\phi^4-h_o\phi \ . \tag{22}$$

For uniform ϕ, the first term is zero. The minimization principle yields the following equation of state determining ϕ:

$$h_o=m_o^2\phi+\lambda_o\phi^3 \ . \tag{23}$$

The phase transition occurs when $m_o=0$, which corresponds to $\tau=0$ in (21). For $h_o=0$, we find that $\phi=0$ if $m_o>0$. This corresponds to $\tau<0$ in (21). For $m_o<0$, $\phi\neq0$. This corresponds to $\tau>0$ and $\phi\neq0$ in (21). It is therefore evident that we obtain usual Landau exponents for our model: $\gamma=1$, $\beta=\frac{1}{2}$, $\alpha=0$, and $\delta=3$ where they are defined as usual.

It is easily seen that the same (mean-field) exponents are obtained even for $n=1$.[27,28] As a matter of fact, we obtain a phase diagram similar to Fig. 3 and the same mean-field exponents as above for all $n\geq0$.[28] It is known[18] that there is a continuous transition at zero field in $d=3$ for $n=1$. Moreover, computer simulation suggest a first-order transition at zero H (Refs. 28 and 29) in $d=4$. Therefore, the transition must move from $H=0, d=3$ to $H=H_c, d=4$. It would be interesting to see how H_c moves away from $H=0$ as d is raised above 3.

By studying the correlation functions, Brezin and Drouffe[27] have demonstrated that our model (1) for $n=1$ is *equivalent* to a ϕ^4 field theory in the long-distance limit. Therefore, we expect the upper critical dimension D_c^H to be 4, and the hyperscaling to be valid for $d\leq D_c^H$. This is certainly true for $n=1$ since our model (1) is equivalent to the usual Ising model in $d=3$ (Ref. 18) for which hyperscaling is generally believed to be valid. As shown herein, hyperscaling is obeyed in $d=1$ for $n=0$. Since the mean-field critical behavior does not usually distinguish between different n's,[28] we expect hyperscaling to be valid also for $n=0$. Therefore, the correlation length exponent ν is $\frac{1}{2}$ for $d\geq D_c^H=4$.

Our identification of $D_c^H=4$ differs from various suggested values of D_c^H: $D_c^H=8$ by Parisi[17] or $D_c^H=\infty$ by Billoire *et al.*,[4] suggesting that our SAS's belong to a different universality class. However, it is gratifying to see that our result is consistent with the conjecture of Durhuus *et al.*[3]

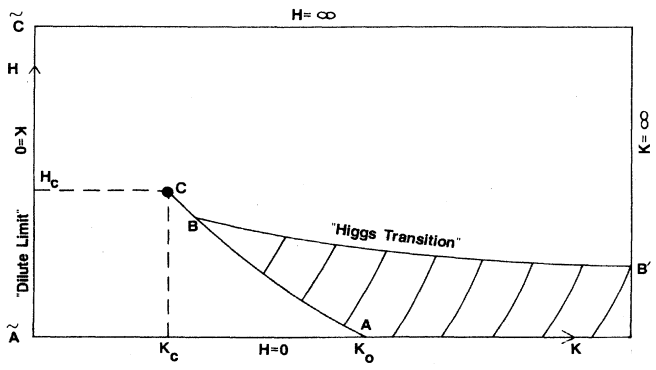

FIG. 3. The transition line AC $[H=H^*(K)]$. Curve BB' indicates a line of possible "Higgs" transitions.

It should be emphasized that our identification of D_c^H, D_f, etc., is done in a mean-field calculation. From (12), it is evident that the free energy ω_n depends on the value of n in general. In other words, it knows about the existence of the longitudinal components $S^{(1)}$ and the $(n-1)$ transverse component $S^{(\alpha)}$, $\alpha=2,\ldots,n$. Even when $m\neq1$, the free energy depends on n due to the term z_n in (12).

However, near the critical point, the expansion (17) depends only on the longitudinal field ϕ. The transverse fluctuations have been thrown away. Therefore, it is not surprising that our equation of state (21) is identical to the equation of state (23) for a one-component ϕ^4-field theory. Our mean-field values of various exponents will not change even if the transverse fields were included. However, to obtain these exponents in $d<D_c^H=4$, the transverse components must be included. In that case, the exponents, in general, will depend on n. Since our main interest is in identifying D_c^H, we have neglected the transverse components in our analysis.

At C, ϕ_P and ϕ_γ are both nonzero. However, the density $\phi_C=m(\partial\hat{\omega}/\partial m)$, $m=m'=1$, of different SAS's cannot be determined from knowing ϕ_P and ϕ_γ. By solving (1) for arbitrary $m=m'$, we find that $\phi_C\simeq0.039$ at C. We will not give the details as they are not very illuminating. The average number of plaquettes $\bar{P}$ of a SAS may be defined by ϕ_P/ϕ_C which is finite at C. However, an infinite SAS must exist at C as the correlation length ξ diverges here. Near the critical point, ξ diverges as $|\tau|^{-\nu}$, $\nu=\frac{1}{2}$. Therefore the radius of gyration R of this growing SAS must also diverge as $\xi\sim|\tau|^{-\nu}$ as $|\tau|\to0$.

The singular part of the free energy is obviously determined by this diverging surface. Since $\phi_P\sim\partial\omega_o/\partial\kappa$, we find that the singular part $\phi_P^{(S)}$ of ϕ_P must behave as

$$\phi_P^{(S)}\sim|\tau|^{d\nu-1} \ . \tag{24}$$

Since the volume occupied by this surface behaves as ξ^d, we find the number of plaquettes P in this surface behaves as

$$P\sim\phi_P^{(S)}\xi^d\sim1/|\tau| \ .$$

In other words,

$$P\sim R^{1/\nu} \ . \tag{25}$$

Therefore, the fractal dimension D_f of SAS is $D_{f=1/\nu}=2$, in our mean-field approximation as conjectured by Durhuus *et al.*[3]

V. DISCUSSION AND SUMMARY

It is known that at zero field, one cannot have a nonzero magnetization[30] for integer n. Our calculation at finite field, where we observe the critical behavior does not violate this result. However, as H decreases, our mean-field result must become invalid due to the intervention of some other transition[7,27–29,31] (for example, the "Higgs" transition along some curve BB', Fig. 3, or the roughening transition, etc.), before we get to $H=0$, $K>K_0$, where M is *not* a sensible order parameter. Our

calculation involving only single parameter M cannot describe the Higgs transition, unless we put some matter field or introduce some other more sensible order parameter, which we have not discovered yet. However, as suggested by Brezin and Drouffe,[27] we can use the preceding method for calculating any gauge-invariant observable. Because of the failure of our mean-field calculation, we predict that there must be an analogous "Higgs" transition line BB' even in our surface model. At present, we do not know the significance of this transition and is currently under investigation.

It should be noted that in deriving (13), we have taken the thermodynamic limit before taking the self-avoidance limit $n \to 0$, whereas the original mapping (4) requires taking $n \to 0$ before $N_B \to \infty$. Such an interchange of limits is, in general, not valid when there are phase transitions. Therefore, one may suspect our mapping in the shaded region bounded by BB' and AC in Fig. 3. One must investigate this issue of limit interchange in order to study the significance of the "Higgs" transition line BB' and the ordered phase below it. This is currently being explored.

Even though one cannot define a plaquette in $d < 2$, we can consider an effective one-dimensional lattice $(2 \times 2 \times N)$ in the limit $N \to \infty$. Consider $H = 0$. It is evident that for $K < K_c = 1$, we have only finite closed surfaces, whereas for $K > K_c$, we have an *infinite* closed surface, covering the whole lattice, a situation similar to the one for SAW's in $d = 1$.[24] It is evident that $\nu = 1$ here. This shows that D_c^L can be regarded as 1, and we suggest that D_c^H need not be twice D_c^L, in general. Furthermore, since the transition at $H = 0$ is continuous in $d = 1$ and first order in mean field, we expect that the transition at $H = 0$ becomes continuous in some low dimensions.

Let us now evaluate the number of (closed) Hamilton surfaces $(H \to 0, K \to \infty)$. Since $M^4 \to 2/K$, we obtain from (13)

$$\omega_0 = -\tfrac{3}{2} + \ln\sqrt{2K} \Longrightarrow \ln(2Kn_p/e^3)^{1/2} , \qquad (26)$$

where we have replaced K by Kn_p (see above) at the end. Let us assume that the number $W_{\rm HS}$ of the Hamilton surfaces grows as $\mu_{\rm HS}^{|S|} = \mu_{\rm HS}^{N_B/2}$, $N_B = 2|S|$, $|\gamma| = 0$. Setting $W_{\rm HS}$ in (5), we find that $\omega_0 \to \ln\sqrt{K}\mu_{\rm HS}$. Comparing this with (26), we find that

$$\mu_{\rm HS} = 2n_p/e^3 = 4(d-1)/e^3 \simeq (d-1)/5 . \qquad (27)$$

It is evident that the connectivity constant μ for a SAS of area $|S|$ on an infinite lattice (whose number $W_{\rm SAS}$ $|S| \sim |S|^{-\theta}\mu^{|S|}$) must be bounded from below by $\mu_{\rm HS}$, and provides a better lower bound than given before[3] for large d.

We should make one final remark. It is commonly believed that there is a close connection between surfaces and branched polymers.[1] Therefore, it seems logical that they both have the same critical behavior. In particular, it is believed that the upper critical dimensions for self-avoiding random surfaces and self-avoiding branches polymers (SABP's) should be the same. As we have argued here, $D_c^H = 4$ for our surfaces. Therefore, one should also expect $D_c^H = 4$ for SABP's. We have shown elsewhere[32] that the original arguments of Lubensky and Isaacson[32] suggesting $D_c^H = 8$ for SABP's is incorrect. We indeed find that $D_c^H = 4$ even for SABP's. As a matter of fact, we find an identical phase diagram for branched polymers.[32] Therefore, there indeed appears to be a deep connection between SARS's and SABP's.

This deep relationship between SABP's and SAS's is further strengthened by their behavior in the dilute limit, i.e., in the limit of only one SABP or one SAS; see (7). In the case of a branched polymer, the transition in this limit turns out to be first order in a mean-field approximation and there is strong evidence that it remains first order for all $d \geq 2$ (Ref. 32). By considering our model for $m = m' = n \to 0$, it can be shown that the transition is indeed first order in our mean-field approximation when considering a single SAS. As a matter of fact, the line of transition AC for any nonzero n and $m = m' = n$ moves out to $\tilde{A}\tilde{C}$ at $K = 0$ in the dilute limit $n \to 0$, as shown in Fig. 3. Similarly, the "Higgs" transition line BB' also moves out to $H \to \infty$ in this limit. The details are presented elsewhere.[34] Therefore, there does exist a very close relationship between SABP's and SAS's.

Let us summarize our results. We consider a very general model of nonplanar SAS's (Ref. 3) and show its equivalence with the $n = 0$ limit of a spin model. This is a first valid $n = 0$ connection between a spin model and SAS's. By solving it in a mean-field approximation, we have argued that our model is equivalent to the usual ϕ^4 theory. We suggest that $D_c^H = 4$ and that $D_f = 2$ for nonplanar surfaces for $d \geq D_c^H$. We also suggest that $D_c^L = 1$. We discuss the full phase diagram. We evaluate the connectivity constant $\mu_{\rm HS}$ for Hamiltonian surfaces in the mean-field approximation which yields a better lower bound for μ in higher dimensions.

[1] For a general review, see J. Froblich, in *Applications of Field Theory to Statistical Mechanics*, Vol. 216 of *Lecture Notes in Physics*, edited by L. Garrido (Springer, Berlin, 1985).

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