Electromagnetic response of disordered superconductors

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We use the exact eigenstate formalism to construct a gauge-invariant theory for correlation functions for disordered superconductors. The density response and the longitudinal and transverse current response at zero temperature are calculated explicitly for Bardeen-Cooper-Schrieffer (BCS) superconductors with arbitrary amounts of disorder. At small frequencies and wave numbers this can be done analytically, and we discuss the coherence length, the penetration depth, and the Anderson-Bogolubov mode at various degrees of disorder. We then calculate the longitudinal dielectric function numerically for larger frequencies and wave numbers. We find that the disorder does not change the absence of low-lying collective modes in charged superconductors. Implications for the quasiparticle inelastic lifetime are discussed.

I. INTRODUCTION

Superconductors differ drastically from normal metals in their transport properties. Specifically, it was the dc conductivity being infinite that first caught Kamerlingh Onnes' attention. It is therefore remarkable that even after Barden, Cooper, and Schrieffer (BCS)[1] developed their microscopic theory, it took some time until the transport theory for superconductors was fully worked out. The difficulty was connected with the problem of gauge invariance, which is not easy to handle in a system where interactions are of qualitative importance. The BCS paper included a description of the transverse electromagnetic response, albeit not a gauge-invariant one, but it could not deal with the longitudinal response. Many authors have contributed to a clarification of this point,[2] but the final solution is usually connected with the names of Anderson,[3] Bogolubov,[4] and, on a more technical level, Nambu.[5] The main result was that gauge invariance is restored by a longitudinal collective mode. This is the Goldstone boson which arises due to a spontaneously broken symmetry in the superconductor. In a hypothetical neutral superconductor this Anderson-Bogolubov (AB) mode is soft and lies in the gap. It can be interpreted either as a first soundlike mode propagating in the condensate, or as bound electron pairs with nonzero momentum. In the physical case of a charged superconductor, the AB mode is "pushed up" to high frequencies by the Higgs mechanism,[6] and becomes indistinguishable from the plasmon. In a real superconductor, there is therefore no collective mode at small frequencies and wave numbers. This holds at low temperatures, which we will be interested in. In the immediate vicinity of T_c, the situation is much more complex, and low-lying collective excitations do exist.[7]

The methods of Anderson and Bogolubov can be exploited analytically in the limit of small frequencies, $\omega \ll \Delta$, and wave numbers, $q\xi \ll 1$, where ξ is the coherence length and Δ is the gap. For many applications, however, it is necessary to know the dielectric function ϵ at larger q and ω. An obvious example is the quasiparticle lifetime due to Coulomb scattering. Prange[8] has used Nambu's Green function technique to reduce the calculation of $\epsilon(q,\omega)$ to the task of performing three Kramers-Kronig transforms, which can be easily done numerically. He has discussed the longitudinal dielectric function in detail, and has emphasized that the question of whether or not there is an s-wave collective state in the gap of a BCS superconductor is of quantitative rather than of qualitative nature. There is no physical principle which prevents the dielectric function from having a zero in the gap, it is just that the numbers work out such that it does not.

Due to the work mentioned previously, the electromagnetic response of clean BCS superconductors is completely known. The same is not true for disordered superconductors, which have become of great interest in recent years.[9] As our understanding of these systems develops, the lack of theoretical knowledge about their electromagnetic response becomes more and more of a stumbling block. A recent strong coupling theory for disordered superconductors[10] neglected inelastic processes for lack of information about the dielectric function. On the other hand, there is some evidence that disorder enhanced Coulomb scattering of quasiparticles is important at strong disorder.[11] A calculation of inelastic lifetimes in superconducting films[12] used the Coulomb propagator of the normal metal, thus neglecting the gap in the electromagnetic response. This is not satisfactory, since it strongly underestimates the lifetimes below T_c. Another field for potential applications of $\epsilon(q,\omega)$ is nonequilibrium superconductivity. Existing theory treats the various pair-breaking rates as phenomenological parameters.[13] In clean systems close to equilibrium, pair breaking due

to both electron-phonon scattering[14] and Coulomb scattering[15] has been calculated microscopically. The electron-phonon contribution is by far the dominant one. This is expected to change in disordered systems,[11] and a calculation of the Coulomb scattering rate for this case would be highly desirable. Again this is not possible without a reliable theory for the dynamically screened Coulomb potential.

It is the purpose of the present paper to improve on this point. We will provide a practical scheme for calculating the dielectric function of arbitrarily disordered superconductors. For explicit calculations we will restrict ourselves to the BCS model, where the interaction is replaced by a separable model potential, and only the gap parameter is kept instead of the full self-energy. It has recently been shown[10,12] that additional self-energy parts are important in disordered superconductors. These effects can be included in the present formalism, and will be addressed in the future.

This paper is organized as follows. In Sec. II we use Anderson's exact eigenstate method[16] and Nambu's Green function formalism to construct a conserving approximation[17] for arbitrary four-point correlation functions for arbitrarily disordered superconductors. This is achieved by essentially rewriting Prange's equations[8] in the exact eigenstate basis. We then solve the resulting integral equation for the density and the current response, which determines the dielectric function, the conductivity tensor, and the magnetic susceptibility. Then correlations are expressed in terms of the corresponding correlation functions for normal conducting electrons. In Sec. III, the response functions and the related dielectric function at small q and ω are calculated analytically for clean, diffusive, and critical (i.e., close to a mobility edge) normal conducting electron dynamics, respectively. In the clean case we recover the well-known results. The diffusive and critical cases allow for a discussion of electromagnetic absorption, the relevant length scales, and the AB mode as a function of disorder. In Sec. IV we calculate the dielectric function numerically outside the long wavelength limit. Finally, we discuss our results with respect to various applications.

II. A CONSERVING APPROXIMATION FOR CORRELATION FUNCTIONS

A. Electromagnetic response and correlation functions

The longitudinal dielectric function $\epsilon(q,\omega)$ of an electron liquid can be written as[18]

$$\epsilon(\mathbf{q},\omega)=1+v(\mathbf{q})\chi_{\rm sc}(\mathbf{q},\omega)\ . \tag{2.1}$$

Here $v(\mathbf{q})$ is the bare electron-electron interaction, and $\chi_{\rm sc}$ is the screened or irreducible polarizability. It is connected with the full polarizability χ by

$$\chi_{\rm sc}(\mathbf{q},\omega)=\chi(\mathbf{q},\omega)/[1-v(\mathbf{q})\chi(\mathbf{q},\omega)]\ , \tag{2.2}$$

but is most easily defined diagrammatically as all contributions to the polarizability which are irreducible with respect to the interaction. The use of $\chi_{\rm sc}$ rather than χ takes care of the most drastic manifestation of the long-range part of the interaction, viz., screening, and $\chi_{\rm sc}$ depends only weakly on the interaction. In fact for normal conducting electrons it is an acceptable approximation [namely random-phase approximation (RPA)] to replace $\chi_{\rm sc}$ by the Lindhard function for the noninteracting system. In a superconductor, the situation is more complicated. $\chi_{\rm sc}$ is dominated by the effective attractive interaction, which can not be neglected without abandoning superconductivity altogether. The problem is then to keep the interaction in $\chi_{\rm sc}$ in such a way that gauge invariance is maintained.

Gauge invariance implies particle number conservation, which enables one to obtain the longitudinal current response or the conductivity σ from the density response. The connection is[18]

$$\sigma(\mathbf{q},\omega)=-ie^2(\omega/q^2)\chi_{\rm sc}(\mathbf{q},\omega)\ . \tag{2.3a}$$

In particular, the real part of the homogeneous longitudinal conductivity, $\sigma'(\omega)$, is given by

$$\sigma'(\omega)=\lim_{q\to 0} e^2(\omega/q^2)\chi_{\rm sc}''(q,\omega)\ . \tag{2.3b}$$

Here

$$\chi_{\rm sc}''(q,\omega)={\rm Im}\chi_{\rm sc}(q,\omega+i0)$$

and

$$\sigma'(\omega)={\rm Re}\sigma(\omega+i0)\ ,$$

are the dissipative parts of $\chi_{\rm sc}$ and σ, respectively.[19]

The analogous correlation functions in the transverse channel determine the magnetic response of the system. The magnetic susceptibility χ_M in terms of the transverse current correlation χ_T reads[18]

$$\chi_M(\mathbf{q},\omega)=\frac{e^2}{c^2q^2-\omega^2}[\chi_T(\mathbf{q},\omega)-n/m]\ . \tag{2.4a}$$

The static magnetic susceptibility is

$$\chi_M(q)=\frac{e^2}{c^2q^2}[\chi_T(q,i0)-n/m]\ , \tag{2.4b}$$

where n and m are electron density and mass, respectively. Notice that for the longitudinal current correlation one has $\chi_L(q,i0)=n/m$. In a normal conductor, longitudinal and transverse response are indistinguishable in the long-wavelength limit. Therefore

$$\chi_T(q,i0)-n/m=O(q^2)\ ,$$

and Eq. (2.4b) yields the Landau diamagnetism. In a superconductor, $\chi_T\neq\chi_L$ even for $q\to 0$, and Eq. (2.4b) describes magnetic screening. The penetration depth λ is defined by

$$\chi_M(q\to 0)=(-1/4\pi)(\lambda q)^{-2}\ . \tag{2.5}$$

B. Gauge-invariant correlation functions

We now proceed to construct gauge invariant expressions for the various correlation functions introduced in the last section. We are interested in disordered systems, and we first consider correlation functions which have *not* been ensemble averaged. In this situation it is convenient to work in real space. We use Nambu's matrix notation,[5] and rewrite the equations given in Ref. 2 for an arbitrary 4-point correlation function as

$$\tilde{\chi}_{\mu\nu}(\mathbf{x}_1\mathbf{x}_2,i\Omega)=-T\sum_{i\omega}\int d\mathbf{x}_1' d\mathbf{x}_1'' d\mathbf{x}_2' d\mathbf{x}_2''\,\mathrm{tr}\gamma_\nu(\mathbf{x}_1''\mathbf{x}_1')G(\mathbf{x}_1'\mathbf{x}_2',i\omega-i\Omega)\tilde{\Gamma}_\mu(\mathbf{x}_2'\mathbf{x}_2\mathbf{x}_2'',i\Omega)G(\mathbf{x}_2''\mathbf{x}_1'',i\omega)\ , \tag{2.6}$$

$$\tilde{\Gamma}_\mu(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3,i\Omega)=\gamma_\mu(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)-T\sum_{i\omega}\int d\mathbf{x}_2' d\mathbf{x}_2''\tau_3 G(\mathbf{x}_1\mathbf{x}_2',i\omega-i\Omega)\tilde{\Gamma}_\mu(\mathbf{x}_2'\mathbf{x}_2\mathbf{x}_2'',i\Omega)G(\mathbf{x}_2''\mathbf{x}_3,i\omega)\tau_3 V(\mathbf{x}_3\mathbf{x}_1)\ . \tag{2.7}$$

These equations are shown graphically in Figs. 1 and 2. Here $\tilde{\chi}$ and $\tilde{\Gamma}$ denote unaveraged correlation functions and vertices, respectively. $\tilde{\Gamma}$, the bare vertex γ, the Green function G, and the Pauli matrix τ_3 are all 2×2 matrices, and tr denotes the trace. We are using the Matsubara formalism here. The corresponding zero-temperature correlations can be obtained in the end by the analytic continuation $i\Omega\to\omega+i0$. V denotes the effective attractive interaction. In what follows we will use for simplicity a pointlike interaction:

$$V(\mathbf{x}_1\mathbf{x}_2)=V\delta(\mathbf{x}_1-\mathbf{x}_2)\ .$$

This allows for an easy solution of the integral equation for Γ_μ. It introduces, however, ultraviolet convergence problems which will be dealt with in Sec. III. The bare vertex γ_ν determines the correlation under investigation. For instance, the screened density correlation is

$$\tilde{\chi}_{\mathrm{sc}}(\mathbf{x}_1\mathbf{x}_2,i\Omega)=\tilde{\chi}_{00}(\mathbf{x}_1\mathbf{x}_2,i\Omega)\ , \tag{2.8a}$$

with the density vertex

$$\gamma_0(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)=\tau_3\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_1-\mathbf{x}_3)\ . \tag{2.8b}$$

Likewise, the transverse current correlation is

$$\tilde{\chi}_T(\mathbf{x}_1\mathbf{x}_2,i\Omega)=\tilde{\chi}_{TT}(\mathbf{x}_1\mathbf{x}_2,i\Omega)\ , \tag{2.9a}$$

with

$$\gamma_T(\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3)=\mathbf{1}\sum_{\mathbf{k},\mathbf{q}}e^{i(\mathbf{k}-\mathbf{q}/2)\cdot(\mathbf{x}_1-\mathbf{x}_3)+i\mathbf{q}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\frac{1}{m}\mathbf{k}\cdot\mathbf{e}_T(\mathbf{q})\ , \tag{2.9b}$$

where $\mathbf{1}$ is the 2×2 unit matrix, and $\mathbf{e}_T(\mathbf{q})$ is a unit vector perpendicular to $\mathbf{q}$. These are all correlation functions we will need for our present purposes. Examples of others include the stress vertex and the corresponding stress correlation function which determines the sound attenuation coefficient.[20]

The structure of Eqs. (2.6) and (2.7) ensures gauge invariance of $\tilde{\chi}$ if the matrix Green function G is calculated in the generalized Hartree-Fock scheme which constitutes the standard theory of superconductivity. The basic principle is that self-energy contributions and vertex corrections have to be treated consistently. This was shown by Nambu in Ref. 5. In the present case, we still have to perform the ensemble average over the disorder. In order not so spoil gauge invariance, we again have to be careful that our procedure is consistent with the corresponding theory for the Green function. The existing theories for G[21–23,10] force us to make the following ap-

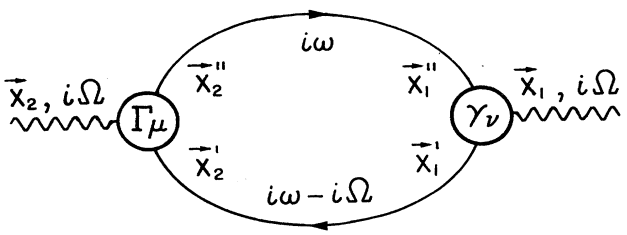

FIG. 1. Graphical representation of minus the correlation function χ, Eq. (2.6). Straight and wavy lines denote Green's functions and the interaction, respectively.

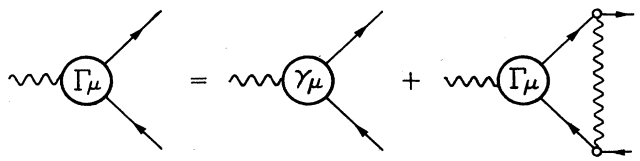

FIG. 2. Graphical representation of the t-matrix equation for the vertex Γ_μ. The small circle represents the matrix τ_3. Labeling is analogous to Fig. 1.

proximation. On the right-hand side of Eqs. (2.6) and (2.7) we factorize the ensemble average $\langle \cdots \rangle_{\rm av}$ by writing $\langle G\tilde{\Gamma}G\rangle_{\rm av} \approx \langle G\langle\tilde{\Gamma}\rangle_{\rm av}G\rangle_{\rm av}$. Diagrammatically this means that we do not allow for a crossing of interaction and impurity potential lines. This is consistent with the treatment of G in both BCS-Gorkov theory,[21] and in more recent theories for disordered superconductors.[22,10] We then have a well-defined class of theories for G (and therewith for the gap Δ and for the transition temperature T_c) which, if used with the resulting expression for χ, give gauge-invariant results. To deal with the remaining average of products of Green functions, we follow Anderson[16] and introduce the eigenfunctions $\psi_n(\mathbf{x})$ and eigenenergies E_n of a fictitious electronic system which includes the disorder but not the interaction. To remain consistent with existing theories for G,[21-23,10] we have to assume that G is diagonal in this "exact eigenstate basis." This approximation has been motivated in Ref. 23. Accordingly, we write

$$G_{nm}(i\omega)=\int d\mathbf{x}\,d\mathbf{y}\,\psi_n^*(\mathbf{x})G(\mathbf{xy},i\omega)\psi_m(\mathbf{y}) \approx \delta_{nm}G_n(i\omega)=\delta_{nm}\int d\epsilon\,\delta(\epsilon-E_n)G(\epsilon,i\omega)\ . \tag{2.10}$$

We note the ensemble averaged correlation function by $\chi=\langle\tilde{\chi}\rangle_{\rm av}$, and the averaged vertex by $\Gamma=\langle\tilde{\Gamma}\rangle_{\rm av}$. Using the translational invariance after the average, we Fourier transform and obtain

$$\chi_{\mu\nu}(\mathbf{q},i\Omega)=-T\sum_{i\omega}\int d\epsilon\,d\epsilon'\sum_{\mathbf{k},\mathbf{p}}F_{\mathbf{kp}}(\mathbf{q};\epsilon,\epsilon')\mathrm{tr}\gamma_\nu(\mathbf{k},\mathbf{q})G(\epsilon,i\omega-i\Omega)\Gamma_\mu(\mathbf{p},-\mathbf{q};i\Omega)G(\epsilon',i\omega)\ , \tag{2.11}$$

$$\Gamma_\mu(\mathbf{p},\mathbf{q};i\Omega)=\gamma_\mu(\mathbf{p},\mathbf{q})+VT\sum_{i\omega}\int d\epsilon\,d\epsilon'\sum_{\mathbf{k}',\mathbf{p}'}F_{\mathbf{k}'\mathbf{p}'}(\mathbf{q};\epsilon,\epsilon')\tau_3G(\epsilon,i\omega-i\Omega)\Gamma_\mu(\mathbf{p}',\mathbf{q};i\Omega)G(\epsilon',i\omega)\tau_3\ . \tag{2.12}$$

Here, F is a correlation function for noninteracting electrons. It is defined in terms of the exact eigenstate wave functions

$$F_{\mathbf{kp}}(\mathbf{q};\epsilon,\epsilon')=\int d\mathbf{x}_1d\mathbf{x}_2d\mathbf{x}_3d\mathbf{x}_4e^{-i(\mathbf{k}-\mathbf{q}/2)\cdot\mathbf{x}_1+i(\mathbf{k}+\mathbf{q}/2)\cdot\mathbf{x}_2} \times e^{i(\mathbf{p}-\mathbf{q}/2)\cdot\mathbf{x}_3-i(\mathbf{p}+\mathbf{q}/2)\cdot\mathbf{x}_4}\sum_{n,m}\langle\psi_n^*(\mathbf{x}_1)\psi_n(\mathbf{x}_3)\psi_m^*(\mathbf{x}_4)\psi_m(\mathbf{x}_2)\delta(\epsilon-E_n)\delta(\epsilon'-E_m)\rangle_{\rm av}\ . \tag{2.13}$$

It has been discussed in Refs. 20, 23, and 10 that to a very good approximation F_{kp} depends only on the difference $\epsilon-\epsilon'$ of its energy arguments, and is proportioned to the absorptive part of the phase-space Kubo function[19] $\Phi''_{\mathbf{kp}}(\mathbf{q},\epsilon-\epsilon')$:

$$F_{\mathbf{kp}}(\mathbf{q};\epsilon,\epsilon')\approx(1/\pi)\Phi''_{\mathbf{kp}}(\mathbf{q};\epsilon-\epsilon')\ . \tag{2.14}$$

As a correlation function for noninteracting electrons, Φ can be calculated by a variety of techniques. It can therefore be assumed to be known, and it contains the information about the disorder in the system. For given Green function G, Eqs. (2.11) and (2.12) then provide a closed system of integral equations for the correlation functions $\chi_{\mu\nu}$. As we have discussed earlier, our derivation makes sure that the result is gauge invariant if a well defined class of theories for G is used. These include BCS-Gorkov theory and the theories developed in Refs. 22 and 10. If in the future it should become possible to avoid the aforementioned noncrossing approximation in calculating G, it will be straightforward to adopt the present theory accordingly.

C. Solution of the integral equation

We will now solve the integral equation (2.12) for the vertex. We first observe that the vertex can be written in the form

$$\Gamma_\mu(\mathbf{p},\mathbf{q};i\Omega)=\gamma_\mu(\mathbf{p},\mathbf{q})+V\delta_\mu(\mathbf{q},i\Omega)\ , \tag{2.15a}$$

where δ_μ obeys the integral equation

$$\delta_\mu(\mathbf{q},i\Omega)=T\sum_{i\omega}\int d\epsilon\,d\epsilon'\,\tau_3G(\epsilon,i\omega-i\Omega)\frac{1}{\pi}\Phi''_{n\mu}(\mathbf{q},\epsilon-\epsilon')G(\epsilon',i\omega)\tau_3 + VT\sum_{i\omega}\int d\epsilon\,d\epsilon'\,\frac{1}{\pi}\Phi''_{nn}(\mathbf{q},\epsilon-\epsilon')\tau_3G(\epsilon,i\omega-i\Omega)\delta_\mu(\mathbf{q},i\Omega)G(\epsilon',i\omega)\tau_3\ . \tag{2.15b}$$

Here,

$$\Phi''_{nn}(\mathbf{q},\omega)=\sum_{\mathbf{kp}}\Phi''_{\mathbf{kp}}(\mathbf{q},\omega)\ , \tag{2.15c}$$

is the density spectrum, and the matrix

$$\Phi''_{n\mu}(\mathbf{q},\omega)=\sum_{\mathbf{kp}}\Phi''_{\mathbf{kp}}(\mathbf{q},\omega)\gamma_\mu(\mathbf{p},\mathbf{q})\ , \tag{2.15d}$$

is a cross correlation between the density and the mode belonging to the vertex γ_μ.

To proceed, we have to specify the matrix γ_μ. We are interested here in the two cases of the transverse current vertex, Eq. (2.9b), and the density vertex, Eq. (2.8b). For the former, we see by inserting the Fourier transform of Eq. (2.9b) in Eq. (2.15d) that Φ''_{nT} vanishes due to symmetry. Therefore, $\delta_T \equiv 0$, and

$$\Gamma_T(\mathbf{p},\mathbf{q};i\Omega) \equiv \gamma_T(\mathbf{p},\mathbf{q}) = \frac{1}{m}\mathbf{p}\cdot\mathbf{e}_T(\mathbf{q}) \ . \tag{2.16}$$

For the transverse current, the vertex corrections vanish due to lack of coupling between longitudinal and transverse modes.

In the case of the density vertex, Eq. (2.8b), we have $\Phi''_{n0} = \tau_3 \Phi''_{nn}$, and the vertex correction is nonzero. We follow Prange[8] and expand δ in a quaternion basis

$$\delta(\mathbf{q},i\Omega) = \sum_{i=0}^{3} \delta_i(\mathbf{q},i\Omega)\tau_i \ , \tag{2.17}$$

where $\tau_0 = 1$, and τ_1, τ_2, τ_3 are the Pauli matrices. We also specify the Nambu Green function. In this paper we use the BCS approximation

$$G(\epsilon,i\omega) = \frac{i\omega + \epsilon\tau_3 + \Delta\tau_1}{(i\omega)^2 - E^2} \ , \tag{2.18}$$

where $E^2 = \epsilon^2 + \Delta^2$. It is obvious that more complicated Green functions[10] could be treated within the same scheme. However, the analysis then gets considerably more complicated, and cannot be carried as far analytically. We now use the fact that

$$\Phi''_{nn}(q,\omega) = \Phi''_{nn}(q,-\omega) \ .$$

It then readily follows that the coefficients δ_0 and δ_1 in Eq. (2.17) obey homogeneous equations and vanish. For the remaining coefficients, we obtain

$$\delta_3(\mathbf{q},i\Omega) = \left[B_+ + \frac{VA^2}{1-VB_-} \right] \left[1 + VB_+ + \frac{V^2A^2}{1-VB_-} \right]^{-1} , \tag{2.19a}$$

$$\delta_2(\mathbf{q},i\Omega) = -A(1+V\delta_3)(1-VB_-)^{-1} \ . \tag{2.19b}$$

Here we have introduced

$$A(\mathbf{q},i\Omega) = i\Omega\Delta \int d\epsilon d\epsilon' \frac{1}{\pi}\Phi''_{nn}(\mathbf{q},\epsilon-\epsilon')T\sum_{i\omega}\frac{1}{NN'} \ , \tag{2.20a}$$

$$B_\pm(\mathbf{q},i\Omega) = -\int d\epsilon d\epsilon' \frac{1}{\pi}\Phi''_{nn}(\mathbf{q},\epsilon-\epsilon')T\sum_{i\omega}\frac{(i\omega - i\Omega)i\omega \pm \epsilon\epsilon' - \Delta^2}{NN'} \ , \tag{2.20b}$$

where $N = (i\omega - i\Omega)^2 - E^2$, and $N' = (i\omega)^2 - E'^2$. Finally, the frequency summations in Eqs. (2.20) are easily done, and we find

$$A(\mathbf{q},i\Omega) = i\Omega\frac{\Delta}{\pi}\int d\epsilon d\epsilon' \, \Phi''_{nn}(\mathbf{q},\epsilon-\epsilon')a_{\epsilon\epsilon'}(i\Omega) \ , \tag{2.21a}$$

$$B_\pm(\mathbf{q},i\Omega) = \frac{1}{\pi}\int d\epsilon d\epsilon' \, \Phi''_{nn}(\mathbf{q},\epsilon-\epsilon')\chi^{\Delta}_{\epsilon,\pm\epsilon'}(i\Omega) \ , \tag{2.21b}$$

with the spectral functions

$$\chi^{\Delta}_{\epsilon\epsilon'}(i\Omega) = \frac{EE' - \Delta^2 + \epsilon\epsilon'}{4EE'}\frac{f(E') - f(E)}{i\Omega + E - E'} - \frac{EE' + \Delta^2 - \epsilon\epsilon'}{4EE'}\frac{1 - f(E') - f(E)}{i\Omega - E - E'} + (i\Omega \to -i\Omega) \ , \tag{2.22a}$$

$$a_{\epsilon\epsilon'}(i\Omega) = \frac{-1}{4EE'}\frac{f(E') - f(E)}{i\Omega + E - E'} - \frac{1}{4EE'}\frac{1 - f(E') - f(E)}{i\Omega - E - E'} + (i\Omega \to -i\Omega) \ . \tag{2.22b}$$

Here, f denotes the Fermi function, and $(i\Omega \to -i\Omega)$ indicates additional terms which differ only in the sign of $i\Omega$. In terms of these integrals, the screened density response can now be written as

$$\chi_{sc}(\mathbf{q},i\Omega) = 2C(\mathbf{q},i\Omega)/[1 + VC(\mathbf{q},i\Omega)] \ , \tag{2.23a}$$

$$C(\mathbf{q},i\Omega) = B_+ + VA^2/(1 - VB_-) \ . \tag{2.23b}$$

Likewise, we have for the transverse current correlation function

$$\chi_T(\mathbf{q},i\Omega) = \frac{2}{\pi}\int d\epsilon d\epsilon' \, \Phi''_T(\mathbf{q},\epsilon-\epsilon')\chi^{i\Delta}_{\epsilon\epsilon'}(i\Omega) \ , \tag{2.24}$$

where Φ_T is the transverse current Kubo function,

$$\Phi''_T(\mathbf{q},\omega) = \sum_{\mathbf{kp}} \frac{1}{m}\mathbf{k}\cdot\mathbf{e}_T(\mathbf{q})\Phi''_{\mathbf{kp}}(\mathbf{q},\omega)\frac{1}{m}\mathbf{p}\cdot\mathbf{e}_T(\mathbf{q}) \ . \tag{2.25}$$

We have now reduced to quadratures the correlation functions which are relevant for the electromagnetic response. Before we proceed to do the integrals, we close this section with a few remarks. (1) For free electrons, the phase-space Kubo function $\Phi_{\mathbf{kp}}$ can be expressed in terms of the Lindhard function. If we use this in Eqs. (2.21), we recover from (2.23) the result of Prange.[8] (2) Within the BCS approximation, Eq. (2.18), the normal-state correlation $\Phi_{\mathbf{kp}}$ is the *only* source of disorder. Equations (2.23) and (2.24) give the response of a superconductor in terms of the response of the corresponding

normal conducting system with the same disorder. (3) $\Phi_{\mathbf{kp}}$ in general, and Φ_{nn} and Φ_T, in particular, have been calculated by a variety of techniques for different disorder regimes. We will make use of this fact in the next section.

III. THE ELECTROMAGNETIC RESPONSE AT SMALL FREQUENCIES AND WAVE NUMBERS

A. Homogeneous density response

In general, the integrals we encountered in the last section have to be performed numerically, and we will do so in Sec. IV. However, at zero temperature for small frequencies and wave numbers, they can be done analytically, and considerable insight is gained that way.

We start with the homogeneous limit, $q=0$. In this case the density spectrum degenerates into a δ function[24]

$$\Phi''_{nn}(q=0,\omega)=\pi N_F\delta(\omega)\ , \tag{3.1}$$

where N_F is the (normal conducting) density-of-states per spin at the Fermi level. Equation (3.1) is an exact result which follows from particle number conservation. If we use Eq. (3.1) in Eqs. (2.21), we obtain for complex frequency z

$$B_+(0,z)=N_F 2\Delta^2 F(z)\ , \tag{3.2}$$

$$A(0,z)=N_F z\Delta F(z)\ , \tag{3.3}$$

where

$$F(z)=\int\frac{d\epsilon}{E(4E^2-z^2)}=\frac{1}{2\Delta^2}+O(z^2)\ . \tag{3.4}$$

For B_-, the leading term is proportional to $\int d\epsilon(1/E)$ which is logarithmically divergent. This is due to the fact that we have used a pointlike attractive potential. We therefore have to cut off the integral at a frequency on the order of the Debye frequency, whence it turns into the integral entering the gap equation. We therefore have to make the identification

$$N_F\int\frac{d\epsilon}{2E}\equiv 1/V\ . \tag{3.5}$$

Then we have

$$B_-(0,z)=\frac{1}{V}+N_F\frac{z^2}{2}F(z)\ . \tag{3.6}$$

Using these results in Eq. (2.23), we make two important observations. In the first place, we find

$$\chi_{\mathrm{sc}}(0,z)\equiv 0\ ,$$

as it has to be due to particle number conservation. This demonstrates the conserving nature of our approximations. In the second place, we see from Eq. (3.6) that

$$1-VB_-(q,z)=O(q^2,z^2)\ .$$

Therefore, the absorptive part of χ_{sc} contains an undamped excitation with linear dispersion. This is the AB mode. Its existence is a direct consequence of particle number conservation.

In the longitudinal channel the homogeneous correlations depend on disorder only via N_F, cf. above. This again is a consequence of particle number conservation. It is also the reason underlying Anderson's theorem for the superconducting T_c, which in BCS-Gorkov approximation is determined by the homogeneous density response. In the transverse channel there is no conservation law, and even the homogeneous correlations depend explicitly on disorder. We treat this case herein after the longitudinal one at nonzero q.

B. Low-frequency longitudinal response

We now consider nonzero wave numbers. Generally, Φ_{nn} can be written as[19]

$$\Phi_{nn}(\mathbf{q},z)=-g(\mathbf{q})/[z+q^2K(\mathbf{q},z)]\ . \tag{3.7}$$

Here, $g(q)$ is the wave number dependent compressibility which obeys $g(q\to 0)=N_F$, and $K(q,z)$ is a current correlation function. The density spectrum is obtained as

$$\Phi''_{nn}(q,\omega)=\mathrm{Im}\Phi_{nn}(q,\omega+i0)\ .$$

We will consider three different cases: For free electrons, one has

$$K(q,z)=-\tfrac{1}{3}v_F^2/z+O(q^2)\quad(\text{clean}) \tag{3.8a}$$

with the Fermi velocity v_F. For diffusive electrons, one has

$$K(q,z)=iD\quad(\text{diffusive})\ , \tag{3.8b}$$

where $D=v_F^2\tau/3$ is the diffusion constant with the collision time τ. Finally, if the disorder increases further, the system approaches an Anderson transition.[25] In this regime one has a critical current spectrum[26]

$$K(q\to 0,z)=D_M(z/\epsilon_F)^{1/3}\tfrac{3}{2}(-i)^{1/3}\quad(\text{critical})\ . \tag{3.8c}$$

Here, one has to take the unique cubic root which yields a positive current spectrum and $D_M=1/2e^2N_F\rho_M$ with Mott's resistivity $\rho_M=3\pi^2/k_Fe^2$. Equations (3.8a), (3.8b), and (3.8c) are valid for $\rho\to 0$, $\rho\lesssim\rho_M$, and $\rho\gg\rho_M$, respectively.[27]

By inspection of the integral in Eq. (2.21b) we see that the function

$$f(\epsilon')=\Phi_{nn}(q,\epsilon'-\epsilon+i0)$$

has either a pole in the lower half plane (clean and diffusive cases) or a branch cut which can be chosen to lie entirely in the lower half plane (critical case). We can then do the integral over ϵ' by considering only one branch cut connected with E'. This way we rewrite Eq. (2.21b) in the static limit as

$$B_-(q,0)=\frac{1}{\pi}\mathrm{Im}\int d\epsilon \int_\Delta^\infty \frac{dxx}{(x^2-\Delta^2)^{1/2}}\frac{1}{x-i\epsilon}\Phi_{nn}(q,ix-\epsilon)\ . \tag{3.9}$$

We now expand in powers of q. The leading term gives again the integral (3.5). The next leading term converges already, and we can do the ϵ integration first to obtain

$$B_-(q,0)=\frac{1}{V}-q^2\frac{N_F}{2}\int_\Delta^\infty \frac{dx}{x(x^2-\Delta^2)^{1/2}}\mathrm{Im}K(q,i2x)+O(q^4)\ . \tag{3.10}$$

Here we have made use of the spectral representation

$$K(q,z)=\int (d\omega/\pi)\mathrm{Im}K(q,\omega+i0)/(\omega-z)\ .$$

Eq. (3.10) expresses B_- in the static long-wavelength limit in terms of the current correlation function of the noninteracting system. Similar expressions can be derived for B_+ and A, but we will not need them.

With the explicit expressions for K given earlier, it is now elementary to do the remaining integral. We find

$$B_-(q\to 0,0)=\frac{1}{V}-N_F\begin{cases} q^2v_F^2/12\Delta^2 & \text{(clean)}\ , \\ \frac{\pi}{2}(Dq^2/2\Delta) & \text{(diffusive)}\ , \\ \frac{D_Mq^2}{\Delta}(\Delta/\epsilon_F)^{1/3}b & \text{(critical)}\ , \end{cases} \tag{3.11}$$

where

$$b=3\sqrt{\pi}\Gamma(\tfrac{1}{3})/2^{8/3}\Gamma(\tfrac{5}{6})=1.99\ldots\ .$$

We now use Eq. (3.11) together with Eqs. (3.2), (3.3), and (3.6) in Eq. (2.23), and find to leading order in q and z

$$C(q,z)=-N_F(qv)^2/[z^2-(qv)^2]\ . \tag{3.12}$$

This clearly shows the AB mode. The "sound" velocity v is given by $v=(\pi/\sqrt{3})\Delta\xi$ with

$$\xi=\begin{cases} \xi_0 \text{ for } \rho/\rho_M\ll 1/k_F\xi_0 \text{ (clean)}\ , & \text{(3.13a)} \\ (\xi_0 l)^{1/2} \text{ for } 1/k_F\xi_0\ll\rho/\rho_M\lesssim 1 \text{ (diffusive)}\ , & \text{(3.13b)} \\ l_\Delta \text{ for } \rho/\rho_M\gg 1 \text{ (critical)}\ . & \text{(3.13c)} \end{cases}$$

Here, $\xi_0=v_F/\pi\Delta$ is the Pippard coherence length, $l=v_F\tau$ is the mean free path, and

$$l_\Delta=[2b(4\pi^2)^{1/3}/\pi^4]^{1/2}(N_F\Delta)^{-1/3}\ .$$

This is the kinematical part of the "sound" velocity. Considering χ_{sc} rather than C, Eq. (2.23), introduces a renormalization of v which is of order N_FV. This renormalization depends on the details of the approximation,[3,8] and should not be taken too seriously.

Prange[8] has noted that $\chi_{sc}=2C$ is a conserving approximation as well as the full expression in Eq. (2.23a). The main difference between C and the full χ_{sc} is, besides the renormalization of v, that C'' has a collective mode contribution at all q, while in χ_{sc} the collective mode ceases to exist for q above a certain value. However, the residue of the pole in C decreases rapidly with increasing q, and this difference is of little practical importance. In the remainder of this paper we will discuss C.

Equation (3.13a) reproduces the well-known result for the clean case. Equation (3.13b) for the diffusive case is a result which one might have expected: the expression for v remains valid if ξ_0 is replaced by the "dirty limit" coherence length $(\xi_0 l)^{1/2}$. A qualitatively similar result has been obtained by Oppermann[28] for a special model with diffusive dynamics. The present result $v^2=\pi\Delta D$ has also been obtained with kinetic equation techniques.[7] l_Δ in Eq. (3.13c) can also be interpreted as a coherence length. $N_F\Delta$ is the number of states per unit volume within an energy Δ around the Fermi surface, or the number density of states responsible for superconductivity. l_Δ is then the spatial extension of these states. Clearly, the coherence length cannot be shorter than l_Δ, so with increasing disorder the decreasing ξ must saturate at this length scale. It is interesting to see that this result agrees with the direct calculation of the coherence length by Kotliar and Kapitulnik,[29] who arrived there on a quite different route.

In a hypothetical neutral superconductor, the full polarizability χ would be given by Eq. (2.2) with $v(q)\equiv -V$. In such a system the AB mode would be observable. In a charged system, the collective excitations are given by the zeros of the dielectric function, Eq. (2.1). There are no such zeros within the gap as we will see. However, the AB mode can still be seen in the conductivity. From Eq. (2.3) we get

$$\sigma'(\omega)=\sigma_0(\pi^2/2)\Delta\delta(\omega) \tag{3.14}$$

for the conductivity in the low-frequency limit. Here, $\sigma_0=v^2 2N_F/\pi\Delta$. Our derivation makes it clear that the δ function in the absorptive part of the conductivity is actually the homogeneous limit of the AB mode. In the diffusive case, we recover the well-known result[30] $\sigma_0=2N_FD=\sigma_n$, the normal-state conductivity. In the critical case, the prefactor saturates at the small value $\sigma_0\sim\sigma_M(\Delta/\epsilon_F)^{1/3}$. We emphasize that Eq. (3.14) gives the zero-frequency contribution only. In the clean case this is all there is. Within the jellium model, σ for a clean superconductor is the same as σ for a clean normal metal, and Eq. (3.14) exhausts the f-sum rule

$$\int (d\omega/\pi)\sigma'(\omega)=n/m\ .$$

The physical interpretation of the δ function is an accelerated supercurrent in a superconductor, and accelerated electrons in a normal metal. Both draw power from a dc field, while there is no loss at finite frequency.

In the presence of disorder, the superconducting conductivity still shows the δ function, but Eq. (3.14) no longer exhausts the f-sum rule. The rest of the spectral weight appears at frequencies above the gap.[31] At nonzero frequency, calculation of σ' amounts to calculating B_+, Eq. (2.21b), with the density vertex replaced by the current vertex. We find

$$\sigma'(\omega)=\frac{e^2}{\pi}2N_F\int d\epsilon\, d\epsilon'\, K''(0,\epsilon-\epsilon')\mathrm{Im}\chi^{\Delta}_{\epsilon\epsilon'}(\omega+i0)\quad(\omega>2\Delta)\ . \tag{3.15}$$

In the diffusive case, Eq. (3.8b), the integrals can be performed, and we recover the Mattis-Bardeen expression.[31] In this case, $\sigma'(\omega)$ vanishes linearly at the gap edge. In the critical case, Eq. (3.8c), we can still extract the leading behavior at the edge, which is

$$\sigma'(\omega)=\tilde{\sigma}_M(\omega/2\Delta-1)^{7/6}\ , \tag{3.16}$$

where $\tilde{\sigma}_M$ is a conductivity scale on the order of σ_M. We see that the critical current spectrum of the normal metal is reflected in the behavior at the absorption threshold. The effect is, however, very small. At frequencies large compared to 2Δ, σ', of course, crosses over to the conductivity of the normal metal.

We close our discussion of the longitudinal response with two remarks. (1) The diffusion pole approximation, Eq. (3.8b), does, of course, violate particle number conservation. This is easily cured, and the virtue of our conserving approximation is fully brought out by using instead the Drude formula

$$K(z)=iD/(1-iz\tau)\ . \tag{3.8'}$$

The difference between this and Eq. (3.8b) is, however, negligible for frequencies $\omega\ll 1/\tau$. Since $\Delta\tau\gg 1$ except for extremely small disorder, and since we are interested in $\chi_{sc}(q,\omega)$ with ω on the order of the gap, the simpler Eq. (3.8b) is completely adequate. (2) We see from Eq. (3.12) or (3.14) that with increasing disorder, there is less and less spectral weight in the collective mode. Since at zero temperature in the absence of elastic pairbreakers the mode cannot be damped, this is how the intuitive expectation is fulfilled that disorder must adversely affect the mode.

C. Static transverse response

We finally consider the static homogeneous transverse current correlation. From Eqs. (2.4b) and (2.5) we see that this is sufficient to determine the magnetic screening length.

We do the integration over ϵ' in Eq. (2.24) the same way as in the longitudinal case and obtain

$$\chi_T(0,i0)=\frac{2}{\pi}\mathrm{Im}\int d\epsilon\int_{\Delta}^{\infty}\frac{dx}{(x^2-\Delta^2)^{1/2}}\frac{x^2-2\Delta^2-ix\epsilon}{x^2+\epsilon^2}\Phi_T(0,ix-\epsilon)\ . \tag{3.17}$$

In the clean case, we have

$$\Phi_T(0,z)=-n/2mz\ ,$$

and Eq. (3.17) gives the well-known result $\chi_T(0,i0)=0$. For the general case, it is convenient to subtract this (vanishing) contribution from the integral in Eq. (3.17). This improves convergence, and we can do the ϵ integral first:

$$X_T(0,i0)=-4\Delta^2\int_{\Delta}^{\infty}\frac{dx}{(x^2-\Delta^2)^{1/2}}\frac{1}{x}\left[\mathrm{Im}\Phi_T(0,2ix)-\frac{n/m}{4x}\right]\ . \tag{3.18}$$

Since Φ_T is a correlation for noninteracting electrons, at $q=0$ it is identical with the longitudinal current correlation, i.e., the conductivity. In the three cases we considered before, we therefore have

$$\Phi_T(0,z)=-\frac{n}{2m}\begin{cases}1/z\quad(\text{clean})\ , & (3.19a)\\ \dfrac{1}{z+i/\tau}\quad(\text{diffusive})\ , & (3.19b)\\ \sigma_M\frac{3}{2}(z/\epsilon_F)^{1/3}(-i)^{1/3}\quad(\text{critical})\ . & (3.19c)\end{cases}$$

We can now perform the remaining integral in Eq. (3.18), and obtain the penetration depth from Eq. (2.5). In the clean case we have $\lambda=\lambda_L$ with the London length

$$\lambda_L=(mc^2/4\pi ne^2)^{1/2} .$$

In the diffusive case, we find

$$\lambda=\lambda_L\frac{1}{\sqrt{2\Delta\tau}}\left[\frac{\pi}{2}-\frac{2\Delta\tau}{1+2\Delta\tau}g(2\Delta\tau)\right]^{-1/2} \quad \text{(diffusive)} , \tag{3.20a}$$

where

$$g(x)=\Theta(1-x)\left[\frac{1+x}{1-x}\right]^{1/2}\ln\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}} +\Theta(x-1)2\left[\frac{x+1}{x-1}\right]^{1/2}\arctan\left[\frac{x-1}{x+1}\right]^{1/2} . \tag{3.20b}$$

This result is more accurate than the usual Chambers-Pippard expression.[30] In the clean limit, both give a linear increase of λ with $1/\Delta\tau$, but the slopes are different by a factor

$$(\pi/2)(1+\pi/4)=2.80\ldots .$$

In the dirty limit, $\Delta\tau\ll 1$, both Eq. (3.20) and the Chambers-Pippard formula give

$$\lambda=\lambda_L\sqrt{2/\pi}(2\Delta\tau)^{-1/2} .$$

As one would expect from the previous section, this increase saturates in the critical regime, where Eq. (3.19c) in Eq. (3.18) gives

$$\lambda=\lambda_L a(\Delta/\epsilon_F)^{-2/3} , \tag{3.21}$$

where

$$a=2^{5/6}/[3B(\tfrac{1}{2},\tfrac{1}{3})]^{1/2}=0.50\ldots .$$

This is the result Kapitulnik and Kotliar[29] obtained by a different method.[32]

IV. NUMERICAL RESULTS, AND DISCUSSION

A. The screened polarizability in the diffusive case

In Sec. III we have considered the case $q\xi\ll 1,\omega\ll\Delta$. In this limit the screened polarizability could be calculated analytically. Outside this regime, we could not find a way to perform the integrals in Eqs. (2.21), so we have to resort to numerical integration. We will concentrate on the diffusive case, which at present is the most relevant. The critical regime is hard to reach experimentally, and presumably physical effects are important there which have been neglected in our simple model.

It is convenient to first calculate the absorptive parts of the functions $A, B_\pm$. The real parts are then obtained by a Kramers-Kronig transform. We introduce the dimensionless quantities $\Omega=\omega/\Delta$, $Q^2=Dq^2/\Delta=q^2\xi^2\pi/3$, $a''=A''/2N_F$, and $b''_\pm=B''_\pm/2N_F$. Taking the imaginary parts of Eqs. (2.22), we find for $\Omega>0$

$$a''(Q,\Omega)=\tfrac{1}{4}Q^2\Omega\Theta(\Omega-2)\left[\int_1^{\Omega-1}\frac{dx}{(x^2-1)^{1/2}[(x-\Omega)^2-1]^{1/2}}[f_a(x)-f_a(1)]+f_a(1)\frac{4}{\Omega+2}K\left[\frac{\Omega-2}{\Omega+2}\right]\right] , \tag{4.1a}$$

with

$$f_a(x)=\frac{1}{\{(x^2-1)^{1/2}+[(x-\Omega)^2-1]^{1/2}\}^2+Q^4}+\frac{1}{\{(x^2-1)^{1/2}-[(x-\Omega)^2-1]^{1/2}\}^2+Q^4} . \tag{4.1b}$$

Likewise,

$$b''_\pm(Q,\Omega)=\tfrac{1}{4}Q^2\Theta(\Omega-2)\left[\int_1^{\Omega-1}\frac{dx}{(x^2-1)^{1/2}[(x-\Omega)^2-1]^{1/2}}[f_\pm(x)-f_\pm(1)]+f_\pm(1)\frac{4}{\Omega+2}K\left[\frac{\Omega-2}{\Omega+2}\right]\right] , \tag{4.2a}$$

$$f_\pm(x)=\frac{x(\Omega-x)\mp(x^2-1)^{1/2}[(x-\Omega)^2-1]^{1/2}+1}{\{(x^2-1)^{1/2}-[(x-\Omega)^2-1]^{1/2}\}^2+Q^4}+\frac{x(\Omega-x)\pm(x^2-1)^{1/2}[(x-\Omega)^2-1]^{1/2}+1}{\{(x^2-1)^{1/2}+[(x-\Omega)^2-1]^{1/2}\}^2+Q^4} . \tag{4.2b}$$

Here Θ denotes the step function, and K is a complete elliptic integral. In the clean case, the corresponding integrals can be expressed in terms of tabulated functions.[8] Here we have not found a way to do so. We have written Eqs. (4.1a) and (4.2a) in a way which facilitates numerical integration. Once this has been performed, the real parts are determined by principal value integrals

$$a'(Q,\Omega)=\Omega\frac{2}{\pi}\int_0^\infty d\omega\frac{a''(Q,\omega)}{\omega^2-\Omega^2} , \tag{4.3a}$$

$$b'_+(Q,\Omega)=\frac{2}{\pi}\int_0^\infty d\omega\frac{\omega b''_+(Q,\omega)}{\omega^2-\Omega^2} , \tag{4.3b}$$

$$b'_-(Q,\Omega)=\frac{1}{2N_FV}+\frac{2}{\pi}\int_0^\infty d\omega\left[\frac{\omega b''_-(Q,\omega)}{\omega^2-\Omega^2}-\frac{\pi/4}{(\omega^2+1)^{1/2}}\right]+\Delta b_- . \tag{4.3c}$$

Here we have made use of the fact that a'' is an even function of frequency, while $b''_{\pm}$ are odd. We have also used Eq. (3.5) to deal with the singular part of b_-. Finally, there is an additional term Δb_-. The reason for the appearance of Δb_- is that in going from Eq. (2.21b) to Eq. (4.3c) some energy integrations have been performed, and in others there has been a change of variable. Consequently, the connection between the logarithmically divergent integral and $1/VN_F$ is no longer given by Eq. (3.5). However, since we are dealing with a contribution to $b_-(0,0)$, the correction must be simply a constant. We therefore can use the analytically tractable case $Q=0$ to determine it. We use the imaginary part of Eq. (3.6) in Eq. (4.3c) to find

$$\Delta b_- = (\tfrac{1}{2})\ln 2 \ . \tag{4.3d}$$

The principal value integral can be easily performed by the same kind of subtraction trick we have used in Eqs. (4.1) and (4.2). We have checked our results by considering (1) the behavior of the spectra close to the threshold, (2) the high-frequency asymptotics, and (3) the minus first moments of a'' and $b''_{\pm}$, all of which can be obtained analytically. We estimate the accuracy of our result for c to be better than 1%, with the largest error occuring for the smallest Q. The six real functions a'', a', $b''_{\pm}$, $b'_{\pm}$ determine C and therewith χ_{sc} via Eqs. (2.23).

In Fig. 3 we plot the real and imaginary parts of

$$c(Q,\Omega+i0)=C(q,\omega+i0)/2N_F$$

as functions of Ω for various values of Q. The δ-function contribution to c'' which corresponds to the collective excitation can clearly be seen at small Q. It causes c' to diverge as $[\Omega-\Omega_p(Q)]^{-1}$. In Table I we list the position $\Omega_p(Q)$ and the residue $r(Q)$ of the pole in $c(Q,\Omega)$. For small Q these agree with Eq. (3.12). At $\Omega=2$ the continuous part of the spectrum sets in. It is readily seen from Eqs. (4.1) and (4.2) that a'' and $b''_{\pm}$ are all discontinuous at the threshold with

$$a''(Q,2)=b''_{+}(Q,2)=b''_{-}(Q,2)=\pi/2Q^2 \ .$$

The corresponding real parts are therefore logarithmically divergent, and

$$c''(Q,\Omega\to 2)\sim \text{const}/[\ln(\Omega-2)]^2 \ .$$

χ'_{sc} shows the same behavior, and it is qualitatively the same as in the clean case.[8] It was noted in Ref. 8 that the continuous behavior at the edge is an important feature, since a discontinuity in C'' would necessarily lead to a zero of $\epsilon(q,\omega)$ in the gap.

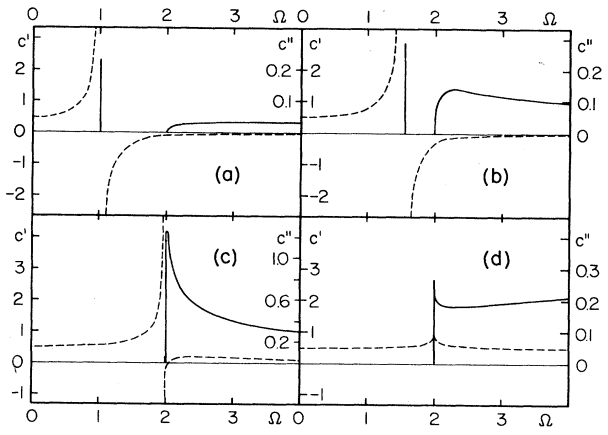

FIG. 3. Real part c' (dashed lines) and imaginary part c'' (fully lines) of the function

$$c(Q,\Omega)=C(q,\omega+i0)/2N_F \ .$$

Q and Ω are defined in the text. Q has been chosen as (a) 0.6, (b) 1.0, (c) 1.8, (d) 3.4. Vertical bars denote the δ-function contribution to c''. Notice the different scales for c' and c'', respectively.

From Eq. (2.1) we see that for ϵ to have a zero in the gap, χ'_{sc} must come very close to zero, since $v(q)$ is large. Explicitly, the condition is

$$\frac{1}{2N_F}\chi'_{sc}(q,\omega)=-(q/\kappa_d)^{d-1} \quad (d=2,3) \ , \tag{4.4}$$

where κ_d is the Thomas-Fermi wave number for a d-dimensional system. Since κ_d is on the order of 1 Å^{-1}, the right-hand side is usually very small for $q\xi\cong 1$. The most favorable case for fulfilling Eq. (4.4) is large disorder and small Fermi energy (or large gap), since this maximizes q for given $q\xi$. From our numerical calculations we conclude that the condition (4.4) can not be fulfilled for frequencies within the gap. This holds even for critical disorder (cf. Sec. IV B), and also for the extremely short coherence lengths of the new oxide superconductors (for which the present theory would probably be inadequate anyway). We conclude that there is no disorder induced zero of the dielectric function, and that the behavior of $\epsilon(q,\omega)$ at arbitrary disorder is qualitatively very similar to that in the clean case.[8] The main differences are (1) the smearing of all singularities except the AB mode and the onset at $\omega=2\Delta$, (2) the reduced spectral weight in the AB mode, and (3) the long high-frequency tail of the spectrum.

TABLE I. Positions $\Omega(Q)$ and residues $r(Q)$ of the pole in $c(Q,\Omega)$.

Q	0.20	0.40	0.60	0.80	1.00	1.20	1.40	1.60	1.80	2.00
$\Omega(Q)$	0.35	0.70	1.02	1.30	1.55	1.74	1.87	1.95	1.99	2.00
$r(Q)$	0.088	0.170	0.234	0.277	0.289	0.271	0.208	0.121	0.049	0.016

B. The screened polarizability in the critical case

Let us also briefly discuss the critical case. The main question is whether this changes the conclusion of the last section that there is no zero of the dielectric constant within the gap. In this context it is important to realize that we must not simply replace the diffusive constant D in the previous section with the critical current spectrum, Eq. (3.8c), for the following reasons. We use Eqs. (2.21) and (2.22) to write the spectra of A, $B_{\pm}$ for arbitrary disorder as

$$A''(q,\omega)=\frac{\omega\Delta}{2}\int_{\Delta}^{\omega-\Delta}\frac{dx}{(x^2-\Delta^2)^{1/2}[(\omega-x)^2-\Delta^2]^{1/2}}(\Phi''_{nn}\{q,(x^2-\Delta^2)^{1/2}-[(\omega-x)^2-\Delta^2]^{1/2}\}$$
$$+\Phi''_{nn}\{q,(x^2-\Delta^2)^{1/2}+[(\omega-x)^2-\Delta^2]^{1/2}\}) , \tag{4.5}$$

and similar expressions for $B''_{\pm}$. We see that close to the absorption threshold, the frequency argument of Φ''_{nn} is on the order of $W=\sqrt{\omega(\omega-2\Delta)}$. At fixed q close to the threshold, we therefore always have a $q\gg(N_F W)^{1/3}$. In this limit the current correlation K is given by[33]

$$K(q,z\to 0)\sim iD_M(q/k_F) , \tag{4.6}$$

rather than by Eq. (3.8c). We see that close to the threshold, the results for the diffusive case still hold if the replacement $D\to D_M q/k_F$ is made. In particular, χ''_{sc} still goes to zero continuously at the edge. We conclude that critical disorder does not change the absence of a zero of $\epsilon(q,\omega)$ within the gap.

C. Consequences for the quasiparticle inelastic lifetime

In this paper, we have developed and demonstrated a practical scheme to calculate the dielectric function of disordered superconductors. As we mentioned already in the Introduction, an important application will be the inelastic Coulomb scattering of quasiparticles, and we close with a few remarks concerning this.

Tewordt[15] has calculated the Coulomb inelastic lifetime τ_{in} of quasiparticles in the clean case at zero temperature. The relevant "scattering potential" is the imaginary part of the Coulomb propagator

$$V(\mathbf{q},\omega)=v(\mathbf{q})/\epsilon(\mathbf{q},\omega) .$$

To calculate χ_{sc} and therewith ϵ, Ref. 15 used the pair approximation for χ''_{sc}, i.e., in our Eq. (2.23b) only B_+ was retained. Simultaneously, $\chi'_{sc}(q,\omega)$ was replaced by $2N_F$. This seems inconsistent, since the pair approximation makes χ''_{sc} discontinuous at $\omega=2\Delta$, hence χ'_{sc} should diverge. However, a closer inspection shows that this approximation is much better than it looks at first sight. For $q\xi<1$, χ'_{sc} differs drastically from $2N_F$, but this region contributes little to τ_{in} for phase space reasons. At larger q, χ'_{sc} rapidly approaches the Lindhard function except for a singularity at 2Δ. This singularity is just a mild kink, however, since χ''_{sc} is continuous at the edge. Hence the replacement of χ'_{sc} by $2N_F$ is justified. If one then replaces the continuous onset of χ''_{sc} by the discontinuity of the pair approximation, one neglects only logarithmic corrections to the frequency or temperature dependence of τ_{in}. Moreover, these corrections would appear only in an exponentially small frequency or temperature regime.

Our investigation of the disordered regime has shown that the above discussion can be carried over to the case of disordered superconductors. Figure 3 shows that at small Q, C is dominated by its pole [Figs. 3(a) and 3(b)]. For $Q\lesssim 2$, the pole rapidly approaches the absorption threshold and looses spectral weight, while the continuous spectrum close to the threshold is strongly enhanced [Fig. 3(c)]. For $Q=3.4$, the sharp features in both c' and c'' have already disappeared [Fig. 3(d)]. We note that actually c'' still has a δ-function contribution, and c' still diverges very close to $\Omega=2$, but this cannot be seen anymore on any reasonable scale. Table I shows the rapid decrease of the pole's residue. As mentioned before, χ_{sc} as given by Eq. (2.23) has no longer a pole at large Q. We conclude that the pair approximation might be sufficient to calculate τ_{in} in the disordered case as well. This holds with the caveat that at strong disorder, the region in q space where the structure of χ' is important, may not be small anymore, and its contribution to τ_{in} may no longer be negligible. This will require a thorough investigation.

There is some experimental evidence that in disordered superconductors the quasiparticle scattering rates are anomalously enhanced and show a nonexponential temperature dependence.[11] A possible explanation for these observations would have been a disorder-generated mode in the gap. Our analysis has shown that there is none, ruling out this possibility. We therefore conclude that the theoretical explanation of these results should be looked for in more conventional disorder-enhanced scattering mechanisms. An obvious candidate is the superconducting analogue of the well-known enhanced Coulomb scattering in normal metals.[34,23] This can be investigated with the help of the results of the present paper. We will address these questions in the future.

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