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Spin diffusion in paramagnetic quantum fluids

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We derive the linearized spin dynamics for paramagnetic quantum liquids. In these systems, the existence of quantum-mechanical exchange in the presence of (an externally induced) broken symmetry in spin space gives rise to a coherent precession of the spins damped by diffusion. Such collective oscillations occur as a result of the precession of the spin current about a macroscopic local exchange field generated either by an external polarizing magnetic field (as in dilute nondegenerate spin-polarized gases) or by the long-lived nonequilibrium polarization (as in degenerate Fermi liquids). We use a unified approach based on the Kadanoff-Baym formulation of the kinetic equations to describe these phenomena. The connection between strongly and weakly interacting paramagnetic systems is made obvious by considering two special cases: (1) a weakly interacting spin-polarized quantum gas and (2) a parametrization of strongly correlated degenerate Fermi liquids in terms a weakly interacting gas of "quasiparticles." Transport coefficients are calculated with a novel method which may be generalizable to systems with no translational invariance.

I. INTRODUCTION

Several experiments have recently shown evidence of spin oscillations in nondegenerate, spin-polarized quantum gases.[1,2] These observations, which have also been supported by more indirect experimental findings,[3,4] go against the usual belief that collective oscillations (spin waves) can only occur in strongly interacting degenerate systems.[5,6] In the original work of Lhuillier and Laloe[7] the phenomenon was explained in terms of interference effects in the course of microscopic collisions. Although this contribution was of great value in the case of dilute Boltzmann gases, it provided little insight into the connections with the known physics of collective modes in strongly interacting systems (Fermi liquids, ferromagnets), which are usually discussed in terms of a macroscopic, phenomenological picture.[6] In a previous publication[2] we have given a brief discussion of collective oscillations in a dilute gas of spin polarized hydrogen at the lowest temperatures stressing the analogy to the spin waves of degenerate Fermi liquids. (After our paper was published we learned that a similar point of view had been sketched earlier by Bashkin.[8]) In fact, at the fundamental level the *possible* occurrence of coherent spin oscillations in ferromagnets, Fermi liquids, spin polarized gases and, in general any paramagnet in a magnetic field can be inferred on the basis of symmetry arguments alone.

The two ingredients which make such effects possible are (i) quantum exchange and (ii) the presence of a broken symmetry in spin space (i.e., a macroscopic spin polarization). In ferromagnets, at sufficiently low temperatures, the exchange interaction between nearest-neighbor spins is sufficiently large to induce such a magnetization spontaneously, in spite of the rotational invariance of the Hamiltonian. In this case, spin waves correspond to low-energy oscillations about the background magnetization, with a characteristic frequency, ω_{sw}, determined by the exchange constant, J ($\omega_{\mathrm{sw}} \propto Jk^2$). On the other hand, in spin-polarized quantum fluids the broken symmetry is induced by an external polarizing magnetic field, while exchange interactions are always important whenever the thermal wavelength becomes greater than the spatial scale of variation of the interparticle potential (of order r_s, the hard core radius). For dilute, nondegenerate gases the exchange energy is typically much smaller than the thermal energy $k_B T$ and does not contribute to thermodynamic properties (in contrast to Fermi liquids). As we will discuss below, at low temperatures the exchange is nonetheless sufficient to lead to collective spin oscillations in the dynamics away from equilibrium.

Whereas symmetry arguments predict the occurrence of collective spin oscillations for both spontaneously (e.g., ferromagnets) or externally induced (e.g., spin-polarized liquids) broken symmetries, the detailed origin of the effects is different. In the former case, the spin density, $\boldsymbol{\sigma}(r,t)$, precesses about a local magnetic field $\propto J\nabla^2\boldsymbol{\sigma}(\mathbf{r},t)$. The spin waves correspond to the linearized modes of the corresponding Bloch equations. For all paramagnetic systems in an external field (spin-polarized systems, Fermi liquids) the local field is proportional to $\boldsymbol{\sigma}(\mathbf{r},t)$ itself, and, clearly, cannot contribute to the precession of $\boldsymbol{\sigma}(r,t)$. In this case the spin oscillations occur as a consequence of precession of the spin current, $\mathbf{J}\nparallel\boldsymbol{\sigma}(\mathbf{r},t)$, about the local molecular field, which reacts on the spin density through the continuity equation. Finally, we note that on symmetry grounds the damping of spin waves in ferromagnets, $\Gamma \propto k^4$, and thus spin waves are well defined excitations at sufficiently long wavelengths.[9] By contrast, the observability of spin oscillations in paramagnetic systems depends on the relative magnitude of the spin wave frequency and damping (which are both proportional to k^2), which can only be obtained through a detailed microscopic calculation.

The aim of this paper and its companion[10] is to present a first-principle derivation of the spin dynamics of quantum liquids, with particular emphasis on dilute spin polarized quantum gases. Our approach is based on the generalization of the Kadanoff-Baym formalism[11] and has a number of advantages over the previously used Boltzmann equation treatments: (i) it applies to both the degenerate as well as nondegenerate regimes, for arbitrary external magnetic fields; (ii) it allows for a natural discussion of Bose condensation (for the Bose gas) or other low-temperature broken symmetry phases; (iii) it also leads to useful ways of parametrizing the physics of strongly correlated systems, even when a Fermi liquid description may not be applicable.

The outline of the paper is as follows: In Sec. II, we introduce the formalism and derive the general kinetic equations for the single-particle Green's function (or equivalently, the distribution function). In Sec. III we introduce a new method for calculating diffusion coefficients. In Sec. IV, we apply the formalism to the case of an interacting gas by developing an expansion in powers of the density (the T-matrix approximation). We also discuss some technical aspects of the problem which appear to have generated controversy.[12,13] Finally, we show in Sec. V how the well known results[14,15] for the diffusion coefficients of strongly correlated Fermi liquids emerge when the single-particle Green's function is parametrized in terms of a "quasiparticle" picture. We conclude with a brief discussion.

II. THE KINETIC EQUATIONS

We are interested in determining the equation of motion for the one-particle density matrix, $\rho_{\alpha\beta}(1)=\frac{1}{2}[n(1)\delta_{\alpha\beta}+\underline{\sigma}(1)\cdot\underline{\tau}_{\alpha\beta}]$, where $n(1)$ and $\underline{\sigma}(1)$ are, respectively, the particle number and spin densities at position r_1 at time t_1 [i.e., $1\equiv(r_1,t_1)$], and $\tau_{\alpha\beta}$ are the Pauli matrices (normalized such that $\tau^2=1$). This will be accomplished by studying the equations of motion for the single particle Green's functions $G_{\alpha\beta}(1,2)$, $G^{<}_{\alpha\beta}(1,2)$, $G^{>}_{\alpha\beta}(1,2)$, and $G^{r}_{\alpha\beta}(1,2)$, defined as usual[11] through the following equations:

$$\begin{aligned}G_{\alpha\beta}(1,2)&=-i\langle T\Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle\\&\equiv-i\Theta(t_1-t_2)\langle\Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle-i\epsilon\Theta(t_2-t_1)\langle\Psi^\dagger_\beta(2)\Psi_\alpha(1)\rangle\\&\equiv\Theta(t_1-t_2)G^{>}_{\alpha\beta}(1,2)+\Theta(t_2-t_1)G^{<}_{\alpha\beta}(1,2)\ ,\end{aligned}\tag{1a}$$

$$G^{r}_{\alpha\beta}=-i\Theta(t_1-t_2)[\langle\Psi_\alpha(1)\Psi^\dagger_\beta(2)\rangle-\epsilon\langle\Psi^\dagger_\beta(2)\Psi_\alpha(1)\rangle]\ .\tag{1b}$$

Here $\Psi^\dagger_\alpha(1)(\Psi_\alpha(1))$ are the creation (annihilation) operators for a particle at space-time point 1, in spin state $\alpha;\Theta(t)=1$ for $t>0$ and zero otherwise; and $\epsilon=+1(-1)$ for bosons (fermions). Throughout this paper we use units for which $\hbar=1$. The averages, denoted by $\langle\ldots\rangle$, are calculated with a density matrix defined by the Hamiltonian

$$H=\sum_{\alpha,\beta}\int dr\,\Psi^\dagger_\alpha(\mathbf{r})\left[\left(-\frac{1}{2m}\nabla^2_r-\mu\right)\delta_{\alpha\beta}-\underline{\mathbf{H}}(\mathbf{r})\cdot\underline{\tau}_{\alpha\beta}\right]\Psi_\beta(\mathbf{r})+\frac{1}{2}\sum_{\alpha,\beta}\int dr\int dr'\,V(\mathbf{r}-\mathbf{r}')\Psi^\dagger_\alpha(\mathbf{r})\Psi^\dagger_\beta(\mathbf{r}')\Psi_\beta(\mathbf{r}')\Psi_\alpha(\mathbf{r})\tag{2}$$

where $V(\mathbf{r}-\mathbf{r}')$ is a short-range interaction potential, bold faced symbols represent vectors in real space, and the *underlining* of a symbol denotes a quantity transforming like a vector under spin rotations.

The desired equation for $\rho_{\alpha\beta}(1)=i\epsilon G^{<}_{\alpha\beta}(1,1^+)$ can then be obtained from the definitions (2) and the Heisenberg equations of motion for the field operators $\Psi^\dagger_\alpha(1)$, $\Psi_\alpha(1)$ in the real time domain. We follow the formalism developed by Kadanoff and Baym[11] in which the resulting equations are supplemented with boundary conditions on the Green's functions in imaginary time which ensure the correct thermodynamics in the long time limit. This approach leads to a natural extension of the well known thermal equilibrium perturbation theory methods to nonequilibrium problems, and provides a framework for formulating approximation schemes, consistent with conservation laws (read Ward identities). The latter characteristic makes this method particularly useful in parametrizing the physics of strongly interacting systems, for which straightforward perturbative approaches are not applicable (see Sec. V).

A detailed discussion of the relevant derivations is given in Appendix A. The resulting equation for $G^{<}_{\alpha\beta}$ is exact and is valid independent of the nature and strength of the interparticle interactions:

$$\begin{aligned}i\partial_T G^{<}(\mathbf{R},T,\mathbf{p},\omega)=Q\Big\{&[E(\mathbf{R},T,\mathbf{p})+\Sigma_c(\mathbf{R},T,\mathbf{p},\omega),G^{<}(\mathbf{R},T,\mathbf{p},\omega)]+[\Sigma^{<}(\mathbf{R},T,\mathbf{p},\omega),G(\mathbf{R},T,\mathbf{p},\omega)]\\&-\frac{i}{2}\{\Gamma(\mathbf{R},T,\mathbf{p},\omega),G^{<}(\mathbf{R},T,\mathbf{p},\omega)\}+\frac{i}{2}\{A(\mathbf{R},T,\mathbf{p},\omega),\Sigma^{<}(\mathbf{R},T,\mathbf{p},\omega)\}\Big\}\ ,\end{aligned}\tag{3a}$$

$$Q=\exp\frac{i}{2}[(\nabla^a_R\nabla^b_p-\nabla^a_p\nabla^b_R)-(\partial^a_T\partial^b_\omega-\partial^a_\omega\partial^b_T)]\tag{3b}$$

where $\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2)/2$, and $T=(t_1+t_2)/2$ are the center of mass coordinate and time, and $G^{<}(\mathbf{R},T,\mathbf{p},\omega)$ is the Fourier transform of $G^{<}(\mathbf{R},T,\mathbf{r},t)$ with respect to the relative coordinates, $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and $t=t_1-t_2$ [see Eq. (A8)]. Above, $[a,b]$ and $\{a,b\}$ are the commutator and anticommutator of operators a and b; while the superscripts a,b introduced in the definition of $\mathcal{Q}$ (3b) specify on which operator (a or b) a given derivative in $\mathcal{Q}[a,b]$ (or $\mathcal{Q}\{a,b\}$) acts. In Eq. (3a) $\mathbf{E}(\mathbf{R},T,\mathbf{p}) =(\epsilon_p-\mu)\mathbf{1}-\underline{\mathbf{H}}(\mathbf{R})\cdot\underline{\boldsymbol{\tau}}+\boldsymbol{\Sigma}_{\mathrm{HF}}(\mathbf{R},T;\mathbf{p})$ ($\epsilon_p=p^2/2m$) can be interpreted as the single-particle energy, where Σ_{HF} is the instantaneous self-consistent (Hartree-Fock) potential through which a particle moves. Finally,

$$\begin{gathered}\boldsymbol{\Gamma}(\omega)=\boldsymbol{\Sigma}^{>}(\omega)-\epsilon\boldsymbol{\Sigma}^{<}(\omega), \quad \mathbf{A}(\omega)=\mathbf{G}^{>}(\omega)-\epsilon\mathbf{G}^{<}(\omega)\ , \\ \mathbf{G}(\omega)=\int\frac{d\omega'}{2\pi}\frac{\mathbf{A}(\omega')}{(\omega-\omega')}, \quad \boldsymbol{\Sigma}^{c}(\omega)=\int\frac{d\omega'}{2\pi}\frac{\boldsymbol{\Gamma}(\omega')}{(\omega-\omega')}\ ,\end{gathered} \tag{4}$$

where the variables $(\mathbf{R},T,\mathbf{p})$ are implied. Physically, $\boldsymbol{\Gamma}$ is the difference (bosons) or sum (fermions) of the scattering rate out of $[\boldsymbol{\Sigma}^{>}(\omega)]$ and the scattering rate into $[\boldsymbol{\Sigma}^{<}(\omega)]$ a given state and represents the damping of that state. This permits an interpretation of $\boldsymbol{\Sigma}^{c}$ (from the Kramers-Kronig relation) as the shift in the single-particle energy produced by collisions with the other particles. As explained in Appendix A, $\mathbf{A}(\omega)$ is the single-particle spectral function and measures the density of states available for the addition or the removal of particles with momentum p and energy ω. Consequently, $\mathbf{G}(\omega)$ represents the real part of the single-particle Green's function.

For the purpose of our discussion, Eqs. (3) can be further simplified by making a "gradient expansion" which is valid whenever the scale of spatial variation of the distribution function is long compared to the de Broglie wavelength λ_T, and the time dependence slow compared to the average single-particle energy, ϵ_0. The rapid precession about the external field can be removed by transforming to the rotating frame ($\mathbf{E}'=\mathbf{E}+\underline{\mathbf{H}}\cdot\underline{\boldsymbol{\tau}}$). To first order in $(1/\epsilon_0)\partial_T$ and $(1/\lambda_T)\nabla_R$, one obtains:

$$\begin{aligned} i\partial_T\mathbf{G}^{<}(\mathbf{R},T,\mathbf{p},\omega)=&[\mathbf{E}'(\mathbf{R},T,\mathbf{p})+\boldsymbol{\Sigma}^{c}(\mathbf{R},T,\mathbf{p},\omega),\mathbf{G}^{<}(\mathbf{R},T,\mathbf{p},\omega)]+[\boldsymbol{\Sigma}^{<}(\mathbf{R},T,\mathbf{p},\omega),\mathbf{G}(\mathbf{R},T,\mathbf{p},\omega)] \\ &+\frac{i}{2}\{\mathbf{E}'(\mathbf{R},T,\mathbf{p})+\boldsymbol{\Sigma}^{c}(\mathbf{R},T,\mathbf{p},\omega),\mathbf{G}^{<}(\mathbf{R},T,\mathbf{p},\omega)\}^{\mathrm{PB}}+\frac{i}{2}\{\boldsymbol{\Sigma}^{<}(\mathbf{R},T,\mathbf{p},\omega),\mathbf{G}(\mathbf{R},T,\mathbf{p},\omega)\}^{\mathrm{PB}} \\ &-\frac{i}{2}\{\boldsymbol{\Gamma}(\mathbf{R},T,\mathbf{p},\omega),\mathbf{G}^{<}(\mathbf{R},T,\mathbf{p},\omega)\}+\frac{i}{2}\{\mathbf{A}(\mathbf{R},T,\mathbf{p},\omega),\boldsymbol{\Sigma}^{<}(\mathbf{R},T,\mathbf{p},\omega)\}\ ,\end{aligned} \tag{5}$$

where, for the sake of compactness we have introduced the *anticommutator Poisson-bracket* of two operators, $\mathbf{a}$ and $\mathbf{b}$:

$$\begin{aligned}\{\mathbf{a},\mathbf{b}\}^{\mathrm{PB}}=&\{\nabla_R\mathbf{a},\nabla_p\mathbf{b}\}-\{\nabla_p\mathbf{a},\nabla_R\mathbf{b}\} \\ &-\{\partial_T\mathbf{a},\partial_\omega\mathbf{b}\}+\{\partial_\omega\mathbf{a},\partial_T\mathbf{b}\}\ .\end{aligned} \tag{6}$$

Terms involving *commutator Poisson-brackets* [defined by replacing anticommutators by commutators in Eq. (6)], as well as $\{\boldsymbol{\Gamma},\mathbf{G}^{<}\}^{\mathrm{PB}}$, and $\{\mathbf{A},\boldsymbol{\Sigma}^{<}\}^{\mathrm{PB}}$, lead to contributions smaller by factors of order λ_T/L than those already included in (5), and will thus be neglected.

In spite of the intimidating form of the equation of motion (5), each term has a rather simple physical interpretation, which will become clearer in the next sections as we study specific examples. We note that all commutator terms are even under time reversal and thus the corresponding effects are reversible (i.e., do not contribute to entropy production). In particular, the first term in Eq. (5) only contributes to the spin dynamics and leads to a precession of the spin of a particle with momentum p about its own molecular field *in addition* to that due to the Larmor term. It is the presence of this additional precession which gives rise at the macroscopic level to coherent spin oscillations if the damping $\boldsymbol{\Gamma}$ is sufficiently small. The associated Poisson-bracket (third) term describes the drift in phase space of a gas of "quasiparticles" governed by a single particle Hamiltonian, $\mathbf{E}'+\boldsymbol{\Sigma}^{c}$.

On the other hand, anticommutators are easily seen to lead to both odd and even contributions under time reversal: more precisely, the odd contribution to $\{\boldsymbol{\Gamma},\mathbf{G}^{<}\}-\{\mathbf{A},\boldsymbol{\Sigma}^{<}\}=\{\boldsymbol{\Sigma}^{>},\mathbf{G}^{<}\}-\{\boldsymbol{\Sigma}^{<},\mathbf{G}^{>}\}$, measures the difference between the scattering rates out of and into a given single-particle state, and is responsible for the relaxation of the system towards thermal equilibrium; while the even contribution can be thought of as describing final-state interference effects,[16] and leads (as we will see later) to reversible macroscopic behavior. Finally, the commutator $[\boldsymbol{\Sigma}^{<},\mathbf{G}]$ and the accompanying drift term can be interpreted as describing the precession of spins (in scattered states) about a virtual molecular field. We will see in the examples considered that in any system with well defined quasiparticles this term must vanish.

As explained in Appendix A, the self-energy matrices, $\boldsymbol{\Sigma}$, are defined in terms of the two-body Green's function,

$$G^{(2)}_{\alpha\beta,\alpha'\beta'}(1,2;1'2')=-\langle T\Psi_\alpha(1)\Psi_\beta(2)\Psi^\dagger_{\beta'}(2')\Psi^\dagger_{\alpha'}(1')\rangle \tag{7}$$

through

$$i\epsilon V(1-2)G^{(2)}_{\alpha\gamma,\alpha'\gamma}(1,2^-;1'2^+)=\Sigma_{\alpha\gamma}(1,2)G_{\gamma\alpha'}(2,1')\ , \tag{8a}$$

$$i\epsilon V(1'-2)G^{(2)}_{\alpha\gamma,\alpha'\gamma}(1,2^-;1'2^+)=G_{\alpha\gamma}(1,2)\Sigma_{\gamma\alpha'}(2,1') \tag{8b}$$

where summation over repeated (spatial or spin) indices is implied throughout. The process of deriving the kinetic theory of a given system reduces to the problem of formulating an appropriate approximation scheme for $\boldsymbol{\Sigma}$. Below we will discuss in some detail two examples: (i) a dilute spin polarized gas in an arbitrary external magnetic field, for which the self-energy, and consequently the kinetic equations can be derived exactly to leading nontrivial order in the density (Sec. IV); (ii) a strongly interacting, degenerate (normal) Fermi liquid in which case the self-energy (or more precisely the Green's function itself) can be parametrized according to Landau's quasiparticle picture (Sec. V).

In the following, rather than adopting the traditional Chapman-Enskog method for solving the transport equations (as done, for example, in Ref. 7 for weakly interacting gases) we will directly derive equations for the diffusion coefficients. These require the solution of integral equations for "vertex functions" well known in the context of other formulations of kinetic theory. Before specializing our discussion to the two examples mentioned above, in the next section we will introduce our general procedure for calculating the spin diffusion coefficients. A similar scheme had been developed independently in Ref. 18.

III. THE EQUATIONS FOR THE DIFFUSION COEFFICIENTS

In a system which is invariant under rotations in spin space, the magnetization, $\underline{\mathbf{M}}$, is conserved (i.e., the total magnetization commutes with the Hamiltonian). As a result, the spin density $\underline{\sigma}(\mathbf{R},T)$ $[\underline{\mathbf{M}}(\mathbf{R},T)=\mu\underline{\sigma}(\mathbf{R},T)$, μ is the magnetic moment] satisfies the continuity equations $\partial_T\sigma_l(\mathbf{R},T)=-\sum_i\nabla_R^i J_l^i(\mathbf{R},T)$, where $J_l^i(\mathbf{R},T)$ is the ith component of the spin current associated with the spin component $\sigma_l(\mathbf{R},T)$ ($l=\pm,z$, where for any vector in spin space, $v, v_\pm=v_x\pm iv_y$). Since J_l^i is not a conserved quantity, it must decay to its steady state value on a microscopic time scale, set by the momentum relaxation time. On longer time scales the spin current is solely determined by the local gradient in the magnetization density which relaxes only by motion of spins over macroscopic distances. The resulting constitutive relation, $J_l^i(\mathbf{R},T)=-D_l\nabla^i\sigma_l(\mathbf{R},T)$, defines the diffusion coefficient, D_l, for the l component of the spin. (Here we will imagine that any uniform precession due to the presence of an external field is removed by transforming to the rotating frame.) For the purpose of our calculation it is useful to integrate this constitutive equation by parts, to obtain (after discarding the surface contributions)

$$\begin{aligned} D_l &= \lim_{T\to\infty}[3\delta\sigma_l(T)]^{-1}\int d^3R\,\mathbf{R}\cdot\mathbf{J}_l(\mathbf{R},T) \\ &= \lim_{T\to\infty}[3\delta\sigma_l(T)]^{-1}\int d^3R R^i\int\frac{d^3P}{(2\pi)^3}e^{+i\mathbf{P}\cdot\mathbf{R}}J_l^i(\mathbf{P},T) \\ &= \lim_{T\to\infty}[3\delta\sigma_l(T)]^{-1}i[\nabla_P\cdot\mathbf{J}_l(\mathbf{P},T)]_{P=0} \end{aligned} \tag{9}$$

where $\delta\sigma_l(T)=\sigma_l(T)-\sigma_l^{\text{eq}}$ is the departure of the average polarization density away from its value in thermal equilibrium, σ_l^{eq}, at time T. Together with the microscopic definition of the spin current in terms of the single-particle Green's function,

$$\mathbf{J}_l(\mathbf{R},T)\equiv\int\frac{d\omega}{2\pi}\frac{d^3k}{(2\pi)^3}\left[\frac{\mathbf{k}}{m}\right]\text{Tr}[\tau_l G^<(\mathbf{R},T,\mathbf{k},\omega)] . \tag{10}$$

Eq. (9) leads to the expression

$$D_l=\lim_{T\to\infty}[3m\delta\sigma_l(T)]^{-1}i \times\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}[\mathbf{k}\cdot\nabla_P\text{Tr}\{\tau_l G^<(P,T,k,\omega)\}]_{P=0} . \tag{11}$$

As usual, it is important to take the $P\to0$ limit before letting $T\to\infty$, to exclude the singular contribution of the static response.[19]

In this section, we will show how the equations of motion [Eq. (5)] can be used to derive integral equations for $\nabla_P\text{Tr}\{\tau_l G^<(\mathbf{P},T,\mathbf{k},\omega)\}\equiv\mathbf{\Lambda}_l(\mathbf{P},T,\mathbf{k},\omega)$ (the vertex functions), which in turn determine the spin diffusion coefficients through Eq. (11). We proceed by first ignoring the coupling between spin and density fluctuations, and we thus assume that the density and other spin-independent quantities take on their uniform, time-independent thermal equilibrium values. This is reasonable both for a dilute spin-polarized gas, in which case density fluctuations decay rapidly on the time scale relevant to the spin dynamics, and in a degenerate Fermi liquid for which the spin polarization is a small fraction of the total density. (This decoupling of density and spin fluctuations is, however, not justified, for example, close to a metal insulator transition where both spin and density fluctuations become localized, in which case the full, nonlinear mode-coupling problem must be considered.)

With this simplification, one can extract from (5) the following equation of motion for the polarization density, $\sigma_l(\mathbf{R},T,\mathbf{k},\omega)\equiv\text{Tr}\{\tau_l G^<(\mathbf{R},T,\mathbf{k},\omega)\}$:

$$\begin{aligned} [\partial_T\boldsymbol{\sigma}(\mathbf{R},T,\mathbf{k},\omega)]_{\text{rot}} = {} & 2\mathbf{\Omega}(\mathbf{R},T,\mathbf{k},\omega)\times\boldsymbol{\sigma}(\mathbf{R},T,\mathbf{k},\omega)-[\Gamma_0(k,\omega)\boldsymbol{\sigma}(\mathbf{R},T,\mathbf{k},\omega)+\mathbf{\Gamma}(\mathbf{R},T,\mathbf{k},\omega)f(\omega)A_0(\mathbf{k},\omega)] \\ & +[(A_0(\mathbf{k},\omega)\mathbf{\Sigma}^<(\mathbf{R},T,\mathbf{k},\omega)+\mathbf{A}(\mathbf{R},T,\mathbf{k},\omega)f(\omega)\Gamma_0(\mathbf{k},\omega)] \\ & -[\nabla_k\Omega_0(\mathbf{k},\omega)\cdot\nabla_R\boldsymbol{\sigma}(\mathbf{R},T,\mathbf{k},\omega)-\nabla_R\mathbf{\Omega}(\mathbf{R},T,\mathbf{k},\omega)\cdot\nabla_k A_0(\mathbf{k},\omega)f(\omega)] . \end{aligned} \tag{12}$$

Above, the subscript "rot" denotes the transformation to the rotating frame; also we made use of the definitions $\mathbf{G}^<=(n1+\underline{\sigma}\cdot\underline{\tau})/2$, $\mathbf{E}+\mathbf{\Sigma}_c+\underline{\mathbf{H}}_0\cdot\underline{\tau}=(\Omega_0 1+\underline{\mathbf{\Omega}}\cdot\underline{\tau})$, $\mathbf{\Sigma}=(\Sigma_0 1+\underline{\mathbf{\Sigma}}\cdot\underline{\tau})$, $\mathbf{\Gamma}=(\Gamma_0 1+\underline{\mathbf{\Gamma}}\cdot\underline{\tau})$, $\mathbf{A}=(A_0 1+\underline{\mathbf{A}}\cdot\underline{\tau})/2$, and of the thermal equilibrium identities $n(\mathbf{R},T,\mathbf{k},\omega)\approx n(\mathbf{k},\omega)=f(\omega)A_0(\mathbf{k},\omega)$, and $\Sigma_0^<(\mathbf{R},T,\mathbf{k},\omega)\approx\Sigma_0^<(\mathbf{k},\omega)=f(\omega)\Gamma_0(\mathbf{k},\omega)$, where $f(\omega)=(e^{\beta\omega}-\epsilon)^{-1}$ is either the Bose-Einstein ($\epsilon=+1$) or Fermi-Dirac ($\epsilon=-1$) distribution functions. [Note that $\Omega_l(\mathbf{R},\mathbf{k},\omega)=\text{Re}\Sigma_l(\mathbf{R},\mathbf{k},\omega)-\delta h_l(\mathbf{r})$ is the real part of the magnetic self-energy in the rotating frame, and $\delta h_l(\mathbf{r})$ is the inhomogeneous contribution to the external field ($\mathbf{H}=\mathbf{H}_0+\delta\mathbf{h}$) which survives the transformation to the rotating frame.] Finally, in writing (12) we have ignored both commutators and anti-commutators that involve the real part of the Green's function, **G**. As already mentioned in the previous section, and as will be shown in detail below, these terms turn out to be negligible for systems of well defined quasiparticles.

In the steady state limit [i.e., for $[\partial_T(\ldots)]_{\rm rot}=0$], and after Fourier transforming with respect to **R** (**P** is the conjugate variable), Eq. (12) leads to the following relations satisfied by $\mathbf{\Lambda}_l(\mathbf{P}=0,T\to\infty,\mathbf{k},\omega)\equiv\mathbf{\Lambda}_l(\mathbf{k},\omega)$ (the vertex functions):

$$\Gamma_0(\mathbf{k},\omega)\mathbf{\Lambda}_z(\mathbf{k},\omega)+A_0(k,\omega)[f(\omega)\nabla_P\Gamma_z(\mathbf{P},T\to\infty,\mathbf{k},\omega)-\nabla_P\Sigma_z^<(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}$$
$$=-i[\sigma_z(\mathbf{k},\omega)\nabla_k\Omega_0(\mathbf{k},\omega)-\Omega_z(\mathbf{k},\omega)\nabla_k n(\mathbf{k},\omega)]\,, \quad (13a)$$

$$[\Gamma_0(\mathbf{k},\omega)-2i\Omega_z(\mathbf{k},\omega)]\mathbf{\Lambda}_+(\mathbf{k},\omega)+2i\sigma_z(\mathbf{k},\omega)[\nabla_P\Omega_+(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}$$
$$+A_0(\mathbf{k},\omega)[f(\omega)\nabla_P\Gamma_+(\mathbf{P},T\to\infty,\mathbf{k},\omega)-\nabla_P\Sigma_+^<(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}$$
$$=-i[\sigma_+(\mathbf{k},\omega)\nabla_k\Omega_0(\mathbf{k},\omega)-\Omega_+(\mathbf{k},\omega)\nabla_k n(\mathbf{k},\omega)]\,. \quad (13b)$$

In writing (13) we have ignored the term involving $\nabla_P A_z$ which is negligible for our purpose (the justification will be given below for each particular example). Finally, we note that all terms in Eq. (13b) are infinitesimally small in the steady state limit (since the transverse polarization vanishes in equilibrium); ultimately we are, however, interested in the steady-state limit of $\mathbf{\Lambda}_+(\mathbf{k},\omega)/\sigma_+(T)$ which is finite.

The remaining task is to formulate a model for the self-energy, which allows us to calculate $\nabla_P\Sigma_l^<$ in terms of $\nabla_P\sigma_l^<$. In the next section we discuss the self energy of a dilute spin-polarized quantum gas to lowest order in the density (the T-matrix approximation). In that case $\nabla_P\Sigma_l^<$ is given by an integral of $\nabla_P\sigma_l^<$ (the vertex function) with a kernel depending only on the known (momentum dependent) scattering cross sections and the equilibrium distribution function. The solutions to the resulting integral Eqs. (13) then determine the diffusion coefficient through Eq. (11). In the next two sections (IV and V) we discuss Eqs. (13) in the context of weakly interacting spin polarized gases and a degenerate normal Fermi liquid, respectively.

IV. THE DILUTE SPIN-POLARIZED GAS

A nondegenerate dilute quantum gas is characterized by two expansion parameters: nr_s^3 (r_s is the scattering length of the interparticle potential) and $n\lambda_T^3$ ($\lambda_T=2\pi\hbar^2\beta/m$ is the thermal de Broglie wavelength). The former parameter measures the average potential energy per particle, and its smallness ensures the validity of the T-matrix approximation. The latter parameter controls the nature of thermal fluctuations. In this paper we will restrict ourselves to $n\lambda_T^3\ll 1$, in which case thermal properties are described by Boltzmann statistics. The interesting problem of the crossover to a Bose condensed regime (which occurs in the degenerate limit $n\lambda_T^3\geq 1$) will be discussed elsewhere. We should note, however, that, strictly speaking, in the degenerate limit the T-matrix approximation is exact *only* to leading order in the density; extending it to higher orders is sometimes possible but requires some care.[10]

We proceed by introducing the T-matrix approximation which amounts to calculating $G^{(2)}_{\alpha\beta,\alpha'\beta'}(1,2;1'2')$, and hence $\Sigma_{\alpha\beta}(1,2)$, by treating exactly the two-particle collisions in the medium of other particles, to lowest order in the density. In the limit of $nr_s^3\ll 1$, the resulting two-body scattering amplitude T can be written in terms of the (off-shell) scattering amplitude in vacuum, $t(k,k';z)$, and differs from the latter through the inclusion of virtual processes which allow for energy and momentum exchange with the medium.[11] We define the many-particle T matrix through[11]

$$T_{\alpha\beta;\alpha'\beta'}(1,2;1',2')$$
$$=V(1-2)\delta(1-1')\delta(2-2')\delta_{\alpha\alpha'}\delta_{\beta\beta'}$$
$$+i\int d3\int d4V(1-2)G_{\alpha\gamma}(1,3)$$
$$\times G_{\beta\delta}(2,4)T_{\gamma\delta;\alpha'\beta'}(3,4;1',2') \quad (14)$$

which is represented graphically in Fig. 1(a). As usual,

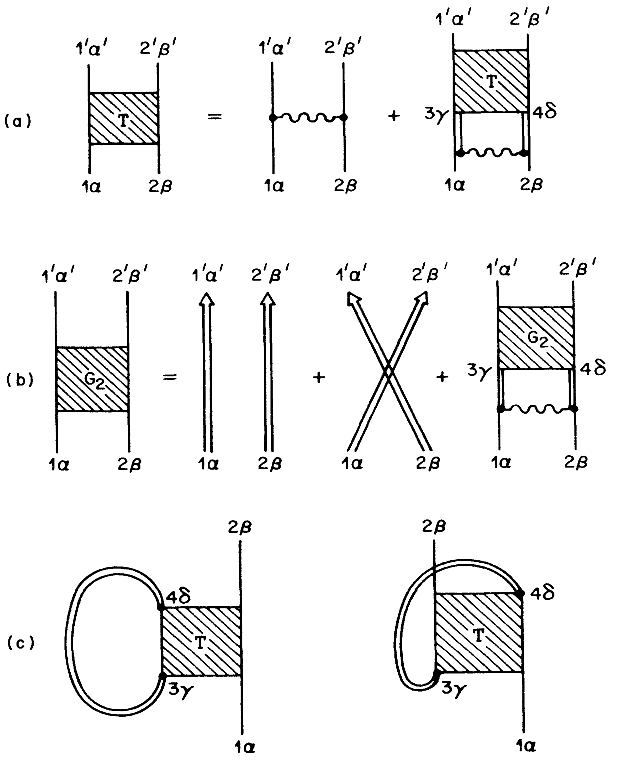

FIG. 1. (a) Diagram representing graphically the integral equation defining the many-body T matrix. (b) The ladder approximation to the two-body Green's function. (c) Within the ladder approximation, the direct and exchange diagram contributing to the one-particle self-energies.

the sum over repeated indices is implicit. We approximate the two-body Green's function retaining only the ladder diagrams shown in Fig. 1(b),

$$\begin{aligned}G^{(2)}_{\alpha\beta,\alpha'\beta'}(1,2;1'2')=&G_{\alpha\alpha'}(1,1')G_{\beta\beta'}(2,2')\\&+\epsilon G_{\alpha\beta'}(1,2')G_{\beta\alpha'}(2,1')\\&+i\int d3\int d4G_{\alpha\gamma}(1,3)G_{\beta\delta}(2,4)\\&\times V(3-4)G^{(2)}_{\gamma\delta,\alpha'\beta'}(3,4;1'2').\end{aligned}\tag{15}$$

By comparing this integral equation with that definition of the T matrix (14), we can easily relate the self-energy defined in Eq. (8) to T,

$$\begin{aligned}\Sigma_{\alpha\beta}(1,2)=i\epsilon\int d3\int d4[&T_{\alpha\gamma;\beta\delta}(1,3;2,4)\\&+\epsilon T_{\alpha\gamma;\delta\beta}(1,3;4,2)]G_{\delta\gamma}(4,3)\ .\end{aligned}\tag{16}$$

As shown in Appendix A, the self-energies may be decomposed as

$$\begin{aligned}\boldsymbol{\Sigma}(\mathbf{r}_1,\mathbf{r}_2;t_1-t_2)=&\boldsymbol{\Sigma}^{\mathrm{HF}}(\mathbf{r}_1,\mathbf{r}_2)\delta(t_1-t_2)\\&+\boldsymbol{\Sigma}^{>}(\mathbf{r}_1,\mathbf{r}_2;t_1-t_2)\Theta(t_1-t_2)\\&+\boldsymbol{\Sigma}^{<}(\mathbf{r}_1,\mathbf{r}_2;t_1-t_2)\Theta(t_2-t_1)\end{aligned}\tag{17}$$

where, in momentum space

$$\begin{aligned}\Sigma^{\gtrless}_{\alpha\beta}(\mathbf{R},T,\mathbf{k},\omega)=\int\frac{d\omega'}{2\pi}\int\frac{d\mathbf{k}'}{(2\pi)^3}\Bigg[&T^{\gtrless}_{\alpha\beta;\beta\delta}\left[\mathbf{R},T,\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}-\mathbf{k}'}{2},\mathbf{k}+\mathbf{k}',\omega+\omega'\right]\\&+\epsilon T^{\gtrless}_{\alpha\gamma;\delta\beta}\left[\mathbf{R},T,\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}'-\mathbf{k}}{2},\mathbf{k}+\mathbf{k}',\omega+\omega'\right]\Bigg]G^{\lessgtr}_{\delta\gamma}(\mathbf{R},T,\mathbf{k}',\omega')\ ,\end{aligned}\tag{18a}$$

$$\begin{aligned}\Sigma^{\mathrm{HF}}_{\alpha\beta}(\mathbf{R},T,\mathbf{k})=\int\frac{d\mathbf{k}'}{(2\pi)^3}\Bigg[&T^{0}_{\alpha\gamma;\beta\delta}\left[\mathbf{R},T,\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}-\mathbf{k}'}{2},\mathbf{k}+\mathbf{k}'\right]\\&+\epsilon T^{0}_{\alpha\gamma;\delta\beta}\left[\mathbf{R},T,\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}'-\mathbf{k}}{2},\mathbf{k}+\mathbf{k}'\right]\Bigg]\int\frac{d\omega'}{2\pi}G^{<}_{\delta\gamma}(\mathbf{R},T,\mathbf{k}',\omega')\end{aligned}\tag{18b}$$

where the spectral representation of the T matrix, which depends explicitly on the center of mass momentum $\mathbf{P}$, is defined in Appendix B [Eq. (B6)]. Above, $\mathbf{T}^0$ is the frequency-independent contribution to the T matrix, while $\mathbf{T}^>$ and $\mathbf{T}^<$ are defined from T (in direct analogy with the definitions of $\boldsymbol{\Sigma}^>$ and $\boldsymbol{\Sigma}^<$ in terms of $\boldsymbol{\Sigma}$ [Eq. (17)]). They are most conveniently calculated in terms of the retarded and advanced part of $\mathbf{T}$ from the optical theorem proved in Appendix B [Eq. (B7)], where we also give the resulting expressions for the self-energies [Eq. (B9)] of the dilute gas.

Finally, we express the many-body T matrix in terms of the (off-shell) scattering amplitude for two particles in the vacuum. This is done by approximating the Green's functions in the integral Eq. (14) for T with the free particle Green's function. The relevant details are given in Appendix C. In the dilute limit, and in an arbitrary external magnetic field, $\underline{\mathbf{H}}(\mathbf{R},T)$, we find

$$\begin{aligned}\mathbf{T}(\mathbf{R},T;\mathbf{k},\mathbf{k}';\mathbf{P},z=\omega+i\eta)=&t(\mathbf{k},\mathbf{k}')11+\int\frac{d^3q}{(2\pi)^3}t(\mathbf{k},\mathbf{q})t^*(\mathbf{k}',\mathbf{q})\\&\times\left\{\left[\left[\omega+i\eta-\frac{P^2}{4m}-\frac{q^2}{m}\right]11-\underline{\mathbf{H}}(\mathbf{R},T)\cdot(\underline{\tau}1+1\underline{\tau})\right]^{-1}+\left[\frac{q^2}{m}-\frac{k'^2}{m}-i\eta\right]^{-1}11\right\}\ .\end{aligned}\tag{19}$$

We used the notation $\mathbf{T}\equiv T_{\alpha\alpha';\beta\beta'}$, $11\equiv\delta_{\alpha\beta}\delta_{\alpha'\beta'}$, $\underline{\tau}1\equiv\underline{\tau}_{\alpha\beta}\delta_{\alpha'\beta'}$, and $1\underline{\tau}\equiv\delta_{\alpha\beta}\underline{\tau}_{\alpha'\beta'}$. A number of points are worth mentioning: (i) when $\underline{\mathbf{H}}(\mathbf{R},T)=0$ and $\omega=P^2/4m+k'^2/m$ expression (19) coincides with the two-particle t matrix in free space; (ii) (19) shows clearly that even to lowest order in the density T differs from t by *virtual processes* which involve exchange of energy and momentum with the environment; (iii) since t is spin independent, all the magnetic field dependence of T comes from the effect of the surrounding medium on the two colliding particles (through the dependence of T on the average frequency, $\omega+\omega'$).

In the dilute limit, a brief examination of Appendix B shows that $\boldsymbol{\Sigma}^{\mathrm{HF}}\sim(nr_s\lambda_T^2)\varepsilon_0$, $\boldsymbol{\Sigma}^>\sim[nr_s^2\lambda_T(1+\epsilon n\lambda_T^3)^2]\varepsilon_0$, $\boldsymbol{\Sigma}^<\sim[n^2r_s^2\lambda_T^4(1+\epsilon n\lambda_T^3)]\varepsilon_0$, $\boldsymbol{\Sigma}^c\sim\Gamma\sim O(\boldsymbol{\Sigma}^>)$, while $\mathbf{G}$ and $\mathbf{A}$ are both of order unity in the density (ε_0 is the average energy per particle). At sufficiently low temperatures $\boldsymbol{\Sigma}^{\mathrm{HF}}+\boldsymbol{\Sigma}^c\sim(nr_s\lambda_T^2)\varepsilon_0$ is much greater than the quasiparticle damping, $\Gamma\sim(nr_s^2\lambda_T)\varepsilon_0$, and in a trivial sense this system shows an analogy with the much more subtle quasiparticle picture introduced for strongly correlated Fermi liquids by Landau. Formally, this analogy can be expressed by ignoring the width of the spectral function A_l to write,

$$G^<(\mathbf{R},T,\mathbf{p},\omega)\approx2\pi f(T,\mathbf{R},\omega)\delta((\omega-\epsilon_p)1+\underline{\mathbf{H}}(\mathbf{R},T)\cdot\underline{\tau}-\boldsymbol{\Sigma}^{\mathrm{HF}}(\mathbf{R},T,\mathbf{p})-\boldsymbol{\Sigma}^c(\mathbf{R},T,\mathbf{p},\omega))\tag{20}$$

and a similar expression (with f replaced by $1+\epsilon f$) for $G^>$. $f(T,\mathbf{R},\omega)$ can be interpreted as a distribution function of "quasiparticles" of energy $\omega_\pm=\epsilon_p\mp|\underline{\mathbf{H}}(\mathbf{R},T)|+\Sigma_\pm(\mathbf{R},T,\mathbf{p},\omega_\pm)$ where $\Sigma_\pm$ denotes the two diagonal components of

$\mathrm{Re}\boldsymbol{\Sigma}=\boldsymbol{\Sigma}^{\mathrm{HF}}+\boldsymbol{\Sigma}^{c}$ in a frame where $\underline{\mathbf{H}}$ lies along the z direction. To leading order in the density $\omega_{\pm}\approx\epsilon_p \mp|\underline{\mathbf{H}}(\mathbf{R},T)|+\Sigma_{\pm}(\mathbf{R},T,\mathbf{p},\epsilon_p\mp|\underline{\mathbf{H}}(\mathbf{R},T)|)$.

The Hartree-Fock and collisional contributions to the self-energy shift $\Sigma_{\pm}$ can be easily calculated in terms of the single-particle t matrix by substituting the expression of T [Eq. (19)] in term of t into the definitions of Σ [Eqs. (18)]. In the nondegenerate limit $n\lambda_T^3 \ll 1$, we show in Appendix C that a simple cancellation of all the virtual processes leads to the expected result

$$\boldsymbol{\Sigma}^{\mathrm{HF}}+\boldsymbol{\Sigma}^{c}=\int\frac{d^3k'}{(2\pi)^3}\int\frac{d\omega'}{2\pi}\mathrm{Re}\left\{\left[t\left[\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}-\mathbf{k}'}{2}\right]+\frac{\epsilon}{2}t\left[\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}'-\mathbf{k}}{2}\right]\right]n(\mathbf{R},T,\mathbf{k}',\omega')\mathbf{1}\right.$$
$$\left.+\frac{\epsilon}{2}t\left[\frac{\mathbf{k}-\mathbf{k}'}{2},\frac{\mathbf{k}'-\mathbf{k}}{2}\right]\underline{\boldsymbol{\sigma}}(\mathbf{R},T,\mathbf{k}',\omega')\cdot\underline{\boldsymbol{\tau}}\right\} \tag{21}$$

where terms of order $(t(\mathbf{k},\mathbf{k}')^3)$ have been neglected. This can be recognized as the Hartree-Fock self energy for a dilute gas of "quasiparticles" interacting through the exact scattering amplitude for two particles in the vacuum. In the nondegenerate limit, Eq. (21) implies a cancellation (independent of temperature and field) between the changes in the single-quasiparticle energy and wave function (self-energy and vertex corrections) which takes place in any approximation scheme which respects conservation laws. Inconsistent treatments have been previously suggested[12,13] and, as expected, lead to unphysical contributions to (21).

As the system becomes degenerate, statistics do not affect the single-particle Hartree-Fock energy which is always the average potential induced by the other particles. However, since the presence of an extra particle in an intermediate state increases (Bosons) or decreases (Fermions) the scattering rate into that state, statistics alter the collisional self-energy. As expected, the additional terms are quadratic in the distribution function and are of opposite sign for fermions and bosons (see Appendix C). This implies that Hartree-Fock and collisional self-energy do not necessarily cancel in the degenerate limit.

In the case of a dilute, nondegenerate gas the (vertex) Eqs. (13) derived in the previous section can be simplified to read

$$\Gamma_0(\mathbf{k},\omega)\boldsymbol{\Lambda}_z(k,\omega)+A_0(\mathbf{k},\omega)[f(\omega)\boldsymbol{\nabla}_P\Gamma_z(\mathbf{P},T\to\infty,\mathbf{k}\omega)-\boldsymbol{\nabla}_P\Sigma_z^{<}(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}=-i\sigma_z(\mathbf{k},\omega)\frac{\mathbf{k}}{m}\ , \tag{22a}$$

$$[\Gamma_0(\mathbf{k},\omega)-2i\Omega_z(\mathbf{k},\omega)]\boldsymbol{\Lambda}_+(\mathbf{k},\omega)+2i\sigma_z(\mathbf{k},\omega)[\boldsymbol{\nabla}_P\Omega_+(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}$$
$$+A_0(\mathbf{k},\omega)[f(\omega)\boldsymbol{\nabla}_P\Gamma_+(\mathbf{P},T\to\infty,\mathbf{k},\omega)-\boldsymbol{\nabla}_P\Sigma_+^{<}(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}=-i\sigma_+(\mathbf{k},\omega)\frac{\mathbf{k}}{m} \tag{22b}$$

with $\boldsymbol{\Omega}(\mathbf{k},\omega)=\varepsilon_k\mathbf{1}+\boldsymbol{\Sigma}^{\mathrm{HF}}(\mathbf{k})+\boldsymbol{\Sigma}^{c}(\mathbf{k},\omega)$. In Eq. (22), the drift term was approximated as $\boldsymbol{\nabla}_k\Omega_0(k,\omega)\approx\boldsymbol{\nabla}_k\varepsilon_k$ as the remaining contributions only lead to small (virial) corrections of order $nr_s^2\lambda_T$ to both thermodynamic and transport properties.[17] Similarly, we have neglected the terms $\Omega_l(k,\omega)\nabla_k n(k,\omega)$ on the right-hand side of Eqs. (13) which can be interpreted as the change in the thermal equilibrium polarization of order $nr_s\lambda_T^2$ due to the exchange field resulting from the interparticle interactions. Also, the contribution $\Gamma_0(\mathbf{k},\omega)\nabla_P A_l(\mathbf{P},T,\mathbf{k},\omega)$ already left out in writing Eqs. (13) in the previous section, is easily seen to be of second order in the density and can indeed be ignored. Finally, we justify neglecting the commutator $[\boldsymbol{\Sigma}^{<},\mathbf{G}]$ in going between Eqs. (5) and (12): to second order in the density, the relevant term takes on the general form

$$\int\frac{d\omega}{2\pi}[\boldsymbol{\Sigma}^{<},\mathbf{G}]=\int\frac{d\omega}{2\pi}\frac{[\boldsymbol{\Sigma}^{<}(\mathbf{R},T,\mathbf{p},\omega),\underline{\mathbf{H}}(\mathbf{R},T)\cdot\underline{\boldsymbol{\tau}})]}{[(\omega-\epsilon_p)^2-H^2(\mathbf{R},T)]}$$
$$\sim\int\frac{d\omega}{2\pi}\delta(\omega-\epsilon_p\pm|\underline{\mathbf{H}}(\mathbf{R},T)|)\mathcal{P}([\omega-\epsilon_p\pm|\underline{\mathbf{H}}(\mathbf{R},T)|][\omega-\epsilon_p\mp|\underline{\mathbf{H}}(\mathbf{R},T)|])^{-1}\ . \tag{23}$$

As a result of the principal value character of $\mathbf{G}$ (denoted by $\mathcal{P}$) and the delta function arising from $\boldsymbol{\Sigma}^{<}$ the right-hand side of Eq. (23) vanishes, as promised.

For the sake of illustration, in Appendix D we solve Eqs. (22) in the limit of s-wave scattering, for a model hard sphere potential. The resulting expressions for the diffusion coefficients read

$$D_z=\tau_D k_B T/m\ , \tag{24a}$$

$$D_{\pm}=\frac{D_z}{1+i\omega_L\tau_D} \tag{24b}$$

where $\tau_D^{-1}\approx(32r_s^2 n/3)\sqrt{\pi/\beta m}$ is the momentum relaxation rate, while $\omega_L\approx-4\pi\hbar r_s\sigma_0/m$ can be interpreted as the precession frequency of the spin current about the equilibrium spin polarization, σ_0; $r_s\approx 0.72$ Å is the s-wave scattering length. While D_z is as usual purely dissipative, the transverse diffusion coefficient, $D_{\pm}$, is complex, signalling a reactive oscillation in $\sigma_{\pm}$. We note that all calculations to date have relied on solutions of the Boltzmann equation based on keeping the first few terms in the Sonine polynomial expansion of the distribution functions.[7] As the convergence of these expansions in the quantum regime is quite slow, we believe that precise, quantitative calculations of the transport coefficients must involve the solution to the integral Eqs. (D9). Such work is presently in progress.[10]

As alluded to in the Introduction, the origin of the reversible component of $D_\pm$ is the precession of the spin current carried by a particle about a momentum-dependent molecular field. From a more microscopic point of view, whenever the thermal wavelength, λ_T, becomes larger than the characteristic scale of variation of the interparticle potential, r_s, exchange in the forward direction becomes dominant. In the course of successive forward scattering collisions the spin carried by a moving particle (which contributes to the spin current) precesses about the coherent spin polarization until it is relaxed by incoherent momentum scattering.

It is worth mentioning that, apart from forward scattering there are also smaller reversible contributions coming from momentum changing "latteral scattering" collisions. In the course of such a collision (i) the spin current is relaxed by the direct and exchange contributions to the cross-section and (ii) the spin of the particles in the final state precess about their average polarization as a result of the interference between direct and exchange processes (final-state interference). This contribution is significant only when the thermal wavelength λ_T is of the order of r_s, the range of the interparticle potential and vanishes in the classical limit (where exchange processes no longer contribute) as well as at the lowest temperature (where the s-wave approximation to the scattering amplitude is valid). In spin-polarized hydrogen these "lateral scattering" contributions become relevant at temperature of order $T \sim 2^\circ$ K.

Macroscopic spin oscillations become observable whenever the precession rate of the spin current is larger than the momentum relaxation rate, in which case the imaginary part of $D_\pm$ is larger than the dissipative (real) part. As discussed in Ref. 2, the long time ($T >> \tau_c$) spin dynamics is determined by the steady state form of the spin current,

$$\underline{\mathbf{J}}^i = -\{D_0/[1+(\mu\sigma_0)^2]\}[\nabla^i\underline{\sigma} + \epsilon\mu\underline{\sigma}_0 \times \nabla^i\underline{\sigma} + \mu^2\underline{\sigma}_0(\underline{\sigma}_0\cdot\nabla^i\underline{\sigma})] \; .$$

From a more macroscopic point of view the reactive behavior can then be traced to the second component of $\underline{\mathbf{J}}^i$, which is normal to the average polarization, $\underline{\sigma}$, *and* its gradient, $\nabla_R\underline{\sigma}$. This is similar to the Hall effect where the Hall current (analogous to the spin current) is perpendicular to the driving electric field (analogous to the gradient $\nabla\underline{\sigma}$).

Finally, we mention that, even though the existence of spin oscillations (damped by diffusion) imply a rapid precession of the spin current during a collision time, τ_c, the resulting mode is hydrodynamic: on the time scale of its oscillation frequency ($\sim \mathrm{Im} D_\pm k^2$) the momentum current has already reached its local equilibrium value.

V. THE SPIN DYNAMICS OF NORMAL FERMI LIQUIDS

Spin waves in degenerate Fermi liquids have been discussed in great detail in the context of electrons in metals and liquid ^{3}He. The theoretical work involves the solution of the Landau-Silin kinetic equations for the quasiparticle distribution function.[5,6,14,15] In this section we will discuss the spin dynamics of degenerate Fermi liquids in the hydrodynamic regime by adopting the point of view developed above for dilute gases. The results are, of course, well known. The merit and main aim of our approach is to emphasize the conceptual connection between the physics of the damped spin waves in these seemingly unrelated systems. As in dilute gases, (hydrodynamic) spin oscillations in Fermi liquids are a consequence of the precession of the spin current carried by a quasiparticle about its own molecular field. This field is, in turn, due to exchange in the course of the forward scattering of two quasiparticles, and is parametrized in terms of the usual Landau Fermi liquid parameters. The latteral scattering contribution to the oscillation is in this case strongly inhibited by the reduction of the final states available in collisions due to the Pauli principle. To our knowledge the first Green's functions derivation of transport in Fermi liquids was given by Wölfle.[20]

The analogy with the case of the weakly interacting gas can be quantified by resorting to Landau's picture in which the physics of a *strongly* correlated Fermi liquid at temperatures and excitation frequencies low compared to the Fermi energy can be understood in terms of the properties of a gas of *weakly interacting* "quasiparticles" in the presence of a set of "molecular fields:" the phase space available for scattering is restricted by the Pauli principle and thus, in the immediate vicinity of the Fermi surface quasiparticles have arbitrarily long lifetimes. Below, we will make use of two of the formal properties of the single-particle Green's function implied by this picture: (i) close to the Fermi surface $\boldsymbol{\Gamma} \ll \mathrm{Re}\boldsymbol{\Sigma}$, and due to the Pauli principle the momentum dependence of $\boldsymbol{\Gamma}$ is weak [i.e., $\nabla_k \boldsymbol{\Gamma}(\mathbf{k},\omega) \sim \boldsymbol{\Gamma}(k_F,\omega)/k_F$]; (ii) even though the main contribution to $\mathrm{Re}\boldsymbol{\Sigma}(\mathbf{k},\omega)$ arises from forward scattering (see below), the contribution from virtual scattering processes gives rise to a strong momentum and frequency dependence of $\mathrm{Re}\boldsymbol{\Sigma}$ for $\omega \approx 0$ and $k \approx k_F$ and thus $\nabla_k \mathrm{Re}\boldsymbol{\Sigma}$ and $\partial_\omega \mathrm{Re}\boldsymbol{\Sigma}$ cannot be ignored.

Following the above discussion, *in the immediate vicinity of the Fermi surface* we parametrize the single-particle Green's function as

$$\mathbf{G}^{\gtrless}(\mathbf{R},T,\mathbf{k},\omega) = \mathbf{A}(\mathbf{R},T,\mathbf{k},\omega) f^{\gtrless}(\mathbf{R},T,\mathbf{k},\omega) \; , \tag{25a}$$

$$\begin{aligned}\mathbf{A}(\mathbf{R},T,\mathbf{k},\omega) &\approx 2\pi\delta((\omega-\varepsilon_k-\mu)\mathbf{1}-\underline{\mathbf{H}}(\mathbf{R})\cdot\underline{\tau}-\mathrm{Re}\boldsymbol{\Sigma}(\mathbf{R},T,\mathbf{k},\omega)) \\ &\approx \pi\{z_+(\omega)[\mathbf{1}+\underline{\nu}(\mathbf{R},T)\cdot\underline{\tau}]\delta(\omega-\mu-|\underline{\mathbf{H}}(\mathbf{R})|-E_+(\mathbf{R},T,\mathbf{k})) \\ &\quad + z_-(\omega)[\mathbf{1}-\underline{\nu}(\mathbf{R},T)\cdot\underline{\tau}]\delta(\omega-\mu+|\underline{\mathbf{H}}(\mathbf{R})|-E_-(\mathbf{R},T,\mathbf{k}))\}\end{aligned} \tag{25b}$$

where we have defined $f^{<}(\omega)\equiv f(\mathbf{R},T,\mathbf{k},\omega)$, $f^{>}(\mathbf{R},T,\mathbf{k},\omega)\equiv 1-f(\mathbf{R},T,\mathbf{k},\omega)$; $z_{\pm}(\omega)=[1-\partial\mathrm{Re}\Sigma_{\pm}\times(\mathbf{R},T,\mathbf{k},\omega)/\partial\omega]^{-1}$ are renormalization functions; $\underline{\nu}$ is a unit vector in the direction of $\underline{\mathbf{H}}(\mathbf{R})+\underline{\boldsymbol{\Omega}}(\mathbf{R},T,\mathbf{k},\omega)$; finally, $E_{\pm}(\mathbf{R},T,\mathbf{k})$ represent the energies of spin-up (+) and spin-down (−) quasiparticles in the rotating frame, and satisfy the implicit equations:

$$E_{\pm}(\mathbf{R},T,\mathbf{k})=\Omega_0(\mathbf{R},T,\mathbf{k},E_{\pm}(\mathbf{R},T,\mathbf{k}))\pm|\underline{\boldsymbol{\Omega}}(\mathbf{R},T,\mathbf{k},E_{\pm}(\mathbf{R},T,\mathbf{k}))|\ . \tag{26}$$

Together with Eqs. (25) this relation defines the local equilibrium quasiparticle distribution functions, $f_{\pm}(\mathbf{R},T,\mathbf{k})\equiv f(\mathbf{R},T,\mathbf{k},\omega=E_{\pm}(\mathbf{k}))$.

Within Fermi liquid theory, a local equilibrium state can be described in terms of a dilute gas of quasiparticles, with quasiparticle number and spin distribution functions $\delta n(\mathbf{R},T,\mathbf{k})$ and $\delta m_z(\mathbf{R},T,\mathbf{k})$, respectively. As usual, the quasiparticle energy (in the rotating frame), $E_{\pm}(\mathbf{R},T,\mathbf{k})$, can be parametrized as

$$\begin{aligned}E_{\pm}(\mathbf{R},T,\mathbf{k})=E_0(\mathbf{k})+\Sigma_{k'}\{&F^{s}(\mathbf{k},\mathbf{k}')\delta n(\mathbf{R},T,\mathbf{k}')\\ &\pm[F^{a}(\mathbf{k},\mathbf{k}')-|\underline{\mathbf{H}}(\mathbf{R})|N(\varepsilon_F)]\\ &\times\delta m_z(\mathbf{R},T,\mathbf{k}')\}/N(\varepsilon_F)\ .\end{aligned} \tag{27}$$

It contains (i) a kinetic energy contribution, $E_0(k)$, which defines the effective mass through $\nabla_k E_0(k)=\mathbf{k}/m_*$; and (ii) "molecular fields," proportional to the number- and spin-densities of other excited quasiparticles which incorporate the effects of (forward) interparticle scattering. The symmetric (s) and antisymmetric (a) Fermi liquid parameters, $F^{s,a}(\mathbf{k},\mathbf{k}')$, can be evaluated on the Fermi surface, and depend solely on the angle between the wavevectors $\mathbf{k}$ and $\mathbf{k}'$. Below, we make use of the standard decomposition in terms of spherical harmonics[21]

$$F^{s,a}(\mathbf{k}\cdot\mathbf{k}')=\frac{4\pi}{N(\varepsilon_F)}\sum_{l,m}F_l^{s,a}Y_{lm}^{*}(\hat{\mathbf{k}}')Y_{lm}(\hat{\mathbf{k}})\ . \tag{28}$$

Also, we limit ourselves to a model in which the momentum relaxing collisions are due to impurity scattering, the dominant mechanism in metals at low temperatures. For the sake of simplicity we will assume that the impurity scattering amplitude, $W(\hat{\mathbf{k}},\hat{\mathbf{k}}')$, is spin independent and isotropic in space, and can thus be decomposed in terms of spherical harmonics

$$W(\hat{\mathbf{k}},\hat{\mathbf{k}}')=\frac{4\pi}{N(\varepsilon_F)}\sum_{lm}\frac{1}{\tau_l(\varepsilon_F)}Y_{lm}^{*}(\hat{\mathbf{k}}')Y_{lm}(\hat{\mathbf{k}}) \tag{29}$$

where $\tau_l(l\geq 0)$ is the scattering time in the channel with angular momentum l, and $N(\varepsilon_F)$ is the quasiparticle density of states at the Fermi energy. The self-energies, $\boldsymbol{\Sigma}^{>}$ and $\boldsymbol{\Sigma}^{<}$, can then be written as

$$\boldsymbol{\Sigma}^{\gtrless}(\mathbf{R},T,\mathbf{k},\omega)=\int\frac{d^3k'}{(2\pi)^3}W(\hat{\mathbf{k}},\hat{\mathbf{k}}')\mathbf{G}^{\gtrless}(\mathbf{R},T,\mathbf{k}',\omega) \tag{30}$$

which in thermal equilibrium reduces, with the help of Eq. (25), to

$$\begin{aligned}\boldsymbol{\Sigma}^{\gtrless}(\mathbf{k},\omega)&=\int\frac{d^3k'}{(2\pi)^3}W(\hat{\mathbf{k}},\hat{\mathbf{k}}')f^{\gtrless}(\omega)\,\mathbf{A}(\mathbf{k}',\omega)\\ &\approx\pi[z_{+}(\omega)(1+\underline{\nu}\cdot\underline{\tau})N_{+}(\mu)\\ &\qquad+z_{-}(\omega)(1-\underline{\nu}\cdot\underline{\tau})N_{-}(\mu)]\tau_0(\varepsilon_F)^{-1}f^{\gtrless}(\omega)\\ &=\boldsymbol{\Gamma}(\mathbf{k},\omega)f^{\gtrless}(\omega)\ .\end{aligned} \tag{31}$$

Above,

$$N_{\pm}(\mu)\simeq\int[d^3k'/(2\pi)^3]\delta(\omega+\mu-[E_{\pm}(\mathbf{k})\pm\underline{\mathbf{H}}\cdot\underline{\nu}])$$

are the densities of spin-up (+) and spin-down (−) quasiparticle states, at a frequency $\omega\ll\mu$ away from the Fermi surface.

We are now in position to examine Eqs. (13) for the vertex functions. We first note that, with the parametrization (25), the omission in (13) of the contributions arising from the commutator $[\boldsymbol{\Sigma}^{<},\mathbf{G}]$ is justified since the argument of the delta function (coming from $\boldsymbol{\Sigma}^{<}$) is removed from the range of the ω integration through the definition of $\mathbf{G}$ as a principal part. Also, by considering the simple model implied by (31) it is easy to see that the contributions $A_0(\mathbf{k},\omega)\nabla_P\Gamma_l(\mathbf{P},T,\mathbf{k},\omega)$ and $\Gamma_0(\mathbf{k},\omega)\nabla_P A_l(\mathbf{P},T,\mathbf{k},\omega)$ to (13) only lead to corrections of order $(H_0/\tau\varepsilon_F^2)\ll 1$ to the diffusion coefficients, and are indeed negligible, as already mentioned above. (Here, H_0 is a measure of the Zeeman energy, τ is of the order of the collision time and ε_F is the Fermi energy.)

We begin with the calculation of the longitudinal diffusion coefficient

$$\begin{aligned}D_z&=\lim_{T\to\infty}[3m\delta\sigma_z(T)]^{-1}i\int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\mathbf{k}\cdot\boldsymbol{\Lambda}_z(\mathbf{k},\omega)\\ &=\lim_{T\to\infty}\left[\frac{iN(\varepsilon_F)k_F}{3m\delta\sigma_z(T)}\right]\int\frac{d\omega}{2\pi}\lambda_z(\omega)\end{aligned} \tag{32}$$

where we have made use of the fact that, since all fluctuations are concentrated close to the Fermi surface, $\boldsymbol{\Lambda}_z$ only depends on the direction of $\mathbf{k}$ and by symmetry must be proportional to the unit vector, $\hat{\mathbf{k}}$: $\boldsymbol{\Lambda}_z(\mathbf{k},\omega)\equiv\delta(E_k-\varepsilon_F)\hat{\mathbf{k}}\lambda_z(\omega)$ where we have used $(E_{\pm}(k)\approx E_k)$. We proceed by (i) taking the dot product of (13a) with $\mathbf{k}$, (ii) dividing the resulting equation by $\boldsymbol{\Gamma}_0(\mathbf{k},\omega)$, and finally (iii) integrating both sides over $\mathbf{k}$ and ω. The integral of the first term (on the left-hand side) is, of course, $N(\varepsilon_F)\int(d\omega/2\pi)\lambda_z(\omega)$. After using the expression

$$\begin{aligned}[\nabla_P\Sigma_z^{<}(\mathbf{P},T\to\infty,\mathbf{k},\omega)]_{P=0}&=\int\frac{d^3k'}{(2\pi)^3}W(\hat{\mathbf{k}},\hat{\mathbf{k}}')\boldsymbol{\Lambda}_z(k',\omega)\\ &=\tau_1^{-1}(\varepsilon_F)\lambda_z(\omega)\hat{\mathbf{k}}\end{aligned} \tag{33}$$

the second term (with a negative sign) reduces to

$$\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} A_0(\mathbf{k},\omega)\mathbf{k}\cdot\nabla_P \Sigma_z^<(\mathbf{P},T\to\infty,\mathbf{k},\omega)|_{P=0}/\Gamma_0(\mathbf{k},\omega) \approx \frac{\tau_0(\varepsilon_F)}{\tau_l(\varepsilon_F)} \left[\int \frac{d\omega}{2\pi}\lambda_z(\omega)\right] k_F N(\epsilon_F) \tag{34}$$

where we have ignored small terms arising from the polarization dependence of $A_0(\mathbf{k},\omega)$ and $\Gamma_0(\mathbf{k},\omega)$.

To calculate the appropriate integral of the right-hand side [of Eq. (13a)], we first make use of the identity [derived from Eq. (26)]

$$\nabla_k E_\pm(\mathbf{k}) z_\pm^{-1}(\mathbf{k},\omega=E_\pm(\mathbf{k})) = \nabla_k(\Omega_0(\mathbf{k},\omega)\pm|\boldsymbol{\Omega}(\mathbf{k},\omega)|)|_{\omega=E(k)} \tag{35}$$

which, after ignoring the small terms proportional to the spin polarization, reduces to $\nabla_k \Omega_0(\mathbf{k},\omega)|_{\omega=E_0(k)} \approx z^{-1}\nabla_k E_0(\mathbf{k}) = z^{-1}\mathbf{k}/m^*$. (Here we have also ignored any further dependence of the renormalization functions on **k** and ω, and on the polarization.) In addition, by neglecting the **k** and ω dependence of Γ (recall that the frequency and momentum dependence of Γ are weak close to the Fermi surface), and with the help of Eq. (27), we can write

$$\begin{aligned}\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \mathbf{k}\cdot[\Omega_z(\mathbf{k},\omega)\nabla_k A_0(\mathbf{k},\omega) f(\omega)]/\Gamma_0(\mathbf{k},\omega) &\approx \int \frac{d^3k}{(2\pi)^3} \frac{h_z(\mathbf{k})}{z\Gamma_0(\mathbf{k},E_0(\mathbf{k}))}[z\mathbf{k}\cdot\nabla_k n(E_0(\mathbf{k}))] \\ &= -\left[1+\left[\frac{F_1^a}{3}\right]\right]\left[\frac{k_F^2\tau_0(\varepsilon_F)}{zm^*}\right][F_0^a\sigma_z(\mathbf{P},T)-2HN(\varepsilon_F)]\ .\end{aligned} \tag{36}$$

Here we have used the relation $\nabla_k n(E_0(\mathbf{k})) \approx -(1+F_1^a/3)\delta(E_0(\mathbf{k})-\varepsilon_F)\mathbf{k}/m^*$;[14] also, the frequency dependence of $\Omega_l(\mathbf{k},\omega)$ is ignored (an approximation already implied above through the neglect of the polarization dependence of the renormalization parameter, z). $\Omega_z(\mathbf{k},\omega)$ is then replaced by the local magnetic field (here given in units of energy), $z^{-1}h_z(\mathbf{k}) = z^{-1}[F_0^a\sigma_z(\mathbf{P},T)-2HN(\varepsilon_F)]$. The relevant integral then becomes

$$\begin{aligned}&\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \mathbf{k}\cdot[A_z(\mathbf{k},\omega)\nabla_k \Omega_0(\mathbf{k},\omega) - \Omega_z(\mathbf{k},\omega)\nabla_k A_0(\mathbf{k},\omega)] f(\omega)/\Gamma_0(\mathbf{k},\omega) \\ &\qquad \approx \left[\frac{1+F_1^a}{3}\right]\left[\frac{k_F^2\tau_0(\varepsilon_F)}{zm^*}\right][(1+F_0^a)\sigma_z(\mathbf{P},T)-2HN(\epsilon_F)] \\ &\qquad = \left[\frac{1+F_1^a}{3}\right]\left[\frac{k_F^2\tau_0(\varepsilon_F)}{zm^*}\right](1+F_0^a)\delta\sigma_z\end{aligned} \tag{37}$$

where the last equality in (37) follows from the formula for the equilibrium magnetization induced by a magnetic field H, $\sigma_0 = 2HN(\varepsilon_F)/(1+F_0^a)$ (we use units in which the magnetic moment is equal to unity).

Finally, Eqs. (13a), (34), and (37) can be combined to give the expected result

$$D_z = (1+F_0^a)v_F^2\tau_{\mathrm{tr}}/3 \tag{38}$$

where $v_F = k_F/m^*$ is the Fermi velocity, and the transport time, τ_{tr}, is defined by $\tau_{\mathrm{tr}}^{-1} = z(m/m^*)(\tau_0^{-1}-\tau_1^{-1})/(1+F_1^a/3)$. Two comments concerning the definition of τ_{tr} are in order: (i) note that the width of the quasiparticle spectral function is $z\Gamma$ and thus the quasiparticle collision time, τ_q, differs from τ by a factor of z (i.e., $\tau_q = \tau/z$); (ii) the additional factor of $(m/m^*)/(1+F_1^a/3)$ reflects the fact that the quasiparticle collisions lead to relaxation to local rather than global equilibrium.[14] The transport time, τ_{tr}, can thus be interpreted as the relaxation time of the quasiparticle momentum. Finally, we note that in the hydrodynamic limit ($k_FL \gg 1$; where $L \sim Q^{-1}$ is the scale of variation of the external perturbation) the higher spherical harmonics can be ignored.

The calculation of the transverse diffusion coefficient involves two additional contributions [the second and third terms on the left-hand side of (13b)], which are ultimately responsible for the reversible spin oscillations. By using definitions (27) and (28) [and after again ignoring the frequency dependence of $\Omega_l(k,\omega)$] we can approximate $\Omega_z(k,\omega) \approx F_0^a\sigma_0/z$, and $[\nabla_P\Omega_+(P,T\to\infty,k,\omega)]_{P=0} \approx F_1^a\Lambda_+(k,\omega)/3z$. The remaining contributions are treated exactly as in the longitudinal case. Following the procedure previously outlined, we obtain the expected result[14]

$$D_+ = \frac{D_z}{1-2i(f_0^a-f_1^a/3)\sigma_0\tau_{\mathrm{tr}}/(1+F_1^a/3)} \tag{39}$$

where $f_l^a \equiv F_l^a/N(\varepsilon_F)$.

The new derivation of the well known results (38) and (39) emphasizes the similarity between the damped spin oscillations discovered in spin polarized *dilute* quantum gases and the analogous effects in degenerate, strongly interacting Fermi liquids. In the latter case the physics can be discussed in terms of a dilute gas of "quasiparticles" interacting with a set of molecular fields which parametrize the main effects of interparticle interactions (through the Fermi liquid parameters). In spite of the conceptual similarities pointed out above, some important differences must, however, be kept in mind: (i) due to the Pauli principle, lateral scattering is inhibited in degenerate liquids; and (ii) in dilute gases complete first principle calculations of any quantities of interest are accessible (at least in principle), while in Fermi liquids the phenomenological Landau parameters must be determined from experiment.

VI. CONCLUSIONS

To put this study in perspective we comment on its relation to other work. The possibility of spin oscillations in spin-polarized quantum gases was first proposed in a brief communication by Bashkin,[8] and independently discussed by Lhuillier and Laloe in an exhaustive study of the Boltzmann equation for polarized quantum gases.[7] In a previous publication[2] we considered this possibility in the context of spin polarized hydrogen and proposed a simple physical picture which stressed the analogy with the damped spin oscillations discussed by Leggett in his work on normal ^{3}He.[6] Our study stimulated the search for "spin waves" in the already ongoing NMR program at Cornell, which ultimately led to the first direct observation[1] of this phenomenon in a spin polarized gas, in the Boltzmann regime. The experimental spectra agreed well with our prediction and calculation of localized spin waves by a magnetic field gradient. Frequency shifts in polarized ^{3}He gas, later observed in Paris,[3] were also given as indirect evidence for the existence of spin oscillations; and anomalous spin echoes (the Leggett-Rice effect) were reported by Gully and Mullin.[4] A comment by Bashkin[12] and several notes by Laloe[13] argued on the basis of perturbation theory in the bare interparticle potential about additional contributions to the spin dynamics not included in earlier work. Those arguments were incorrect for two reasons: (i) even in dilute systems perturbation theory in the bare potential is ill behaved, and the correct low density expansion can only involve physical scattering amplitudes which already incorporate the effect of the hard core; (ii) more importantly, in Ref. 12 the dissipative and reactive contributions were not treated on equal footing. The present paper arose as a necessity to clarify these misconceptions, as well as to put the analogy with Fermi liquid theory presented in Ref. 2 on firmer ground. An incomplete discussion of the analogy with Fermi liquid theory has also been presented in Ref. 17.

The analysis of the spin dynamics we have given in this paper constitutes the basis of Ref. 2. The approach, based on the Kadanoff-Baym formulation of kinetic theory is general and covers condensed as well as uncondensed phases of Fermi and Bose systems, in both degenerate and nondegenerate regimes. In particular, this paper (i) provides a general derivation of the kinetic equations for weakly interacting spin-polarized quantum gases which expands on our earlier work to include the presence of a strong magnetic field; (ii) shows that the cancellation of unphysical "off mass-shell" contributions to the spin dynamics predicted in previous work[12] was a result of an inconsistent treatment of conservation laws; (iii) gives a derivation of spin dynamics a strongly correlated degenerate Fermi liquid using a simple parametrization of the self energy based on Landau Fermi-liquid theory, and clarifies the analogies between the spin oscillations (damped by diffusion) in dilute gases and Fermi liquids; (iv) presents a novel derivation of (spin) diffusion coefficients.

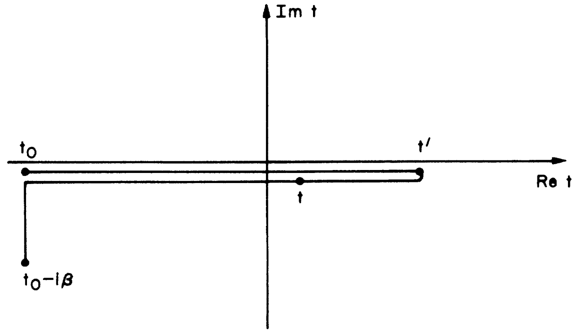

FIG. 2. Contour of integration in the complex time domain introduced by Kadanoff and Baym. Equilibrium theory which involves integration between t_0 and $t_0-i\beta$ appear in this way as a simple limit of the nonequilibrium theory.

We leave for a future publication (i) a complete numerical solution of the integral Eqs. (28) for realistic interparticle potentials and an evaluation of transport coefficients for dilute quantum gases; (ii) a detailed discussion of the crossover between degenerate and nondegenerate regimes; and (iii) a generalization of the kinetic equations to a Bose condensed weakly interacting gas. This approach also has potential applications to the study of transport in random media as well as in the newly discovered "heavy fermion" systems. Investigations in these directions are in progress.

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APPENDIX A: KINETIC EQUATIONS

Using the commutation relations of the creation and annihilation operators, $\Psi_\alpha^+(1)$ and $\Psi_\alpha(1)$, $\Psi_\alpha(r,t)\Psi_\beta^+(r't)-\epsilon\Psi_\beta^+(r't)\Psi_\alpha(rt)=\delta(r-r')\delta_{\alpha\beta}$ and their Heisenberg equations of motion, $i\partial\Psi_\alpha(1)/\partial t=[\Psi_\alpha(1),H]$ one obtains two equations of motion relating the one- and two-particle Green's functions [defined in Eq. (1) and (7) in the text]

$$i\frac{\partial G_{\alpha\beta}(rt;r't')}{\partial t}=\delta(1-1')\delta_{\alpha\beta}-\left[\left[\frac{\nabla_r^2}{2m}+\mu\right]\delta_{\alpha\beta}+H(r)\cdot\tau_{\alpha\gamma}\right]G_{\gamma\beta}(rt;r't)$$
$$-i\epsilon\int dr''V(r-r'')G^{(2)}_{\alpha\gamma;\beta\gamma}(rt,r''t;r't',r''t^{+})\ , \tag{A1}$$

$$-i\frac{\partial G_{\alpha\beta}(rt;r't')}{\partial t'}=\delta(1-1')\delta_{\alpha\beta}-\left[\left[\frac{\nabla_{r'}^2}{2m}+\mu\right]\delta_{\gamma\beta}+H(r')\cdot\tau_{\gamma\beta}\right]G_{\alpha\gamma}(rt;r't')$$
$$-i\epsilon\int dr''V(r-r'')G^{(2)}_{\alpha\gamma;\beta\gamma}(rt;r''t';r't',r''t')\ . \tag{A2}$$

(Above, $\epsilon=+1$ and $\epsilon=-1$ for Bosons and Fermions, respectively; also, throughout the Appendices the underlining or boldfacing of vectors is dropped much of the time, and, as in the text, summation over repeated indices is implied.) After subtracting Eq. (A2) from Eq. (A1), and using the definition of the self-energies in terms of the two-body Green's function [Eqs. (8)], we find the equation of motion of G with respect to the center of mass time variable, T,

$$i\frac{\partial G_{\alpha\beta}(1,1')}{\partial T}=\int d2\{[\epsilon_{\alpha\gamma}(1,2)+\Sigma_{\alpha\gamma}(1,2)]G_{\alpha\beta}(2,1')-G_{\alpha\gamma}(1,2)[\epsilon_{\gamma\beta}(2,1')+\Sigma_{\gamma\beta}(2,1')]\} \tag{A3}$$

where we have defined

$$\epsilon_{\alpha\beta}(1,2)=-\left[\left[\frac{\nabla_{r_1}^2}{2m}+\mu\right]\delta_{\alpha\beta}+H(r_1)\cdot\tau_{\alpha\beta}\right]\delta(t_1-t_2)\delta(r_1-r_2)\ . \tag{A4}$$

Similarly, two equations of motions can be derived for the retarded Green's function. Their sum and difference provide the time evolution with respect to the center of mass (T) and relative (t) times

$$i\frac{\partial G^r_{\alpha\beta}(1,1')}{\partial T}=\int ds\{[\epsilon_{\alpha\gamma}(1,2)+\Sigma^r_{\alpha\gamma}(1,2)]G^r_{\gamma\beta}(21')+G^r_{\alpha\gamma}(1,2)[\epsilon_{\gamma\beta}(2,1')+\Sigma^r_{\gamma\beta}(21')]\}\ , \tag{A5a}$$

$$i\frac{\partial G^r_{\alpha\beta}(1,1')}{\partial t}=\delta(1-1')\delta_{\alpha\beta}+\int d2\{[\epsilon_{\alpha\gamma}(1,2)+\Sigma^r_{\alpha\gamma}(12)]G^r_{\gamma\beta}(21')+G^r_{\alpha\gamma}(1,2)[\epsilon_{\gamma\beta}(21)+\Sigma^r_{\gamma\beta}(21)]\} \tag{A5b}$$

where the retarded self energy is defined in terms of $\Sigma^<$ and $\Sigma^>$ as

$$\Sigma^r(1,1')=\Theta(t-t')[\Sigma^>(1,1')-\Sigma^<(1,1')]\ . \tag{A6}$$

In Eq. (A3), we must specify the limits of the time integrations consistent with the boundary conditions Green's functions and the self-energies. We choose the Kadanoff-Baym contour, $C[t_0\rightarrow t'\rightarrow t\rightarrow t_0-i\beta]$ (see Fig. 2), for which the nonequilibrium many-body perturbation theory is formally identical to that developed for equilibrium systems (the latter being set along the imaginary time axis, $t_0\rightarrow t_0-i\beta$). This procedure is appealing in that it ensures that all approximation schemes are consistent with conservation laws (e.g., fluctuations about thermal equilibrium states are governed at low frequencies and long wavelengths by the laws of hydrodynamics, and excitation spectra in the collisionless regime are consistent with Ward identities).

We calculate the integration along the contour C by separating each analytic part of Σ and G and taking the limit $t_0\rightarrow\infty$. For example, for $t_1'>t_1$ [which leads to the equations for $G^<(1,1')$]

$$\int_C\Sigma_{\beta\gamma}(1,2)G_{\gamma\beta}(2,1')d2=\int_{t_0}^{t_1}(d2)\Sigma^>_{\beta\gamma}(12)G^<_{\gamma\beta}(2,1)+\int_{t_1}^{t_1'}(d2)\Sigma^<_{\alpha\gamma}(1,2)G^<_{\gamma\beta}(21')+\int_{t_1'}^{t_0-i\beta}(d2)\Sigma^<_{\alpha\gamma}(2,1')$$
$$\rightarrow\int_{-\infty}^{+\infty}(d2)[\Sigma^>_{\alpha\gamma}(12)-\Sigma^<_{\alpha\gamma}(12)]G^<_{\alpha\gamma}(21')\Theta(t_1-t_2)$$
$$+\int_{-\infty}^{+\infty}(d2)\Sigma^<_{\alpha\gamma}(1,2)[G^<_{\gamma\beta}(2,1')-G^>_{\gamma\beta}(21')]\Theta(t_1'-t_2)\ . \tag{A7}$$

Similar expressions are obtained for all terms appearing in Eq. (A3).

Next we perform a Fourier transform of the equation of motion in the difference variables $r=r_1-r_1'$ and $t=t_1-t_1'$ using the following conventions:

$$G^>(R,T;p,\omega)=i\int dr\,dt\,\exp[i(\omega t-p\cdot r)]G^>(R,t,r,t)\ , \tag{A8a}$$

$$G^<(R,T;p,\omega)=i\epsilon\int dr\,dt\,\exp[i(\omega t-p\cdot r)]G^<(R,t,r,t) \tag{A8b}$$

for the Green's functions, and

$$\Sigma^{>}(RT;p\omega)=+i\int dr\,dt\,\exp[i(\omega t-p\cdot r)]\Sigma^{>}(RT;rt)\ , \tag{A8c}$$

$$\Sigma^{<}(RT;p\omega)=+i\epsilon\int dr\,dt\,\exp[i(\omega t-p\cdot r)]\Sigma^{<}(RT;rt) \tag{A8d}$$

for the self energies. To calculate the Fourier transform of products of operators we use the identity

$$\int dr_2dt_2\int d(r_1-r_2)\int d(t_1-t_2)a(r_1t_1,r_2t_2)b(r_2t_2,r_1't_1')\exp i[\omega(t_1-t_1')-k\cdot(r_1-r_1')]=\mathcal{Q}a(RTp\omega)b(RTp\omega) \tag{A9a}$$

where the translation operator, $\mathcal{Q}$, is defined as

$$\mathcal{Q}=\exp(i/2)[(\nabla_R^a\nabla_p^b-\nabla_p^a\nabla_R^b)-(\partial_T^a\partial_\omega^b-\partial_\omega^a\partial_T^b)]\ . \tag{A9b}$$

It is then straightforward to evaluate the Fourier transforms of the operators which appear in Eq. (A3) and (A7). For example

$$\begin{aligned}\int_{-\infty}^{+\infty}(dt)e^{i\omega(-t)}\Theta(t)[G_{\gamma\beta}^{<}(-t)-G_{\gamma\beta}^{>}(-t)]&=\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\frac{G_{\gamma\beta}^{>}(\omega')-\epsilon G_{\gamma\beta}^{<}(\omega')}{(\omega-\omega'-i\eta)}\\&=G_{\gamma\beta}(\omega)+iA_{\gamma\beta}(\omega)/2\end{aligned} \tag{A10a}$$

and

$$\int_{-\infty}^{+\infty}(dt)e^{i\omega t}\Theta(t)[\Sigma_{\alpha\gamma}^{>}(t)-\Sigma_{\alpha\gamma}^{<}(t)]=\Sigma_{\alpha\gamma}^{c}(\omega)-i\Gamma_{\alpha\gamma}(\omega)/2\ . \tag{A10b}$$

Above, the variables R,T, and p are implicit, and the definitions of G, A, Σ^c, and Γ are given explicitly in Eq. (4). Finally, using the convention (A7) for the time integration, and the definitions (A8)–(A10) for the Fourier transform, we obtain the kinetic equation (5) for $\mathbf{G}^{<}$, and a similar equation for $\mathbf{G}^{>}$

$$\begin{aligned}i\frac{\partial\mathbf{G}^{>}(RT;p\omega)}{\partial T}\equiv i\partial_T\mathbf{G}^{>}(RT;p\omega)=\mathcal{Q}\Big\{&\mathbf{E}(RT;p)+\mathbf{\Sigma}^{c}(RT;p\omega),\mathbf{G}^{>}(RT;p\omega)]+[\mathbf{\Sigma}^{>}(RT;p\omega),\mathbf{G}(RT;p\omega)]\\&-\frac{i}{2}\{\mathbf{\Gamma}(RT;p\omega),\mathbf{G}^{>}(RT;p\omega)\}+\frac{i}{2}\{\mathbf{A}(RT;p\omega),\mathbf{\Sigma}^{>}(RT;p\omega)\}\Big]\ .\end{aligned} \tag{A11}$$

Equations (5) and (A11) form a closed set of coupled differential equations which, together with appropriate boundary conditions, determine $\mathbf{G}^{>}$ and $\mathbf{G}^{<}$ completely. It is also useful to write the equation for the retarded Green's function:

$$i\partial_T\mathbf{G}^{r}(R,T,p,\omega)=\mathcal{Q}[-\mathbf{H}\cdot\tau+\mathbf{\Sigma}^{r}(R,T,p,\omega),\mathbf{G}^{r}(T,T,p,\omega)]\ , \tag{A12}$$

$$\omega\mathbf{G}^{r}(R,T,p,\omega)=\mathbf{1}+\mathcal{Q}\{[\epsilon_p-(1/8m)\nabla_R^2]\mathbf{1}-\mathbf{H}\cdot\tau-\dot{\mathbf{\Sigma}}^{r}(R,T,p,\omega),\mathbf{G}^{r}(R,T,p,\omega)\}/2 \tag{A13}$$

which are obtained from the Fourier transform of Eqs. (A5) and (A6).

APPENDIX B: OPTICAL THEOREM FOR THE *T*-MATRIX

We begin by rewriting the integral equations [Eq. (9) in the text] as

$$T(x,x';t_1-t_1')=V(r_1-r_2)\delta(x-x')\delta(t_1-t_1')+\int_C dt''\int dx''g(x,x'';t_1-t'')V(r_1-r_2)T(x'',x';t''-t_1') \tag{B1a}$$

or, alternatively

$$\int_C dt''\int dx''[\delta(t_1-t'')\delta(x-x'')-g(x,x'';t_1-t'')V(r_1-r_2)]T(x'',x';t''-t_1')=V(r_1-r_2)\delta(x-x')\delta(t_1-t_1')\ . \tag{B1b}$$

Above we have defined the simplifying notation

$$T_{\alpha\beta;\alpha'\beta'}(12;1'2')=T(x,x';t_1-t_2)\delta(t_1-t_1')\delta(t_2-t_2')\ , \tag{B2a}$$

$$G_{\alpha\beta}(1,2)G_{\alpha'\beta'}(1',2')=g(x,x';t_1-t_2)\delta(t_1-t_1')\delta(t_2-t_2') \tag{B2b}$$

where the variables $x\,(x')$ represent the collection of coordinates and spin indices r_1,r_2,α,β $(r_1',r_2',\alpha',\beta')$; the time integrations are carried out along the Kadanoff-Baym contour (Fig. 2).

To evaluate $T^{>}(\tau)$ $(\tau>0)$ or $T^{<}(\tau)$ $(\tau<0)$, we proceed as in Appendix A, by separating each analytic contribution along C. For example, we find $T^{<}$ from Eq. (B1a)

$$
\begin{aligned}
T^{<}(x,x';t_1-t_1')=\int dx''V(r_1-r_2)\Bigg[&\int_{t_0}^{t_1}(dt'')g^{>}(x,x'';t_1-t'')T^{<}(x'',x';t''-t_1')\\
&+\int_{t_1}^{t_1'}(dt'')g^{<}(x,x'';t_1-t'')T^{<}(x'',x';t''-t_1')\\
&+\int_{t_1'}^{t_0-i\beta}(dt'')g^{<}(x,x'';t_1-t'')T^{>}(x'',x';t''-t_1')\Bigg]\\
=\int(dx'')V(r_1-r_2)\Bigg[&\int_{-\infty}^{+\infty}(dt'')[g_r(x,x'';t_1-t'')T^{<}(x'',x';t''-t_1')\\
&+g^{<}(x,x'';t_1-t'')T^{a}(x'',x';t''-t_1')]\Bigg]
\end{aligned}
\tag{B3a}
$$

where the retarded part of g is defined by Eq. (A6) while $T^a(\tau)=[T^{>}(\tau)-T^{<}(\tau)]\Theta(-\tau)$. A similar decomposition along the contour C for the instantaneous part of Eq. (B1b) yields

$$
\begin{aligned}
&\int dx''\int_{-\infty}^{+\infty}(dt'')[\delta(t_1-t'')\delta(x-x'')-g^r(x,x'';t_1-t'')V(r_1-r_2)]T^r(x'',x';t''-t_1')\\
&\qquad=V(r_1-r_2)\delta(x-x')\delta(t_1-t_1')\ .
\end{aligned}
\tag{B3b}
$$

Eq. (B3a) can be rewritten as

$$
\begin{aligned}
&\int(dx'')\int_{-\infty}^{+\infty}(dt'')[\delta(x-x'')\delta(t_1-t'')-V(r_1-r_2)g^r(x,x'';t_1-t'')]T^{<}(x'',x';t''-t_1')\\
&\qquad=\int(dx'')\int_{-\infty}^{+\infty}(dt'')V(r_1-r_2)g^{<}(x,x'';t_1-t'')T^a(x'',x';t''-t_1)\ .
\end{aligned}
\tag{B4}
$$

Finally, by replacing $V(r_1-r_2)$ on the right-hand side of (B4) by the right-hand side of (B3b), and factorizing the operator $\int dt''[\delta(x-x'')\delta(t_1-t'')-V(r_1-r_2)g^r(x,x'';t_1-t'')]$ on both sides of the resulting equation, we find the generalized Optical Theorem

$$
T^{<}(x,x';t_1-t_1')=\int_{-\infty}^{+\infty}(dt'dt'')\int(dy'dy'')T^r(x,y';t_1-t')g^{<}(y',y'';t'-t'')T^a(y'',x';t''-t_1')\ .
\tag{B5}
$$

To obtain the momentum space representation of (B5) we define the Fourier transform of T with respect to $\rho=r_1-r_2$, $\rho'=r_1'-r_2'$, $r=\frac{1}{2}[(r_1+r_2)-(r_1'+r_2')]$, $\tau=t_1-t_2'$ (with associated Fourier space variables, k,k',P, and ω, respectively) as

$$
T(R,T;k,k';P,\omega)=i\int d\tau\,d\rho\,d\rho'\,dr\exp[i(\omega\tau-\mathbf{P}\cdot\mathbf{r}-\mathbf{k}\cdot\boldsymbol{\rho}+\mathbf{k}'\cdot\boldsymbol{\rho}')]T(r_1r_2r_1'r_2';\tau)
\tag{B6}
$$

where $R=(r_1+r_2+r_1'+r_2')/4$ and $T=(t_1+t_1')/2$. Since the Fourier transforms of T^r and T^a with respect to τ are $T(\ldots;iz=\omega+i\eta)$ and $T(\ldots;iz=\omega-i\eta)$, respectively, the spectral representation of (B6) is given by

$$
-iT^{\gtrless}(R,T;k,k';P,\omega)=\int\frac{d^3qd^3q'}{(2\pi)^6}T(R,T;k,q;P,\omega+i\eta)g^{\gtrless}(R,T;q,q';P,\omega)T(R,T;q',k';P,\omega-i\eta)
\tag{B7}
$$

while the corresponding transform of g is

$$
g^{\gtrless}(R,T;q,q';P,\omega)=-i(2\pi)^3\delta(q-q')\int\frac{d\omega}{2\pi}G^{\gtrless}(R,T;q+P/2,\omega'+\omega/2)G^{\gtrless}(RT;-q+P/2;-\omega'+\omega/2)\ .
\tag{B8}
$$

The optical theorem in the momentum representation can be now used to calculate the self energies, $\Sigma^{\gtrless}$, in the T-matrix approximation. Substituting Eqs. (B7) and (B8) into the defining Eq. (13a) we obtain, after simple manipulations

$$
\begin{aligned}
\Sigma_{\alpha\beta}^{\gtrless}(R,T;p,\omega)=\sum_{\text{spins}}&\int dq\,dk\,dk'\,d\omega'\,d\omega_1d\omega_1'(2\pi)\delta(\omega+\omega'-\omega_1-\omega_1')(2\pi)^3\delta(p+q-k-k')\\
&\times T_{\alpha\xi,\gamma\delta}\left[R,T;\frac{p-q}{2},\frac{k-k'}{2};p+q,\omega+\omega'+i\eta\right]\\
&\times\Bigg[T_{\gamma'\delta';\beta\xi}\left[R,T;\frac{k-k'}{2},\frac{p-q}{2};p+q,\omega+\omega'-i\eta\right]\\
&\qquad+\epsilon T_{\gamma'\delta';\xi\beta}\left[R,T;\frac{k-k'}{2}\,\frac{q-p}{2};p+q,\omega+\omega'-i\eta\right]\Bigg]\\
&\times G_{\gamma\gamma'}^{\gtrless}(R,T;k,\omega_1)G_{\delta\delta'}^{\gtrless}(R,T;k',\omega_1')G_{\xi\xi}^{\lessgtr}(R,T;q,\omega')\ .
\end{aligned}
\tag{B9}
$$

Above we made use of the short hand notations $dk\equiv d^3k/(2\pi)^3$ and $d\omega\equiv d\omega/(2\pi)$, and summation over repeated spin indices is implied. Note that in the nondegenerate limit ($n\lambda_T^3\ll1$; n is the density), $G^{<}$ is of order n while $G^{>}\sim O(1)$. It follows that $\Sigma^{<}$ and $\Sigma^{>}$ are of order n^2 and n, respectively.

APPENDIX C: EXPRESSION OF THE MANY-BODY *T*-MATRIX IN TERMS OF THE TWO-PARTICLE SCATTERING MATRIX IN THE VACUUM

From the Fourier representation of the equation defining T [Eq. (B1a)] and the convention (B6) we can write

$$T(p,p';P,z)=V(p-p')11+\int d^3k\, d^3k'/(2\pi)^6 V(p-k)g(k,k';P,z)T(k'p';Pz)\ . \tag{C1}$$

The notation is the same as that given in connection with Eq. (19). In the dilute limit, we approximate the Green's functions (representing the propagation of particles in intermediate states) by their free particle form, in which case g reduces to

$$g(k,k';P,z)=\frac{(2\pi)^3\delta(k-k')}{\left[\dfrac{z-k^2}{m}-\dfrac{P^2}{4m}\right]11+\underline{\mathbf{H}}\cdot(\underline{\tau}1+1\underline{\tau})}\ . \tag{C2}$$

On the other hand, the scattering amplitude for two particles in vacuum, $t(p,p')$, is defined by the integral equation

$$t(p,p')=V(p-p')+\int\frac{d^3k}{(2\pi)^3}V(p-k)t(k,p')/(e_p'-e_k+i\eta)\ , \tag{C3}$$

where $e_k=k^2/m$. The potential V can now be expressed in terms of the scattering amplitude t by using

$$V(p-p')=t(p,p')+\int d^3q/(2\pi)^3 t(p,q)t(q,p')/(e_q-e_p-i\eta) \tag{C4}$$

as can be verified by substituting (C4) into (C3), and using the generalized optical theorem for t. The second term on the right-hand side of (C4) may be rewritten as

$$\begin{aligned}\int\frac{d^3q}{(2\pi)^3}t(p,q)t(q,p')\{(z-e_p)[(z-e_q)(e_q-e_p-i\eta)]^{-1}-(z-e_q)^{-1}\}\\ =\int\frac{d^3q}{(2\pi)^3}t(p,q)t(q,p')(z-e_p)[(z-e_q)(e_q-e_p-i\eta)]^{-1}-\int\frac{dq^3}{(2\pi)^3}V(p-q)t(q,p')/(z-e_q)\\ -\int\frac{d^3q\,d^3k}{(2\pi)^6}V(p-k)t(k,q)t(q,p')[(e_q-e_k-i\eta)(z-e_q)]^{-1}\end{aligned} \tag{C5}$$

where we have substituted $t(p,q)$ [as given by the right-hand side of Eq. (C3)] into the corresponding term of Eq. (C4). Substituting this result into (C4) we find

$$V(p-p')=\int\frac{d^3k}{(2\pi)^3}\left[\delta(p-k)-\frac{V(p-k)}{z-e_k}\right]\left[t(k,p')+\int\frac{d^3q}{(2\pi)^3}t(k,q)t(q,p')(z-e_k)[(e_q-e_k-i\eta)(z-e_q)]^{-1}\right]\ . \tag{C6}$$

We then conclude that

$$U(p,p';z)=t(p,p')+\int\frac{d^3q}{(2\pi)^3}t(p,q)t(q,p')[(z-e_q)^{-1}+(e_q-e_k-i\eta)^{-1}] \tag{C7}$$

is a solution of the integral equation

$$U(p,p';z)=V(p-p')+\int\frac{d^3k}{(2\pi)^3}V(p-k)U(p,p';z)/(z-\varepsilon_k)\ . \tag{C8}$$

Finally, by identifying (C8) with (C1) and (C2) we find that the many body T matrix can be expressed in terms of U as

$$T(p,p';P,z)=U\left[p,p';\frac{z-P^2}{4m}11+\underline{\mathbf{H}}\cdot(\underline{\tau}1+1\underline{\tau})\right] \tag{C9}$$

which, from (C7), leads immediately to Eq. (14) of the text.

Equation (14) provides the decomposition of the many body T matrix into a frequency independent contribution

$$T^0(p,p')=t(p,p')+\int\frac{d^3q}{(2\pi)^3}t(p,q)t(q,p')/(e_q-e_{p'}-i\eta) \tag{C10}$$

and retarded and advanced components, $T^r(pp';\omega)=T(p,p';z=\omega+i\eta)$ and $T^a(pp';\omega)=T(p,p';z=\omega-i\eta)$, respectively. From (C10) and the optical theorem we find

$$T^0(p,\pm p)=\mathrm{Re}[t(p,\pm p)]+\mathcal{P}\int\frac{d^3q}{(2\pi)^3}t(p,q)t(q,\pm p)/(\epsilon_q-\epsilon_p)\ . \tag{C11}$$

($\mathcal{P}$ is the principal part of the integral.) Using this result we can evaluate the Hartree-Fock self-energy [Eq. (18b)], $\Sigma^{\mathrm{HF}}=\Sigma_0^{\mathrm{HF}}+\Sigma_1^{\mathrm{HF}}$, in terms of the scattering amplitude of two particles in the vacuum, $t(p,p')$:

$$\begin{aligned}\Sigma_0^{\mathrm{HF}}(R,T,p)=&\int\frac{d^3q}{(2\pi)^3}\left[t_R\left[\frac{p-q}{2},\frac{p-q}{2}\right]+\frac{\epsilon}{2}t_R\left[\frac{p-q}{2},\frac{q-p}{2}\right]\right]n(R,T,q)\mathbf{1}\\&+\frac{\epsilon}{2}\int\frac{d^3q}{(2\pi)^3}t_R\left[\frac{p-q}{2},\frac{q-p}{2}\right]\underline{\boldsymbol{\sigma}}(R,T,q)\cdot\underline{\boldsymbol{\tau}}\ ,\end{aligned} \tag{C12a}$$

$$\Sigma_1^{\mathrm{HF}}(R,T,p)=\int\frac{d^3q}{(2\pi)^3}\left[t_+(p-q)+\frac{\epsilon}{2}t_-(p-q)\right]n(R,T,q)\underline{\mathbf{1}}+\frac{\epsilon}{2}\int\frac{d^3q}{(2\pi)^3}t_-(p-q)\underline{\boldsymbol{\sigma}}(R,T,q)\cdot\underline{\boldsymbol{\tau}} \tag{C12b}$$

where

$$\begin{aligned}t_\pm(p-q)&=P\int\frac{d^3k}{(2\pi)^3}t\left[\frac{p-q}{2},k\right]t\left[k,\pm\frac{(p-q)}{2}\right]\Big/\left[e_k-\frac{(p-q)^2}{4m}\right]\\&=2\int\frac{d^3k\,d^3k'}{(2\pi)^6}t\left[\frac{p-q}{2},\frac{k-k'}{2}\right]t\left[\frac{k-k'}{2},\pm\frac{p-q}{2}\right](2\pi)^3\delta(k+k'-p-q)/[(k-k')^2/2m-(p-q)^2/2m]\end{aligned} \tag{C12c}$$

$$=\int\frac{d^3kd^3k'}{(2\pi)^6}t\left[\frac{p-q}{2},\frac{k-k'}{2}\right]t\left[\frac{k-k'}{2},\pm\frac{p-q}{2}\right](2\pi)^3\delta(k+k'-p-q)/(\varepsilon_k+\varepsilon_{k'}-\epsilon_p-\epsilon_q)\ . \tag{C12d}$$

Using the known retarded and advanced T matrix, we can also evaluate the remaining self-energies, $\Sigma^<$, $\Sigma^>$, and Σ^c. In the dilute, nondegenerate limit $\Sigma^<\sim O(r_s^6n^2)$ (and can be ignored), while the latter two quantities are related by

$$\Sigma^c_{\alpha\beta}(R,T,p,\omega)\approx\int\frac{d\omega'}{2\pi}\Sigma^>_{\alpha\beta}(R,T,p,\omega')/(\omega-\omega')\ . \tag{C13}$$

The self-energy, $\Sigma^>_{\alpha\beta}(R,T,p,\omega')$, can be calculated from Eq. (B9), by (i) replacing $T_{\alpha\gamma,\beta\delta}(R,T,(k-k')/2,(p-q)/2,\omega+\omega'+i\eta)$ by the scattering amplitude for two particles in the vacuum, $t((k-k')/2,(p-q)/2)\delta_{\alpha\beta}\delta_{\gamma\delta}$, and (ii) making use of the "quasiparticle" (Hartree-Fock) Green's functions:

$$\mathbf{G}^<(R,T,k,\omega)\approx2\pi\delta((\omega-\epsilon_k)\mathbf{1}-\underline{\mathbf{H}}(R)\cdot\underline{\boldsymbol{\tau}})f(\omega)\ , \tag{C14a}$$

$$\mathbf{G}^>(R,T,k,\omega)\approx2\pi\delta((\omega-\epsilon_k)\mathbf{1}-\underline{\mathbf{H}}(R)\cdot\underline{\boldsymbol{\tau}})\ . \tag{C14b}$$

It is easy to see that the corrections to the approximations implied by both (i) and (ii) are a factor of (r_s/λ_T) smaller than the leading contributions (in hydrogen $r_s/\lambda_T<1$ for temperatures, $T<2$ K). This quasi-particle approximation restricts $\Sigma^>$ on shell where the many-body T matrix coincides with the two-particles t matrix in vacuum. Straightforward algebra leads to

$$\begin{aligned}\Sigma^c(R,T,p,\omega)=&\tfrac{1}{2}\int\frac{d^3q\,d^3k\,d^3k'}{(2\pi)^9}(2\pi)^3\delta(p+q-k-k')\\&\times\left[Q_+\left[\frac{p-q}{2},\frac{k-k'}{2}\right](1+\underline{\boldsymbol{\nu}}\cdot\underline{\boldsymbol{\tau}})/(\omega+\epsilon_q-\varepsilon_k-\varepsilon_{k'}+|\underline{\mathbf{H}}|)\right.\\&\left.+Q_-\left[\frac{p-q}{2},\frac{k-k'}{2}\right](1-\underline{\boldsymbol{\nu}}\cdot\underline{\boldsymbol{\tau}})/(\omega+\epsilon_q-\varepsilon_k-\varepsilon_{k'}-|\underline{\mathbf{H}}|)\right]\end{aligned} \tag{C15a}$$

where

$$\begin{aligned}Q_\pm\left[\frac{p-q}{2},\frac{k-k'}{2}\right]=&\left\{\left[|t((p-q)/2,(k-k')/2)|^2+\frac{\epsilon}{2}t\left[\frac{p-q}{2},\frac{k-k'}{2}\right]t\left[\frac{k-k'}{2},\frac{q-p}{2}\right]\right]n(R,T,q)\right.\\&\left.\pm\frac{\epsilon}{2}t\left[\frac{p-q}{2},\frac{k-k'}{2}\right]t\left[\frac{k-k'}{2},\frac{q-p}{2}\right]\underline{\boldsymbol{\sigma}}(R,T,q)\cdot\underline{\boldsymbol{\nu}}\right\}\ .\end{aligned} \tag{C15b}$$

Above, $\underline{\boldsymbol{\nu}}\equiv\underline{\boldsymbol{\nu}}(R)$ is the direction of the external magnetic field at position R.

Finally, the cancellation of the Σ_1^{HF} [Eq. (C12b)] and Σ^c quasiparticle energy shifts [Eq. (21)], and consequently, to the "molecular field" contribution to the spin dynamics, can be obtained immediately by evaluating Σ^c on "mass shell." Using the notation introduced in connection with Eq. (20) we can write

$$\begin{aligned}\omega_\pm(R,T,p) &\approx [\epsilon_p \mp |\underline{\mathbf{H}}(R)|] + (\tfrac{1}{4})\mathrm{Tr}\{(1\pm\underline{\nu}\cdot\underline{\tau})[\mathbf{\Sigma}^{\mathrm{HF}}(R,T,p)+\mathbf{\Sigma}^c(R,T,p,\epsilon_p \mp |\underline{\mathbf{H}}(R)|]\} \\ &\equiv \epsilon_p \mp |\underline{\mathbf{H}}(R)| + (\tfrac{1}{4})\mathrm{Tr}[(1\pm\underline{\nu}\cdot\underline{\tau})\mathbf{\Sigma}_0^{\mathrm{HF}}(R,T,p)] \ . \end{aligned} \tag{C16}$$

This is the same cancellation which occurs in the derivation of the diffusion coefficients given in Appendix D.

APPENDIX D: TRANSPORT COEFFICIENTS IN A DILUTE GAS

The Boltzmann gas provides a nice illustration of the approach we have used to evaluate transport coefficients since the results can be compared to other approximate methods.

Using the expressions for the self-energies derived in the previous appendices we easily find the integrals defining $\nabla_P\underline{\mathbf{\Gamma}}$ and $\nabla_P\underline{\mathbf{\Sigma}}^<$ in terms of the vertex function $\underline{\mathbf{\Lambda}}$.

$$\nabla_P\underline{\mathbf{\Gamma}}(p,\omega)=\frac{\epsilon}{2}\int dq\, dk\, dk' t\left[\frac{p-q}{2},\frac{k-k'}{2}\right] t\left[\frac{k-k'}{2},\frac{q-p}{2}\right]\underline{\mathbf{\Lambda}}(q)(2\pi)^3\delta(p+q-k-k')(2\pi)\delta(\omega+\epsilon_q-\varepsilon_k-\varepsilon_k') , \tag{D1a}$$

$$\begin{aligned}\nabla_P\underline{\mathbf{\Sigma}}^<(p,\omega)=\int dq\, dk\, dk' \Bigg[&\left| t\left[\frac{p-q}{2},\frac{k-k'}{2}\right]\right|^2 n(k)\underline{\mathbf{\Lambda}}(k')+\frac{\epsilon}{2} t\left[\frac{p-q}{2},\frac{k-k'}{2}\right] t\left[\frac{k-k'}{2},\frac{q-p}{2}\right] \\ &\times[n(k)\underline{\mathbf{\Lambda}}(k')+\underline{\mathbf{\Lambda}}(k)n(k')+i\underline{\sigma}(k)\times\underline{\mathbf{\Lambda}}(k')+i\underline{\mathbf{\Lambda}}(k)\times\underline{\sigma}(k')]\Bigg](2\pi)^3\delta(p+q-k-k') \\ &\times(2\pi)\delta(\omega+\epsilon_q-\varepsilon_k-\varepsilon_k') \ . \end{aligned} \tag{D1b}$$

In Eqs. (D1) we have used the short-hand notation dq for the momentum integration $d^3q/(2\pi)^3$. As alluded in the text, $\Sigma^<$ gives rise to some reactive contributions caused by interference in the final state of a collision, which do not contribute to the entropy production. Similarly, the deviations of the molecular field from its equilibrium value are

$$\nabla_{\mathbf{P}}\underline{\mathbf{\Omega}}(p)=\frac{\epsilon}{2}\int dq\, t\left[\frac{p-q}{2},\frac{q-p}{2}\right]\underline{\mathbf{\Lambda}}(q) \ . \tag{D2}$$

All other functions entering the integral equation for the vertex function [Eq. (13)] involve only equilibrium quantities. For example,

$$\begin{aligned}\Gamma_0(p,\omega)&=\Gamma_d(p,\omega)+\Gamma_{\mathrm{ex}}(p,\omega) \\ &=\int dq\, dk\, dk' \left[\left| t\left[\frac{p-q}{2},\frac{k-k'}{2}\right]\right|^2+\frac{\epsilon}{2} t\left[\frac{p-q}{2},\frac{k-k'}{2}\right] t\left[\frac{k-k'}{2},\frac{q-p}{2}\right]\right] \\ &\quad\times n(q)(2\pi)^3\delta(p+q-k-k')(2\pi)\delta(\omega+\epsilon_q-\varepsilon_k-\varepsilon_{k'}) , \end{aligned} \tag{D3}$$

$$\Omega_z(p,\omega)=\frac{\epsilon}{2}\int dq\, t\left[\frac{p-q}{2},\frac{q-p}{2}\right]\sigma_z(q) \ . \tag{D4}$$

On mass shell, the direct and exchange contribution to the collision rate Γ_0, are expressed in term of scattering cross sections as

$$\Gamma_d(p)=\int dq\frac{|\mathbf{p}-\mathbf{q}|}{m}\sigma^d_{|(p-q)/2|}n(q) , \tag{D5a}$$

$$\Gamma_{ex}(p)=\frac{\epsilon}{2}\int dq\frac{|\mathbf{p}-\mathbf{q}|}{m}\sigma^{\mathrm{ex}}_{|(p-q)/2|}n(q) \tag{D5b}$$

where

$$\sigma^d_{|k|}=\left[\frac{m}{4\pi}\right]^2\int d\Omega_{k'}|t(k,k')|^2 , \tag{D6a}$$

$$\sigma^{\mathrm{ex}}_{|k|}=\left[\frac{m}{4\pi}\right]^2\int d\Omega_{k'}t(k,k')t(k',-k) \ . \tag{D6b}$$

At sufficiently low temperature, the t matrix can be parametrized by a scattering length r_s, defined here as

$$t(k)\simeq-\frac{4\pi r_s}{m}(1-ikr_s)\ . \tag{D7}$$

In this limit, the cross sections $\sigma^d\simeq\sigma^{\text{ex}}$ are approximately equal to $\sigma_0=4\pi r_s^2$. In general, the computation of Γ_0 reduces to a quadrature,

$$\Gamma_0(p)=\frac{\sqrt{2}n}{(\pi m\beta)^{1/2}}\tilde{\Gamma}\left[\left(\frac{\beta}{2m}\right)^{1/2}p\right] \tag{D8a}$$

where

$$\tilde{\Gamma}(v)=2\frac{e^{-v^2}}{v}\int e^{-u^2}\sinh(2uv)u^2\sigma\left[\left(\frac{m}{2\beta}\right)^{1/2}u\right]du\ . \tag{D8b}$$

In the s-wave scattering length approximation, this can be expressed in close form as

$$\tilde{\Gamma}(v)=\sigma_0[e^{-v^2}+(\sqrt{\pi}/2)(1+2v^2)\mathrm{erf}(v)] \tag{D8c}$$

where erf is the error function.

We now turn to the solution of the integral equation for the vertex function. Integrating Eq. (13) over frequency we obtain from Eq. (D1)–(D4) the integral equations for the vertex functions Λ_z and Λ_+

$$\Gamma_0(p)\mathbf{\Lambda}_z(p)+\int dq\ K_z(p,q)\mathbf{\Lambda}_z(q)=\frac{-i\mathbf{p}}{m}\sigma_z(p)\ , \tag{D9a}$$

$$[\Gamma_0(p)-2i\Omega_z(p)]\mathbf{\Lambda}_+(p)+2i\int dq\ K_0(p,q)\mathbf{\Lambda}_+(q)+\int dq\ K_+(p,q)\mathbf{\Lambda}_+(q)=\frac{-i\mathbf{p}}{m}\sigma_+(p) \tag{D9b}$$

where the dynamics of the scattering processes is contained in the kernels K_z, K_0, and K^+:

$$\begin{aligned}K_z(p,q)=-\int du\ dv\Bigg\{\left|t\left(\frac{p-u}{2},\frac{v-q}{2}\right)\right|^2&+\frac{\epsilon}{2}\left[t\left(\frac{p-u}{2},\frac{v-q}{2}\right)t\left(\frac{v-q}{2},\frac{u-p}{2}\right)\right.\\&\left.+t\left(\frac{p-u}{2},\frac{q-v}{2}\right)t\left(\frac{q-v}{2},\frac{u-p}{2}\right)\right]\Bigg\}\sigma_z(v)(2\pi)^3\delta(p+u-v-q)(2\pi)\delta(\epsilon_p+\epsilon_u-\epsilon_v-\epsilon_q)\\&+\frac{\epsilon}{2}\int dk\ dk'\,t\left(\frac{p-q}{2},\frac{k-k'}{2}\right)t\left(\frac{k-k'}{2},\frac{q-p}{2}\right)n(p)(2\pi)^3\delta(p+q-k-k')(2\pi)\delta(\epsilon_p+\epsilon_q-\varepsilon_k-\varepsilon_{k'})\ ,\end{aligned} \tag{D10a}$$

$$K_0(p,q)=\frac{\epsilon}{2}t\left(\frac{p-q}{2},\frac{q-p}{2}\right)\sigma_z(p)\ , \tag{D10b}$$

$$\begin{aligned}K_+(p,q)=K_z(p,q)-\frac{i\epsilon}{2}\int dq\ dk\ dk'&\left[t\left(\frac{p-u}{2},\frac{v-q}{2}\right)t\left(\frac{v-q}{2},\frac{u-p}{2}\right)-t\left(\frac{p-u}{2},\frac{q-v}{2}\right)t\left(\frac{q-v}{2},\frac{u-p}{2}\right)\right]\\&\times\sigma_z(v)(2\pi)^3\delta(p+q-k-k')(2\pi)\delta(\epsilon_p+\epsilon_q-\varepsilon_k-\varepsilon_{k'})\ .\end{aligned} \tag{D10c}$$

As anticipated, the collision kernels K_+ and K_z differ only by the presence of final-state interference in the transverse magnetization.

Solving the integral equations (D9) requires only the knowledge of the scattering amplitude t. Equations (D9) become a scalar integral equation when the s-wave channel dominates the scattering, an approximation which is always correct at sufficiently low temperature. We postpone the discussion of the numerical results for realistic potential to a future publication.

It is instructive to recover the Chapman-Enskog approximation from this formalism. If we assume the vertex function can be expanded in a polynomial series:

$$\mathbf{\Lambda}_z(\mathbf{p})=\mathbf{p}\sigma_z(p)\sum_{n=0}^{\infty}c_n^zS_n(p)\ , \tag{D11a}$$

$$\Lambda^+(p)=\mathbf{p}\sigma^+(p)\sum_{n=0}^{\infty}c_n^+S_n(p) \tag{D11b}$$

where we have chosen the Sonine polynomials as orthogonal basis [$S_0=1$, $S_1(p)=\beta p^2/2m-\frac{5}{2},\dots$]. There is no small parameter in this expansion, and its convergence is slow when r_s and λ_T (the thermal wavelength) are comparable.

In the scattering length approximation, the diffusion coefficients are estimated keeping only the lowest order term. From Eq. (D9) we find, after integration over p and q

$$c_z = -\frac{i3}{mn\,32r_s^2(\pi/m\beta)^{1/2}} \ . \tag{D12}$$

Substituting Eqs. (D11a) and (D12) in Eq. (11), we find

$$D_z = \frac{kT}{m}\tau_D \tag{D13}$$

where

$$\tau_D^{-1} = \tfrac{32}{3} r_s^2 n \left[\frac{\pi}{m\beta} \right]^{1/2} \ . \tag{D14}$$

The calculation of the transverse diffusion coefficient $D_{\pm}$ is very similar, and leads to the result given in Eq. (24b). These values of the transport coefficients agree with the results of Lhuillier and Lalöe. They are expected to be accurate at very low temperature, but are inadequate when λ_T and r_s are comparable where the full integral Eqs. (D9) must be solved.

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