Quasispin-operator description of the Josephson tunnel junction and the Josephson plasma frequency

T. Unnerstall and A. Rieckers
Institut für Theoretische Physik der Universität Tübingen, Auf der Morgenstelle 14, D-7400 Tübingen, Federal Republic of Germany
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This work deals with the description of the Josephson junction by means of quasispin operators. Several approximations of previous such treatments known as the pseudo-angular-momentum theory are dispensed with, and special care is taken to describe the charge imbalance across the junction in a physically proper way. The ensuing microscopic theory can be evaluated rigorously up to a very small error, yielding a closed system of dynamical equations for the macroscopic behavior of the junction which exhibit additional terms as compared to the usual Josephson relations. These terms leave the Josephson frequency practically unchanged, but are shown to yield a correction to the frequency of the Josephson plasma mode on the order of 1%.

I. INTRODUCTION

Ever since Josephson published the main results of his famous dissertation on two weakly coupled superconductors,[1] there have been numerous investigations concerned with the theoretical explanation of the various Josephson effects. The scope of these works extends from essentially phenomenological treatments such as Anderson's number-phase theory[2] or Feynman's two-level description[3] to theories which offer a microscopic explanation, deducing the junction dynamics from the microscopic interactions within and between the two superconductors.

Within the latter approaches one can distinguish between those treatments which rely on perturbation theory and Green's-function methods to evaluate the microscopic model[4,5] and the so-called pseudo-angular-momentum approach (PAM),[6-10] which uses an operator formalism based on Anderson's quasispin description of superconductivity.[11] For single superconductors, this formalism can be shown to be asymptotically exact in the low-temperature limit. Relations between the two directions have been discussed by Rogovin *et al.*[8] PAM can only deal with the pair processes, but it is appealing because of its relative simplicity and the fact that it is not restricted to the perturbation regime.[12] In its usual formulation, however, it faces various drawbacks. One problem is that not all relevant commutators are treated rigorously (see Ref. 9 for a discussion); furthermore, one of the fundamental operators of the theory S_z has been incorrectly interpreted as to measure the whole charge imbalance across the junction (see Ref. 10 and Sec. III).

In the present work we develop a microscopic theory of the Josephson junction which is similar to the PAM in that it is formulated by means of the quasispin operators, but which complies with the above criticisms: It dispenses with approximations on the commutators and gives a refined account of the charge imbalance. In addition, it does not use a further approximation made by PAM which we feel has not been discussed thoroughly enough in the literature. The BCS Hamiltonians of the two superconductors are usually neglected, and at least in nonequilibrium situations this does not seem to be justified (see Sec. IV). Thus, working in the Heisenberg picture and taking the BCS ground state at $t=0$, we evaluate our microscopic model without any assumptions over and above the model Hamiltonian.

In this way the dynamical equations we arrive at to describe the macroscopic behavior of the junction are a direct and basically exact consequence of the microscopic theory.[13] Compared to the classical Josephson relations, they exhibit several additional terms. These terms are shown not to affect the Josephson frequency relation, but to give a corrected formula for the frequency of the Josephson plasma mode;[4] a quantative estimate shows the correction to be on the order of 1%. This frequency has acquired renewed importance in the theory of macroscopic quantum phenomena.[14]

In short, the plan of the paper is as follows. In Sec. II we establish our notation and develop the microscopic model of the junction. Section III is devoted is to a rather detailed discussion of the proper account of the charge imbalance between the superconductors, thereby completing the model. In Sec. IV we derive the dynamical equations describing the macroscopic behavior of the system, commenting on several approximations made in previous treatments. Finally, Sec. V deals with the physical content of these equations, and in particular discusses the correction to the Josephson plasma frequency our theory leads to.

To conclude this introduction, we remark that we have not included the coupling of the Josephson junction to electromagnetic modes in the junction cavity, which has led PAM to predict a slight pulling of the Josephson frequency itself. This point will be considered in a future publication.[15]

II. MICROSCOPIC MODEL

We shall only describe the most important features of the model; background information can be found in Ref.

10. As mentioned above, we use Anderson's quasispin operators to describe the superconducting properties of each of the two junction constituents (which we assume to be identical), i.e.,

$$\sigma_{\mathbf{k}}^{+}=C_{\mathbf{k}\uparrow}^{+}C_{-\mathbf{k}\downarrow}^{+}, \quad \sigma_{\mathbf{k}}^{-}=(\sigma_{\mathbf{k}}^{+})^{*} , \qquad (1)$$

$$\sigma_{\mathbf{k}z}=\tfrac{1}{2}(C_{\mathbf{k}\uparrow}^{+}C_{\mathbf{k}\uparrow}^{-}+C_{-\mathbf{k}\downarrow}^{+}C_{-\mathbf{k}\downarrow}^{-}-1) ,$$

where $C_{\mathbf{k}s}^{\pm}$ are the usual creation and annihilation operators for an electron with momentum $\mathbf{k}$ and spin s. The σ operators obey spin-commutation rules and are defined for all $\mathbf{k}$ whose associated energies $\epsilon(\mathbf{k})$ lie within a certain region around the Fermi energy μ:

$$\mathbf{k}\in\{\mathbf{k}:\epsilon(\mathbf{k})\in\Lambda=[\mu-\hbar\omega_D,\mu+\hbar\omega_D]\} ; \qquad (2)$$

we denote the number of these momenta—which is finite for a finite-volume superconductor—with $|\Lambda|$.

In this notation, the BCS Hamiltonian of a superconductor in the strong-coupling model[16] can be written as

$$H_{\mathrm{BCS}}=\sum_{\mathbf{k}}\epsilon(2\sigma_{\mathbf{k}z}+1_{\mathbf{k}})-\frac{g}{|\Lambda|}\sum_{\mathbf{k},\mathbf{k}'}\sigma_{\mathbf{k}}^{+}\sigma_{\mathbf{k}'}^{-} , \qquad (3)$$

where ϵ is the average of the kinetic energies $\epsilon(\mathbf{k})$ in Λ; the BCS ground state is

$$|\phi\rangle=\otimes_{\mathbf{k}\in\Lambda}[U(\mathbf{k})+V(\mathbf{k})e^{i\phi}\sigma_{\mathbf{k}}^{+}]|0\rangle_{\mathbf{k}} . \qquad (4)$$

Introducing as usual the operators

$$r_{\Lambda}^{\pm}=\frac{1}{|\Lambda|}\sum_{\mathbf{k}}\sigma_{\mathbf{k}}^{\pm} , \qquad (5a)$$

$$r_{\Lambda z}=\frac{1}{|\Lambda|}\sum_{\mathbf{k}}\sigma_{\mathbf{k}z} \qquad (5b)$$

for the right superconductor R, and a similar set $s_{\Lambda}^{\pm},s_{\Lambda z}$ (defined by means of spin operators $\sigma_{\mathbf{q}}^{\pm},\sigma_{\mathbf{q}z}$) for the left superconductor S, we have

$$H_{\mathrm{BCS}}^{R}=|\Lambda|[\epsilon_R(2r_{\Lambda z}+1)-gr_{\Lambda}^{+}r_{\Lambda}^{-}] , \qquad (6a)$$

$$H_{\mathrm{BCS}}^{S}=|\Lambda|[\epsilon_S(2s_{\Lambda z}+1)-gs_{\Lambda}^{+}s_{\Lambda}^{-}] . \qquad (6b)$$

The operators (5) can be interpreted as follows. r_{Λ}^{+} creates *condensed* (i.e., ground) pairs; it is the microscopic analog of the "macroscopic wave function" of the phenomenological theories, as can be seen from the relation

$$\langle\phi_R|r_{\Lambda}^{+}|\phi_R\rangle=\left[\frac{1}{|\Lambda|}\sum_{\mathbf{k}}U(\mathbf{k})V(\mathbf{k})\right]e^{-i\phi_R} \qquad (7a)$$

$$=\frac{\Delta}{g}e^{-i\phi_R} , \qquad (7b)$$

where Δ is the modulus of the order parameter.[17] Accordingly, we interpret $c_{\Lambda}^{R}=r_{\Lambda}^{+}r_{\Lambda}^{-}$ as the observable for the density of condensed pairs.

Remark. It is often not noted that in the present formalism one needs to distinguish between electrons, only formally described as pairs by the operators $\sigma_{\mathbf{k}}^{\pm}$, and *condensed* pairs, described by $r_{\Lambda}^{\pm}$. Only the latter contribute to the condensation energy (= binding energy), as is exhibited by the BCS Hamiltonians (6). Remembering that within the strong-coupling model

$$V(\mathbf{k})^2=U(\mathbf{k})^2\approx\tfrac{1}{2} \qquad (8)$$

for all $\mathbf{k}$,[10] the density of pairs in the BCS ground state is found to be

$$\left\langle\phi\left|\frac{1}{|\Lambda|}\sum_{\mathbf{k}}\sigma_{\mathbf{k}}^{+}\sigma_{\mathbf{k}}^{-}\right|\phi\right\rangle=\tfrac{1}{2} , \qquad (9a)$$

their total number $\frac{1}{2}|\Lambda|$, the density of bound pairs (Cooper pairs)

$$\langle\phi|r_{\Lambda}^{+}r_{\Lambda}^{-}|\phi\rangle=\tfrac{1}{4} , \qquad (9b)$$

their total number $\frac{1}{4}|\Lambda|$.

The operator $r_{\Lambda z}$ is a measure of the total number of electrons in the region Λ; it is related to the number operator N_R as

$$N_R=2|\Lambda|r_{\Lambda z}+|\Lambda|1 \qquad (10)$$

and is defined such that

$$\langle\phi|r_{\Lambda z}|\phi\rangle=0\Longrightarrow\langle\phi|N_R|\phi\rangle=|\Lambda| . \qquad (11)$$

In order to illustrate the physical meaning of these operators further, we consider the state $|\Psi_{(\pm)}\rangle=2r_{\Lambda}^{(\pm)}|\phi\rangle$, in which one Cooper pair is added to (removed from) the region Λ (the numerical factor 2 serves to normalize the state). Using (4) and (5) we get

$$\langle\Psi_{(\pm)}|r_{\Lambda z}|\Psi_{(\pm)}\rangle=(\pm)\frac{1}{|\Lambda|}\Longrightarrow\langle\Psi_{(\pm)}|N_R|\Psi_{(\pm)}\rangle=(\pm)2+|\Lambda| \qquad (12)$$

as expected. Consequently, then, the operator

$$z_{\Lambda}=\tfrac{1}{2}(r_{\Lambda z}-s_{\Lambda z}) \qquad (13)$$

measures the difference of the particle densities *within* Λ between R and S.

Thus having described the individual superconductors, we now turn to the interaction H_I between them. As is well known, it consists of a tunneling part H_T and an electrostatic interaction part H_C,

$$H_I=H_T+H_C . \qquad (14)$$

As regards H_T, we follow PAM in taking the Wallace-Stavn Hamiltonian[18]

$$H_T=\frac{\lambda}{|\Lambda|}\sum_{\mathbf{k},\mathbf{q}}\sigma_{\mathbf{k}}^{+}\sigma_{\mathbf{q}}^{-}+\sigma_{\mathbf{q}}^{+}\sigma_{\mathbf{k}}^{-} \qquad (15a)$$

$$=|\Lambda|\lambda(r_{\Lambda}^{+}s_{\Lambda}^{-}+s_{\Lambda}^{+}r_{\Lambda}^{-}) \qquad (15b)$$

$$=|\Lambda|w_{\Lambda} \qquad (15c)$$

which describes the tunneling of condensed pairs.

In this context, we introduce the operator

$$j_{\Lambda}=\frac{2e}{\hbar}(-i)\lambda(r_{\Lambda}^{+}s_{\Lambda}^{-}-s_{\Lambda}^{+}r_{\Lambda}^{-}) , \qquad (16)$$

whose well-known interpretation as the Josephson tunnel current can be seen directly from the definition, or equally well from the relation

$$j_\Lambda = 2e\frac{d}{dt}r_{\Lambda z} = 2e\frac{i}{\hbar}[H_T, r_{\Lambda z}] \ .$$

The electrostatic interaction H_C between the superconductors clearly depends on the charge imbalance; since the treatment of this concept gave rise to a severe criticism of PAM, we need to be particularly careful on this point and devote a separate section to its discussion.

III. THE CHARGE IMBALANCE

A. Theoretical descriptions and their physical meaning

It is obvious that the voltage is a key concept for the understanding and explanation of the Josephson effects. Many authors have included the voltage into their theory by simply subtracting a term $\mu_R N_R + \mu_S N_S$ from whatever Hamiltonian they work with (i.e., using a "reduced dynamics"), interpreting

$$eV = \mu_R - \mu_S \ . \tag{17a}$$

Equivalently, one can use different kinetic energies ϵ_R, ϵ_S in the BCS Hamiltonians (6) (Ref. 19) and set

$$eV = \epsilon_R - \epsilon_S \ . \tag{17b}$$

PAM, on the other hand, proposed[7] to make the physical origin of the voltage (charge imbalance on the junction capacity) more apparent by using

$$H_C = \frac{Q^2}{2C} = |\Lambda| 2K z_\Lambda^2 \tag{18}$$

with

$$K = e^2\frac{|\Lambda|}{C} \ , \tag{19}$$

and setting (we denote expectation values by dropping the Λ index; e.g., z denotes the expectation value of z_Λ)

$$V = \frac{Q}{C} = \frac{2e}{C}|\Lambda| z = \frac{2K}{e} z \ . \tag{20}$$

One might think now that these are just two different ways of describing the same physical quantity; this is, however, *not* the case. The two treatments are, in fact, both incomplete, in a complementary way. To see this, let us look at the electron distribution in a single superconductor, say R, (see Fig. 1). The total number of electrons is thus

$$N_{\rm tot} = \int_0^\infty f_R(E)dE \tag{21a}$$

$$= N(\mu_R) + \int_{\mu_R - \hbar\omega_D}^{\mu_R + \hbar\omega_D} f_R(E)dE \tag{21b}$$

$$= N(\mu_R) + N^R(\Lambda) \ , \tag{21c}$$

i.e., it depends on the value of μ_R *as well as* on the distribution within Λ (which is centered around μ_R). Since $V \sim \Delta N_{\rm tot}$, we now see the following.

(1) PAM only takes into account the part $N(\Lambda)$, since all operators are defined in restriction to Λ; the definition (20) does not see a difference $\mu_R \neq \mu_S$. In equilibrium,

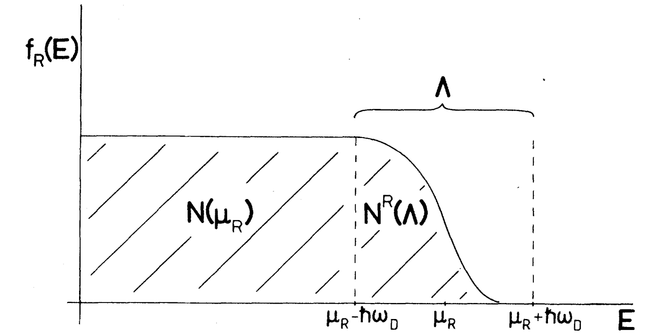

FIG. 1. Electron distribution in a single superconductor (R). (Energy E in arbitrary units.)

i.e., in the BCS ground state, $|\mathrm{BCS}\rangle = |\phi_R\rangle \otimes |\phi_S\rangle$, one has $r_z = s_z = 0$ [see (11)], so that $z = V = 0$, even when $\mu_R \neq \mu_S$. This fact has been pointed out by DiRienzo and Young in Ref. 10. The PAM definition (20) does, however, see differences in particle number (therefore in charge) stemming from nonequilibrium distributions in Λ_R, Λ_S. To illustrate this, we consider the state

$$|\Psi\rangle = 4 r_\Lambda^+ s_\Lambda^- |\mathrm{BCS}\rangle \ ,$$

in which, compared to equilibrium, one Cooper pair has been transferred from the left to the right superconductor. Using (12), one shows that

$$\langle \Psi | z_\Lambda | \Psi \rangle = \frac{1}{|\Lambda|} \neq 0 \ ,$$

which, from Eq. (20), implies $V = (2e/C) \neq 0$.

(2) The definition

$$V = \frac{1}{e}(\mu_R - \mu_S) = \frac{1}{e}(\epsilon_R - \epsilon_S) \ , \tag{22}$$

on the contrary, only takes into account the part $N(\mu)$. This is sufficient for the equilibrium situation [where $N^R(\Lambda) = N^S(\Lambda)$, since $r_z = s_z = 0$], but does not cover differences in the distributions within Λ between R and S.

B. Evaluating discussion of the two approaches

In the following discussion, we shall concentrate on the ac Josephson effect; a consistent account of the dc effect introduces different problems (necessity of a current-source reservoir[20,21]) which are of no relevance to our present purposes.

In Ref. 10 the fact that for equilibrium distributions $z = 0$ is said to be a fundamental inconsistency of PAM. The argument presented there is, however, not fully conclusive. The reason is that the dynamics of states and operators is not treated correctly. Taking as the junction Hamiltonian $H = H_T + H_C$ with

$$V = \frac{2K}{e} z \neq 0 \tag{23}$$

and $|\mathrm{BCS}\rangle$ as the initial state, one obtains in an interaction picture

$$|\mathrm{BCS}\rangle_t = e^{(i/\hbar)H_C t}|\mathrm{BCS}\rangle = \left|\phi_R - \frac{eV}{\hbar}t\right\rangle \otimes \left|\phi_S + \frac{eV}{\hbar}t\right\rangle , \tag{24}$$

while the operators evolve according to

$$H_T(t) = e^{(i/\hbar)H_C t} H_T e^{-(i/\hbar)H_C t} .$$

In Ref. 10 it is now stated that

$$z(t) = {}_t\langle \mathrm{BCS}|z_\Lambda|\mathrm{BCS}\rangle_t \equiv 0 , \tag{25}$$

which would indeed be inconsistent with (23). But (25) is incorrect, since the time dependence of z_Λ is not taken into account.

In fact, in the Heisenberg picture we have

$$\frac{d}{dt}2ez_\Lambda(t) = \frac{i}{\hbar}[H_T + H_C, z_\Lambda](t) = j_\Lambda(t) , \tag{26}$$

showing that in the ac effect, where $j(t)$ oscillates, one cannot possibly have $z(t) \equiv 0$ within PAM formalism. Physically, it is clear that the ac current causes the charge imbalance and hence the voltage to oscillate around an average value, which is what one measures as the dc voltage across the junction ($\langle\ \rangle_T$ denotes average over time):

$$V(t) = V^{\mathrm{dc}} + V^{\mathrm{ac}}(t), \quad \langle V^{\mathrm{ac}}(t)\rangle_T = 0 . \tag{27}$$

This phenomenon has already been pointed out in Ref. 2, and we believe one should try to account for it in the theoretical description; the assumption in Ref. 10 of an ideal battery keeping the voltage constant seems artificial to us (e.g., in this case there could be no Shapiro steps). The important question in our context is now whether this ac voltage corresponds to (i) oscillating μ-values (oscillating Λ-zones) in the two superconductors with fixed equilibrium distributions within Λ_R, Λ_S, or to (ii) oscillating distributions within fixed Λ-zones (fixed μ-values).

The assumption (i) would mean that R and S are in strict equilibrium (entailing $r_z = s_z = z = 0 \forall t$) during the ac effect, relaxing to it at a rate faster than the Josephson time period of $\hbar/2\, eV \approx 10^{-11}$ sec. While we believe this picture to be quite implausible on physical grounds, it is certainly inconsistent with the whole setup of Sec. II, in particular with (26). $z(t)$ oscillates and is not identically zero. Thus, (26) corresponds to conception (ii).

An essentially equivalent but more general argument can be formulated in the Schrödinger picture. If one defined all operators over the *whole* range of momenta, as done in Ref. 10, (i) would correspond to a state vector of the junction

$$|\psi(t)\rangle = |\phi_R\rangle_{n(t)} \otimes |\phi_S\rangle_{-n(t)} , \quad n(t) = n_0 + n_1\cos(\omega t) , \tag{28}$$

where $|\phi\rangle_n$ is the charge-imbalance state of Ref. 10. But (28) would leave the density of condensed pairs ($\sim\Delta$) invariant, which is not the case in the true dynamics [see Eq. (2.20) and Sec. V of Ref. 22]. Thus, (28) cannot be the state of the junction over time and hence (i) not the correct physical picture.

We summarize what we have established so far. The oscillating voltages in the ac effect are due to nonequilibrium distributions within fixed Λ-zones in R and S. Other than (22) and the theory developed in Refs. 10 and 22, PAM fully captures this feature with its operator z_Λ, since this observable measures just the deviation from the equilibrium distributions in the superconductors. Nevertheless, we think that Ref. 10 points to a decisive problem for PAM; this becomes apparent if we now turn to the different question of how to account for the *dc part* in (27).

If one follows PAM in setting

$$V(t) = \frac{2K}{e}z(t) \Longrightarrow V^{\mathrm{dc}} = \frac{2K}{e}\langle z(t)\rangle_T \neq 0 , \tag{29}$$

one is obliged to claim that the two superconductors are *permanently* (far) away from equilibrium. More precisely the distributions oscillate around *nonequilibrium* distributions.

This physical picture is incompatible with the familiar relation (22), which implicitly assumes that the superconductors are (aside from the charge oscillations discussed above, i.e., "in the mean") in equilibrium with different equilibrium parameters μ. In fact,the very use of μ_R, μ_S (as well as a reduced dynamics) presupposes R and S to be always at least *close* to equilibrium. This assumption is indeed the basis of physical discussions of the Josephson effects,[4] which we believe to be valid.

Hence, while we cannot say that the PAM relation (29) is inconsistent—*a priori*, nothing prevents us from considering (nonequilibrium) states in which $\langle z(t)\rangle_T \neq 0$ holds—it is not correct on physical grounds.[23] In this sense, we agree with Ref. 10 in stating that the operator z_Λ cannot describe the *permanent* charge imbalances considered there, i.e., dc voltages.

In conclusion of this discussion, we think it gives a correct account of the real physical situation and makes use of the virtues of both approaches, (20) and (22), if we cover both parts of the charge imbalance, describing dc voltages with (22) and ac voltages with (20):

$$N(\mu_R) - N(\mu_S) \rightarrow V^{\mathrm{dc}} = \frac{1}{e}(\epsilon_R - \epsilon_S) = \frac{1}{e}\Delta\epsilon ,$$

$$N^R(\Lambda) - N^S(\Lambda) \rightarrow V^{\mathrm{ac}} = \frac{2K}{e}z .$$

Thus, (27) is written as

$$V = V^{\mathrm{dc}} + V^{\mathrm{ac}} = \frac{1}{e}(\Delta\epsilon + 2Kz) . \tag{30}$$

In this way, our final model Hamiltonian becomes

$$\begin{aligned} H_\Lambda &= H^R_{\mathrm{BCS}} + H^S_{\mathrm{BCS}} + H_T + H_C \\ &= |\Lambda|[(\epsilon_R + \epsilon_S)1 + 2(\epsilon_R r_{\Lambda z} + \epsilon_S s_{\Lambda z}) \\ &\quad - g(r^+_\Lambda r^-_\Lambda + s^+_\Lambda s^-_\Lambda) \\ &\quad + \lambda(r^+_\Lambda s^-_\Lambda + s^+_\Lambda r^-_\Lambda) + 2Kz^2_\Lambda] . \end{aligned} \tag{31}$$

We have chosen all constants in such a way that H_Λ is strictly extensive $[O(|\Lambda|)]$, as seems to be the most reasonable behavior for large particle numbers.

IV. THE MACROSCOPIC DYNAMICAL EQUATIONS

To evaluate our model in terms of the macroscopic behavior of the junction means to determine the dynamics of global observables (i.e., operators which are summed over all $\mathbf{k}$). Throughout this section for mathematical convenience we choose density variables (with respect to $|\Lambda|$) for our description of the system. The operators

$$r_\Lambda^\pm, r_{\Lambda z}, s_\Lambda^\pm, s_{\Lambda z} \tag{32}$$

form a closed set of such variables under commutation with H_Λ. This can be seen explicitly from the Heisenberg equations of motion $\{\hbar(d/dt)r_{\Lambda z}=i[H_\Lambda, r_{\Lambda z}]$, etc.$\}$

$$\hbar\frac{d}{dt}r_{\Lambda z}=(-i)\lambda(r_\Lambda^+ s_\Lambda^- - s_\Lambda^+ r_\Lambda^-)=\frac{\hbar}{2e}j_\Lambda , \tag{33a}$$

$$\hbar\frac{d}{dt}s_{\Lambda z}=-\frac{\hbar}{2e}j_\Lambda , \tag{33b}$$

$$\hbar\frac{d}{dt}r_\Lambda^+ =2i\epsilon_R r_\Lambda^+ +2ig r_\Lambda^+ r_{\Lambda z}-2i\lambda s_\Lambda^+ r_{\Lambda z} +iK(r_{\Lambda z}-s_{\Lambda z})r_\Lambda^+ , \tag{33c}$$

$$\hbar\frac{d}{dt}s_\Lambda^+ =2i\epsilon_S s_\Lambda^+ +2ig s_\Lambda^+ s_{\Lambda z}-2i\lambda r_\Lambda^+ s_{\Lambda z} -iK(r_{\Lambda z}-s_{\Lambda z})s_\Lambda^+ , \tag{33d}$$

$$\hbar\frac{d}{dt}r_\Lambda^- =\hbar\left[\frac{d}{dt}r_\Lambda^+\right]^*, \quad \hbar\frac{d}{dt}s_\Lambda^- =\hbar\left[\frac{d}{dt}s_\Lambda^+\right]^* . \tag{33e}$$

These equations are valid up to $[O(1/|\Lambda|)]$, since all commutators between the operators (32) are $[O(1/|\Lambda|)]$. If we now take expectation values with the BCS ground state $|\mathrm{BCS}\rangle$ at $t=0$, it is important to note two mathematical facts.

(1) In permutation-invariant product states like $|\mathrm{BCS}\rangle$ with (8) the expectation values of products of operators (32) factorize up to $[O(1/|\Lambda|)]$, as easy computation shows.

(2) In the limit of large numbers $|\Lambda|$, such states remain product states under Schrödinger dynamics which are generated by mean-field Hamiltonians like (31).[24]

In light of this and the fact that $|\Lambda|$ is typically on the order of 10^{10}, the error due to the passage from (33) to the corresponding equations for expectation values (forming a classical, nonlinear set of differential equations) is extremely small. In the thermodynamic limit, $|\Lambda|\to\infty$, these arguments become entirely rigorous.[25]

The resulting system of equations then is (33) without Λ indices. In this system, r_z+s_z is a constant of the motion, reflecting the conservation of particles. In addition, a little algebra shows that

$$\hbar\frac{d}{dt}[c^R(t)-c^S(t)]=-\frac{\hbar}{e}j(t)(r_z+s_z)(t) . \tag{34}$$

For our initial-state $|\mathrm{BCS}\rangle$ it is $(r_z+s_z)=0$ and hence the difference between the Cooper pair densities of R and S is zero for all times. Remembering from the definitions that

$$r^+(t)=r(t)e^{i\phi_R(t)} , \tag{35a}$$

$$s^+(t)=s(t)e^{i\phi_S(t)} , \tag{35b}$$

we then have as a complete set of physical variables

$$c(t)=c^R(t)=r^2(t)=c^S(t)=s^2(t) \quad \text{Cooper pair density} ,$$

$$z(t)=\tfrac{1}{2}[r_z(t)-s_z(t)] \quad \text{difference of particle densities within } \Lambda , \tag{36}$$

$$\Delta\phi(t)=\phi_R(t)-\phi_S(t) \quad \text{phase difference between } R \text{ and } S .$$

With these, Josephson current (density) and tunneling energy (density) take their familiar form

$$j(t)=(-i)\frac{2e}{\hbar}[r^+(t)s^-(t)-s^+(t)r^-(t)] =\frac{4e}{\hbar}\lambda c(t)\sin\Delta\phi(t) , \tag{37a}$$

$$w(t)=\lambda[r^+(t)s^-(t)+s^+(t)r^-(t)] =2\lambda c(t)\cos\Delta\phi(t) . \tag{37b}$$

Rewriting the system (33) (without Λ indices) in terms of the variables (36) leads to our final dynamical equations:

$$\hbar\frac{d}{dt}c(t)=-\frac{1}{e}j(t)z(t) , \tag{38a}$$

$$\hbar\frac{d}{dt}z(t)=\frac{\hbar}{2e}j(t)=2\lambda c(t)\sin\Delta\phi(t) , \tag{38b}$$

$$\hbar\frac{d}{dt}\Delta\phi(t)=2\Delta\epsilon+4Kz(t)+4gz(t) -4\lambda\cos\Delta\phi(t)z(t) . \tag{38c}$$

Before we look at this description of the behavior of the Josephson junction more closely, we would like to add a few comments, comparing our treatment with PAM.

(1) As mentioned in the Introduction, the authors using PAM have usually neglected the BCS Hamiltonians of R and S. This is justified in equilibrium, *if one uses reduced dynamics*. Then, i.e., in the state $|\mathrm{BCS}\rangle$, the first two terms in (33c) and (33d) coming from the BCS Hamiltonians give the same result $(2i\mu_R r_\Lambda^+)$ as the reduced dynamics (cf. Ref. 26). But since the junction generally does not operate in equilibrium (see Sec. III), we do not see a reason *not* to include the free Hamiltonians of R and S. They lead to the terms $2\Delta\epsilon+4gj(t)z(t)$ in (38c), missing in the usual PAM equations,[7] the first one being of course connected with the problems discussed in Sec. III.

(2) Since we have neither approximated any of the commutators nor set $\dot{c}(t)\equiv 0$ as it is often done, Eqs. (38) can be considered to be [up to the very small error discussed below Eq. (33)] an exact consequence of the model Hamiltonian (31). This accounts also for further differences between (38) and PAM treatments.

V. THE JOSEPHSON PLASMA FREQUENCY

Using (30), the dynamical equations describing the junction behavior are

$$(c(t)+z^2(t))^{\cdot}=0 , \tag{39a}$$

$$2e\dot{z}(t)=j(t)=\frac{4e\lambda}{\hbar}c(t)\sin\Delta\phi(t) , \tag{39b}$$

$$\dot{\Delta\phi}(t)=\frac{2e}{\hbar}[V^{\rm dc}+V^{\rm ac}(t)]+\frac{4g}{\hbar}z(t) -\frac{4\lambda}{\hbar}\cos\Delta\phi(t)z(t) . \tag{39c}$$

In order to discuss the physical content of these equations, and in particular the corrections they predict as compared to the usual Josephson relations[27]

$$\dot{c}(t)=0 , \tag{40a}$$

$$2e\dot{z}(t)=j(t)=j_0\sin\Delta\phi(t) , \tag{40b}$$

$$\Delta\dot{\phi}(t)=\frac{2e}{\hbar}V^{\rm dc} , \tag{40c}$$

we have to take a look at typical magnitudes of the constants occurring. Taking junction parameters from Ref. 28,

$$|\Lambda|\approx 10^{10}=\text{number of states } \mathbf{k} , \quad C\approx 10^{-8}\ \text{F}=\text{capacity} , \tag{41a}$$

$$I_0\approx 5\times 10^{-3}\ \text{A}=\text{critical current} , \quad g\approx 2\Delta\approx 2\times 10^{-3}\ \text{eV},\quad c(0)=\tfrac{1}{4} , \tag{41b}$$

we deduce

$$K=\frac{e^2}{C}|\Lambda|\approx 0.1\ \text{eV}$$

$$I_0\approx\frac{4e}{\hbar}\lambda c(0)|\Lambda|\Longrightarrow\lambda\approx 0.2\times 10^{-8}\ \text{eV}\Longrightarrow H_T\approx 10\ \text{eV} .$$

Turning now first to our constant of motion

$$c(t)+z^2(t) \tag{42}$$

(reminiscent of, but slightly different from, the "cooperation number" in Ref. 8), we see that $c(t)$, measuring the Cooper pair density and hence the modulus of the order parameter, is not exactly constant over time. However, at a dc voltage of 10^{-3} V one has[29] $|z(t)|\leq 10^{-6}$ during the ac effect, so that the usual assumption $c(t)\equiv c_0$ is very well justified indeed. Nevertheless, it is interesting to note the physical meaning of (42): The farther one is out of equilibrium, the smaller the Cooper pair density becomes; the condensation is indeed an equilibrium effect.

If we consider the other differences between (39) and (40), it seems to be a satisfying feature of the new equations that they explicitly exhibit the fact that in the ac effect, the dc voltage is superimposed by an ac voltage

$$V^{\rm ac}(t)=2\frac{K}{e}z(t)$$

stemming from the ac current $j(t)$ itself.[2] Nevertheless, the Josephson frequency is unaltered, since the time average of $V^{\rm ac}(t)$ and hence $z(t)$ is zero at a frequency on the order of 10^{11} Hz; therefore, for the ac effect (39c) effectively reduces to (40c),[30]

$$\langle\dot{\Delta\phi}(t)\rangle_T=\frac{2e}{\hbar}V^{\rm dc} .$$

Thus leaving the Josephson frequency relation (almost) unchanged, the additional terms in (39c) do have a considerable effect on the Josephson *plasma* frequency. Here $V^{\rm dc}=0$, so that we have [with $c(t)\approx c_0$ and $g\gg\lambda$]

$$\ddot{\Delta\phi}(t)=\frac{4}{\hbar}(K+g)\dot{z}(t) =\frac{8\lambda}{\hbar^2}c_0(K+g)\sin\Delta\phi(t) , \tag{43}$$

giving for the plasma frequency the corrected formula

$$\omega_J=\left[\frac{2eI_0}{\hbar C}\left[1+\frac{g}{K}\right]\right]^{1/2} . \tag{44}$$

The correction

$$\frac{g}{K}\approx\frac{\Delta}{e^2}\frac{C}{|\Lambda|}$$

depends on the capacity and size of the junction. More precisely, since $C\sim A$ and $|\Lambda|\sim Ax$, where A is the area and x the thickness of the superconducting films, one has $g/K\sim 1/x$, so the correction increases with decreasing film thickness [which we have assumed to be $x\approx 1000$ Å in (41a)].[31] The physical origin of this correction may be that the attractive force prevents part of the charge from oscillating across the junction, leading to a reduced effective capacity.

For the parameters (41), we get

$$\omega_J\approx 6\times 10^{10}\ \text{Hz},\quad \frac{g}{K}\approx 10^{-2} .$$

The accuracy of the experimental measurements of ω_J by Dahm *et al.*[28] does not exceed 10%. It would be interesting to see whether they can be improved up to a point where experimental verification or falsification of our theoretical prediction is possible or to measure the plasma frequency in junctions with very thin films.

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[30]We neglect here the fact that the time average of the (very small) last term in (39c) is not zero; it leads to a correction of (40c) which is, however, completely negligible. This point is discussed further in Ref. 15.
[31]This feature is shared with the correction of the Josephson frequency predicted by PAM (see Ref. 6).