Interface response and rescaling approach to the eigenvectors of layered composite systems. II. Triple-layer slab

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We consider the eigenproblem of a composite system of three different layer subsystems, assumed to be coupled at their interfaces. General formulas for the eigenvectors and eigenvalues (characteristic equation) of the total system, as well as its interface response-rescaling parameters, are derived and expressed in terms of matrix elements of the *individual subsystem* response functions and the interface coupling parameters.

I. INTRODUCTION

In previous papers we have developed the interface-response[1] as well as the interface-rescaling[2] approaches to the calculation of eigenvectors of composite systems. Then, the unified interface response-rescaling treatment was presented in paper I of this series in solving the eigenvalue problem of a double-layer slab.[3] The basis for the calculation of fundamental characteristics (eigenvectors, characteristic equation, interface-rescaling parameters) such composite systems resides in the knowledge of the *individual* response functions for the constituent subsystems. The present paper deals with triple-layer structure, for which we shall derive similar formulas, expressing the fundamental characteristics of the composite system by way of the response functions of the *subsystems.* Recently, there is a strong demand from experimentalists for knowing these fundamental characteristics of layered composite systems (e.g., to intepret spin-wave resonance spectra observed in sandwich structures consisting of iron layers inserted at different positions within nickel films[4]).

In the following presentation we restrict ourselves to the case of finite one-dimensional systems only. However, the application of our method to realistic finite three-dimensional systems (layered structures) is straightforward. Due to translational invariance parallel to surfaces of the system, one can perform the usual Fourier transformation in the two in-plane directions, and all the quantities involved become functions of the in-plane wave vector $\vec{k}_{\parallel}$. Since this is the only difference which shows up, the dependence on $\vec{k}_{\parallel}$, for simplicity, will not be explicitly written down in our presentation.

The outline of this paper is as follows: in Sec. II we chose a simple model of triple-layer structure formulating its eigenvalue problem in matrix form, then we derive the formulas for eigenvectors (Sec. III) and eigenvalues (i.e., characteristic equation, Sec. IV). The results obtained for the eigenvectors enables us to express the interface rescaling parameters (Sec. V) defined before[2] in terms of single matrix elements of the inverse matrices of individual subsystems. The explicit forms of these inverse matrices for *homogeneous* subsystems are presented in Sec. VI, which concludes the paper.

II. THE EIGENPROBLEM

We are interested in a one-dimensional composite system of three alternatively disposed subsystems at interaction by way of the interfaces. With each of the subsystems separately described by a matrix $\overleftrightarrow{H}_1$, $\overleftrightarrow{H}_2$, or $\overleftrightarrow{H}_3$, respectively, the eigenvalue problem of the system as a whole can be expressed in the following form:

$$\overleftrightarrow{h}\overleftrightarrow{u} \equiv \begin{bmatrix} \overleftrightarrow{H}_1 & \overleftrightarrow{V}_{12} & \overleftrightarrow{0}_{N\times(L-N-R)} \\ \overleftrightarrow{V}_{21} & \overleftrightarrow{H}_2 & \overleftrightarrow{V}_{23} \\ \overleftrightarrow{0}_{(L-N-R)\times N} & \overleftrightarrow{V}_{32} & \overleftrightarrow{H}_3 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_N \\ u_{N+1} \\ \vdots \\ u_{N+R} \\ u_{N+R+1} \\ \vdots \\ u_L \end{bmatrix} = \overleftrightarrow{0}_{L\times 1}\,, \tag{2.1}$$

where

$$\overleftrightarrow{V}_{12}=\begin{bmatrix}\overleftrightarrow{0}_{(N-1)\times(R-1)} & \overleftrightarrow{0}_{(N-1)\times 1}\\ -\rho & 0\end{bmatrix},\quad \overleftrightarrow{V}_{21}=\begin{bmatrix}0 & -\rho\\ \overleftrightarrow{0}_{(R-1)\times(N-1)} & \overleftrightarrow{0}_{(R-1)\times 1}\end{bmatrix}=(\overleftrightarrow{V}_{12})^T ,$$

$$\overleftrightarrow{V}_{23}=\begin{bmatrix}\overleftrightarrow{0}_{(R-1)\times(L-N-R-1)} & \overleftrightarrow{0}_{(R-1)\times 1}\\ -\gamma & 0\end{bmatrix},$$

$$\overleftrightarrow{V}_{32}=\begin{bmatrix}0 & -\gamma\\ \overleftrightarrow{0}_{(L-N-R-1)\times(R-1)} & \overleftrightarrow{0}_{(L-N-R-1)\times 1}\end{bmatrix}=(\overleftrightarrow{V}_{23})^T .$$

ρ and γ are coupling parameters between the respective subsystems and the one-column matrix $\overleftrightarrow{u}$ is the eigenvector of matrix $\overleftrightarrow{h}$ describing the system as a whole [in the notation of Eq. (2.1), the eigenvalue E corresponding to $\overleftrightarrow{u}$ is included in the diagonal elements of $\overleftrightarrow{h}$]. Let us denote by l the spatial variable of Eq. (2.1); it takes the discrete values $l=1,2,\ldots,L$. It proves convenient to introduce distinct indices for each range of the subspaces corresponding to the three subsystems. We denote them as follows:

$$l\equiv n=1,2,\ldots,N \quad \text{in the subspace } D_1 , \tag{2.2a}$$

$$l\equiv r=N+1,N+2,\ldots,N+R \quad \text{in the subspace } D_2 , \tag{2.2b}$$

$$l\equiv p=N+R+1,N+R+2,\ldots,L \quad \text{in the subspace } D_3 . \tag{2.2c}$$

Moreover, we introduce the concept of "interface subspace," and

$$l=N,N+1,N+R,N+R+1 \quad \text{in the interface subspace } M , \tag{2.3a}$$

$$l=1,2,\ldots,L \quad \text{in the space } D \text{ as a whole} . \tag{2.3b}$$

Thus, the interface subspace M is "projected out" as that part of the space D where all the interactions connecting the subsystems take place.

We now proceed to define the following two matrices:

$$\overleftrightarrow{h}\equiv\overleftrightarrow{H}_0+\overleftrightarrow{V}_I;\quad \overleftrightarrow{H}_0=\begin{bmatrix}\overleftrightarrow{H}_1 & \overleftrightarrow{0}_{N\times R} & \overleftrightarrow{0}_{N\times(L-N-R)}\\ \overleftrightarrow{0}_{R\times N} & \overleftrightarrow{H}_2 & \overleftrightarrow{0}_{R\times(L-N-R)}\\ \overleftrightarrow{0}_{(L-N-R)\times N} & \overleftrightarrow{0}_{(L-N-R)\times R} & \overleftrightarrow{H}_3\end{bmatrix},$$

$$\overleftrightarrow{V}_I=\begin{bmatrix}\overleftrightarrow{0}_{N\times N} & \overleftrightarrow{V}_{12} & \overleftrightarrow{0}_{N\times(L-N-R)}\\ \overleftrightarrow{V}_{21} & \overleftrightarrow{0}_{R\times R} & \overleftrightarrow{V}_{23}\\ \overleftrightarrow{0}_{(L-N-R)\times N} & \overleftrightarrow{V}_{32} & \overleftrightarrow{0}_{(L-N-R)\times(L-N-R)}\end{bmatrix}. \tag{2.4}$$

The matrix $\overleftrightarrow{H}_0$ describes the *unperturbed* system (isolated subsystems) whereas $\overleftrightarrow{V}_I$ is a perturbation which, when imposed on $\overleftrightarrow{H}_0$, makes it the matrix of the initially considered combination of three mutually coupled subsystems. Clearly, $\overleftrightarrow{V}_I$ has nonzero elements only in the interface subspace M. We shall also need the inverse matrices:

$$\begin{gathered}\overleftrightarrow{G}_0=\overleftrightarrow{H}_0^{-1};\overleftrightarrow{g}=\overleftrightarrow{h}^{-1}\\ \overleftrightarrow{G}_1=\overleftrightarrow{H}_1^{-1};\overleftrightarrow{G}_2=\overleftrightarrow{H}_2^{-1};\overleftrightarrow{G}_3=\overleftrightarrow{H}_3^{-1} .\end{gathered} \tag{2.5}$$

The general properties of inverse matrices lead to the following relation:

$$\overleftrightarrow{G}_0(DD)=\begin{bmatrix}\overleftrightarrow{G}_1(D_1D_1) & \overleftrightarrow{0}_{N\times R} & \overleftrightarrow{0}_{N\times(L-N-R)}\\ \overleftrightarrow{0}_{R\times N} & \overleftrightarrow{G}_2(D_2D_2) & \overleftrightarrow{0}_{R\times(L-N-R)}\\ \overleftrightarrow{0}_{(L-N-R)\times N} & \overleftrightarrow{0}_{(L-N-R)\times R} & \overleftrightarrow{G}_3(D_3D_3)\end{bmatrix}. \tag{2.6}$$

As the essential result of the present work, we shall prove that it suffices to have available the inverse matrices of the *individual subsystems* $\overleftrightarrow{G}_1,\overleftrightarrow{G}_2,\overleftrightarrow{G}_3$ and the matrix $\overleftrightarrow{V}_I$ that couples them into a whole in order to be able to determine the eigenvalues E and the eigenvectors $\overleftrightarrow{u}$ of the matrix $\overleftrightarrow{h}$, i.e., the matrix of the triple-layer system. Since we proceed throughout using matrical notation, we now have to introduce some new matrices. They will play an important role in the proof. Thus, we define the following matrix:

$$\overleftrightarrow{A}(DD)=\overleftrightarrow{V}_I(DD)\overleftrightarrow{G}_0(DD)\ . \tag{2.7}$$

By (2.4) and (2.6), $\overleftrightarrow{A}(DD)$ possesses nonzero elements only in the four rows belonging to the interface subspace. These elements are $A(N;D_2)$, $A(N+1;D_1)$, $A(N+R;D_3)$, and $A(N+R+1;D_2)$. We denote the *rectangular* matrix consisting of these four rows by $\overleftrightarrow{A}(MD)$:

$$\overleftrightarrow{A}(MD)=\begin{bmatrix} \overleftrightarrow{0}_{1\times N} & \overleftrightarrow{A}^{(MD)}_{12} & \overleftrightarrow{0}_{1\times(L-N-R)} \\ \overleftrightarrow{A}^{(MD)}_{21} & \overleftrightarrow{0}_{1\times R} & \overleftrightarrow{0}_{1\times(L-N-R)} \\ \overleftrightarrow{0}_{1\times N} & \overleftrightarrow{0}_{1\times R} & \overleftrightarrow{A}^{(MD)}_{33} \\ \overleftrightarrow{0}_{1\times N} & \overleftrightarrow{A}^{(MD)}_{42} & \overleftrightarrow{0}_{1\times(L-N-R)} \end{bmatrix}, \tag{2.8}$$

where

$$\overleftrightarrow{A}^{(MD)}_{12}=(-\rho G_2(N+1,N+1)\quad -\rho G_2(N+1,N+2)\quad \cdots \quad -\rho G_2(N+1,N+R))\ ,$$
$$\overleftrightarrow{A}^{(MD)}_{21}=(-\rho G_1(N,1)\quad -\rho G_1(N,2)\quad \cdots \quad -\rho G_1(N,N))\ ,$$
$$\overleftrightarrow{A}^{(MD)}_{33}=(-\gamma G_3(N+R+1,N+R+1)\quad -\gamma G_3(N+R+1,N+R+2)\quad \cdots \quad -\gamma G_3(N+R+1,L))\ ,$$
$$\overleftrightarrow{A}^{(MD)}_{42}=(-\gamma G_2(N+R,N+1)\quad -\gamma G_2(N+R,N+2)\quad \cdots \quad -\gamma G_2(N+R,N+R))\ .$$

It will prove useful to perform a further "projection," leaving only those elements of (2.8) which bear strictly on the interface subspace. This leads us to the 4×4 matrix $\overleftrightarrow{A}(MM)$:

$$\overleftrightarrow{A}(MM)=\begin{bmatrix} 0 & -\rho G_2(N+1,N+1) & -\rho G_2(N+1,N+R) & 0 \\ -\rho G_1(N,N) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\gamma G_3(N+R+1,N+R+1) \\ 0 & -\gamma G_2(N+R,N+1) & -\gamma G_2(N+R,N+R) & 0 \end{bmatrix}. \tag{2.9}$$

In turn, (2.9) will serve to define yet another matrix—the key to our further considerations—of the form

$$\overleftrightarrow{\Delta}(MM)=\overleftrightarrow{I}_4+\overleftrightarrow{A}(MM)\ , \tag{2.10}$$

where $\overleftrightarrow{I}_4$ is the 4×4 unit matrix.

III. THE EIGENVECTORS

Let us write the eigenvectors of the unperturbed system $\overleftrightarrow{H}_0$ in the form of the $L\times1$ (column) matrix $\overleftrightarrow{U}_0(D)$. In a separate paper,[1] we have shown that the eigenvectors $\overleftrightarrow{u}(D)$ of the *perturbed* system can be expressed by way of the eigenvectors of the unperturbed system $\overleftrightarrow{U}_0(D)$ as follows:

$$[\overleftrightarrow{u}(D)]^T=[\overleftrightarrow{U}_0(D)]^T-[\overleftrightarrow{U}_0(M)]^T\overleftrightarrow{\Delta}^{-1}(MM)\overleftrightarrow{A}(MD)\ , \tag{3.1}$$

where $[\ ]^T$ stands for the transposed matrix and $\overleftrightarrow{U}_0(M)$ is the 4×1 (column) matrix ["projected" out of the $L\times1$ column matrix $\overleftrightarrow{U}_0(D)$] containing elements belonging to the "interface" subspace only. Furthermore, it turns out that if the perturbed system is finite (as is the case here) the formula (3.1) can be reduced to the simpler form:[1]

$$[\overleftrightarrow{u}(D)]^T=-||\overleftrightarrow{\Delta}(MM)||[\overleftrightarrow{U}_0(M)]^T\overleftrightarrow{\Delta}^{-1}(MM)\overleftrightarrow{A}(MD)\ . \tag{3.2}$$

Equation (3.2) is just the formula that will serve us to determine the eigenvectors of the triple-layer system. At this point, we wish to note that the presence of the determinant $||\overleftrightarrow{\Delta}(MM)||$ in (3.2) is only apparent since it cancels out with the denominator of the inverse matrix $\overleftrightarrow{\Delta}^{-1}(MM)$.

Since the unperturbed system $\overleftrightarrow{H}_0$ consists of three noninteracting subsystems, the eigenvectors $\overleftrightarrow{U}_0(D)$ fall into *three* classes, each of which belongs to one of the eigenvector bases, $\overleftrightarrow{H}_1$, $\overleftrightarrow{H}_2$, or $\overleftrightarrow{H}_3$. Writing, quite generally,

$$\overleftrightarrow{U}_0(D)=\begin{bmatrix} \overleftrightarrow{U}_0(D_1) \\ \overleftrightarrow{U}_0(D_2) \\ \overleftrightarrow{U}_0(D_3) \end{bmatrix} \tag{3.3}$$

we find that the eigenvectors belonging to each of these classes can be written in the form of vectors possessing nonzero elements only in one of the subspaces D_1, D_2, or D_3:

$$\begin{bmatrix} \overleftrightarrow{U}_0(D_1) \\ \overleftrightarrow{0}_{R\times1} \\ \overleftrightarrow{0}_{(L-N-R)\times1} \end{bmatrix},\quad \begin{bmatrix} \overleftrightarrow{0}_{N\times1} \\ \overleftrightarrow{U}_0(D_2) \\ \overleftrightarrow{0}_{(L-N-R)\times1} \end{bmatrix},\quad \begin{bmatrix} \overleftrightarrow{0}_{N\times1} \\ \overleftrightarrow{0}_{R\times1} \\ \overleftrightarrow{U}_0(D_3) \end{bmatrix}. \tag{3.4}$$

In particular, we are justified in choosing the vector

$$[\overleftrightarrow{U}_0(M)]^T=(U_0(N)\quad 0\quad 0\quad 0) \tag{3.5}$$

as the vector $\overleftrightarrow{U}_0(M)$ of Eq. (3.2). The 4×4 matrix $\overleftrightarrow{\Delta}(MM)$, with regard to Eqs. (2.9) and (2.10), has the form

$$\overleftrightarrow{\Delta}(MM)=\begin{bmatrix} 1 & -\rho G_2(N+1,N+1) & -\rho G_2(N+1,N+R) & 0 \\ -\rho G_1(N,N) & 1 & 0 & 0 \\ 0 & 0 & 1 & -\gamma G_3(N+R+1,N+R+1) \\ 0 & -\gamma G_2(N+R,N+1) & -\gamma G_2(N+R,N+R) & 1 \end{bmatrix} . \tag{3.6}$$

On insertion of (2.8) and (3.5) into Eq. (3.2) we get

$$u_l=U_0(N)||\overleftrightarrow{\Delta}(MM)||\times\begin{cases}\rho\Delta^{-1}(N,N+1)G_1(N,n) \quad \text{for } l=n\in D_1 & (3.7\text{a})\\ [\rho\Delta^{-1}(N,N)G_2(N+1,r)+\gamma\Delta^{-1}(N,N+R+1)G_2(N+R,r)] \quad \text{for } l=r\in D_2 & (3.7\text{b})\\ \gamma\Delta^{-1}(N,N+R)G_3(N+R+1,p) \quad \text{for } l=p\in D_3 . & (3.7\text{c})\end{cases}$$

Now, on calculating from (3.6) the inverse matrix elements $\overleftrightarrow{\Delta}^{-1}(MM)$ occurring in (3.7) and on omitting the constant factor $U_0(N)$, we finally arrive at the following expressions for the eigenfunctions:

$$u_l=\begin{cases}\rho^2G_1(N,n)\{G_2(N+1,N+1)+\gamma^2G_3(N+R+1,N+R+1)\\ \qquad\times[G_2(N+1,N+R)G_2(N+R,N+1)\\ \qquad-G_2(N+1,N+1)G_2(N+R,N+R)]\} \quad \text{for } l=n\in D_1\\ \rho G_2(N+1,r)[1-\gamma^2G_2(N+R,N+R)G_3(N+R+1,N+R+1)]\\ +\rho\gamma^2G_2(N+R,r)[G_2(N+1,N+R)G_3(N+R+1,N+R+1)] \quad \text{for } l=r\in D_2\\ \rho\gamma G_3(N+R+1,p)G_2(N+1,N+R) \quad \text{for } l=p\in D_3 .\end{cases} \tag{3.8}$$

These functions are as yet not normalized. For practical purposes, they still have to be multiplied by an appropriate constant.

IV. THE EIGENVALUES

The response function $\overleftrightarrow{g}$ of the system as a whole can also be expressed by way of the "unperturbed" response functions of the subsystems and the matrix elements of the perturbation applying the following relation first proposed by Dobrzynski:[5]

$$\overleftrightarrow{g}(DD)=\overleftrightarrow{G}_0(DD)-\overleftrightarrow{G}_0(DM)\overleftrightarrow{\Delta}^{-1}(MM)\overleftrightarrow{A}(MD) . \tag{4.1}$$

The expressions obtained from (4.1) for the matrix elements $\overleftrightarrow{g}(DD)$ are lengthy and we refrain from writing them out explicitly. It is, however, essential to note that the poles of $\overleftrightarrow{g}(DD)$ are given by the equation $||\overleftrightarrow{\Delta}(MM)||=0$. With regard to Eq. (3.6) this leads to

$$[1-\gamma^2G_2(N+R,N+R)G_3(N+R+1,N+R+1)][1-\rho^2G_1(N,N)G_2(N+1,N+1)] \\ -\rho^2\gamma^2G_2(N+1,N+R)G_1(N,N)G_3(N+R+1,N+R+1)G_2(N+R,N+1)=0 , \tag{4.2}$$

whence one can extract the eigenvalues E of our problem.

Obviously, using Eq. (4.2), the formulas (3.8) for the eigenfunctions can be expressed in different, equivalent forms. The latter turn out to be the forms (we refrain from adducing them) we would arrive at had we used as unperturbed eigenvector—in place of Eq. (3.5)—one of the following expressions:

$$[\overleftrightarrow{U}_0(M)]^T=(0 \;\; U_0(N+1) \;\; U_0(N+R) \;\; 0) \text{ or } (0 \;\; 0 \;\; 0 \;\; U_0(N+R+1)) . \tag{4.3}$$

Since all alternative approaches lead to mutually equivalent results, one is justified in choosing the procedure involving the simplest calculations. We believe this is what we have done in the course of the present investigation.

V. INTERFACE RESPONSE-RESCALING PARAMETERS

In our earlier paper,[2] we introduced the concept of interface rescaling permitting the reduction of the eigenvalue problem of a composite system to that of one of its component subsystems. The procedure involves the calculation of certain new quantities which we refer to as interface rescaling parameters. They have the property of transferring *complete* information on one of the subsystems into the adjacent subsystem through the interface between the two. In particular, we have shown[3] that the interface rescaling parameters can be expressed in terms of elements of the response function of the "information-giving" subsystem; we refer to the parameters thus calculated as *interface response-rescaling parameters.* In the present section, we shall calculate a pair of these rescaling parameters, namely, the pair governing information transfer through the interface separating the subsystem $\overleftrightarrow{\mathrm{H}}_1$ from the other two. Moreover, the characteristic equation of the system as a whole will be shown to be particularly well represented by way of the product of the two parameters.

We define the rescaling parameters as follows: $R(H_1 \overleftarrow{|} H_2H_3)$—the parameter which rescales the subsystem $\overleftrightarrow{\mathrm{H}}_1$ with respect to the subsystem $(\overleftrightarrow{\mathrm{H}}_2\overleftrightarrow{\mathrm{H}}_3)$—satisfies the equation

$$\rho u_{N+1} = R(H_1 \overleftarrow{|} H_2H_3) u_N \ , \tag{5.1}$$

whereas $R(H_1 \overrightarrow{|} H_2H_3)$—the parameter rescaling the subsystem $(\overleftrightarrow{\mathrm{H}}_2\overleftrightarrow{\mathrm{H}}_3)$ with respect to $\overleftrightarrow{\mathrm{H}}_1$—fulfills the inverse relation

$$\rho u_N = R(H_1 \overrightarrow{|} H_2H_3) u_{N+1} \ . \tag{5.2}$$

To calculate the rescaling parameters, we make use of the relation (3.1). In particular, when applied to the interface subspace, (3.1) takes the form

$$[\overleftrightarrow{u}(M)]^T = [\overleftrightarrow{\mathrm{U}}_0(M)]^T - [\overleftrightarrow{\mathrm{U}}_0(M)]^T \overleftrightarrow{\Delta}^{-1}(MM)\overleftrightarrow{\mathrm{A}}(MM) \ . \tag{5.3}$$

By the definition (2.10), Eq. (5.3) changes into

$$[\overleftrightarrow{u}(M)]^T = [\overleftrightarrow{\mathrm{U}}_0(M)]^T \overleftrightarrow{\Delta}^{-1}(MM) \ . \tag{5.4}$$

On insertion of Eq. (3.5) into (5.4) we finally obtain

$$u_M = U_0(N)\Delta^{-1}(N,M) \ , \tag{5.5}$$

whence, with regard to the definitions (5.1) and (5.2), we immediately obtain

$$R(H_1 \overleftarrow{|} H_2H_3) = \rho\Delta^{-1}(N,N+1)/\Delta^{-1}(N,N) \ , \tag{5.6a}$$

$$R(H_1 \overrightarrow{|} H_2H_3) = \rho\Delta^{-1}(N,N)/\Delta^{-1}(N,N+1) \ . \tag{5.6b}$$

The relations (5.6a) and (5.6b) [or, equivalently, (5.1) and (5.2)] show that the product of a pair of rescaling parameters "transporting" information through an interface in both directions fulfills the following equation:

$$R(H_1 \overleftarrow{|} H_2H_3) R(H_1 \overrightarrow{|} H_2H_3) = \rho^2 \ . \tag{5.7}$$

We now proceed to show that (5.7) is, in fact, equivalent to the characteristic equation (4.2) and thus determines the eigenvalues of the system. By (3.6) and (5.6a), we have

$$R(H_1 \overleftarrow{|} H_2H_3) = \frac{\rho^2}{\Gamma_{23}}[\Gamma_{23}G_2(N+1,N+1) + \gamma^2 G_2(N+1,N+R)G_2(N+R,N+1)G_3(N+R+1,N+R+1)] \ , \tag{5.8}$$

where

$$\Gamma_{23} \equiv 1 - \gamma^2 G_2(N+R,N+R)G_3(N+R+1,N+R+1) \ .$$

We note that (as it could have been expected) the parameter $R(H_1 \overleftarrow{|} H_2H_3)$ is expressed by way of the coupling interface parameter ρ and only those inverse matrix elements of the information-giving subsystems $\overleftrightarrow{\mathrm{G}}_2$ and $\overleftrightarrow{\mathrm{G}}_3$, which belong to the interface subspace. On writing the following "partial" rescaling parameters for the transfer of information between *individual* subsystems:

$$R(H_1 \overrightarrow{|} H_2) = \rho^2 G_1(N,N), \quad R(H_1 \overleftarrow{|} H_2) = \rho^2 G_2(N+1,N+1) \ , \tag{5.9a}$$

$$R(H_2 \overrightarrow{|} H_3) = \gamma^2 G_2(N+R,N+R), \quad R(H_2 \overleftarrow{|} H_3) = \gamma^2 G_3(N+R+1,N+R+1) \tag{5.9b}$$

Eq. (5.8) can be expressed as well in the form

$$R(H_1 \overleftarrow{|} H_2H_3) = \gamma^2 \frac{R(H_1 \overleftarrow{|} H_2) - \rho^2 R(H_2 \overleftarrow{|} H_3) \, ||\overleftrightarrow{\mathrm{G}}_2(MM)||}{\gamma^2 - R(H_2 \overrightarrow{|} H_3) R(H_2 \overleftarrow{|} H_3)} \ , \tag{5.10}$$

where the determinant

$$||\overleftrightarrow{\mathrm{G}}_2(MM)|| \equiv \left|\left|\begin{matrix} G_2(N+1,N+1) & G_2(N+1,N+R) \\ G_2(N+R,N+1) & G_2(N+R,N+R) \end{matrix}\right|\right| \ .$$

On the other hand, we note that transfer of information regarding $\overleftrightarrow{\mathrm{H}}_1$ into $\overleftrightarrow{\mathrm{H}}_2$ is equivalent to its transfer into the double (two-layer) subsystem $(\overleftrightarrow{\mathrm{H}}_1\overleftrightarrow{\mathrm{H}}_2)$. Thus, we can write [by Eq. (5.9a)]

$$R(H_1 \overrightarrow{|} H_2H_3) \equiv R(H_1 \overrightarrow{|} H_2) = \rho^2 G_1(N,N) \ . \tag{5.11}$$

Now, on insertion of (5.11) and (5.8) into Eq. (5.7) we arrive at the characteristic equation (4.2). This proves that Eq. (5.7) is an alternative form of the characteristic equation of the triple-layer system as a whole. We have written out Eq. (5.10) in full in order to prove that the *total* rescaling parameter can be expressed in terms of the *partial* rescaling parameters.

VI. EXAMPLE OF APPLICATION: HOMOGENEOUS SUBSYSTEMS

The preceding formulas immediately enable us to express, in closed form, the eigenfunctions, the characteristic equation and the rescaling parameters of a triple-layer slab once the explicit-response functions of its *individual* subsystems are known; strictly speaking, we need only know those matrix elements (of the individual-response matrices) which belong to the interface subspace. In solid-state physics, when dealing with the properties of multilayer structures, it is in most cases sufficient to assume that the individual subsystems are spatially (internally) homogeneous and that the sole inhomogeneity of their characteristics is located on the surfaces. Systems of this kind are characterized (in the approximation of nearest-neighbor interactions) by a tridiagonal matrix of the following form:

$$\overleftrightarrow{\mathrm{H}}_i(a_i,b_i)=\begin{bmatrix} x_i-E_L-a_i & -\beta_i & & & \\ -\beta_i & x_i-E_L & -\beta_i & & \\ & & \ddots & & \\ & & -\beta_i & x_i-E_L & -\beta_i \\ & & & -\beta_i & x_i-E_L-b_i \end{bmatrix}_{L_i\times L_i} \tag{6.1}$$

involving the surface parameters a_i and b_i describing the conditions on the surfaces of the ith subsystems. The inverse of a matrix (6.1) is found to be

$$G_i(l,l')=\frac{1}{W(a_i,b_i)}\times\begin{cases}\left[\sin(lk_i)-\frac{a_i}{\beta_i}\sin[(l-1)k_i]\right] \\ \times\left[\sin[(L_i-l'+1)k_i]-\frac{b_i}{\beta_i}\sin[(L_i-l')k_i]\right] & \text{for } l'\geq l \\ \left[\sin(l'k_i)-\frac{a_i}{\beta_i}\sin[(l'-1)k_i]\right] \\ \times\left[\sin[(L_i-l+1)k_i-\frac{b_i}{\beta_i}[\sin(L_i-l)k_i]\right] & \text{for } l'\leq l\end{cases} \tag{6.2}$$

where

$$W(a_i,b_i)\equiv(\beta_i\sin k_i)\{\beta_i\sin[(L_i+1)k_i] -(a_i+b_i)\sin(L_ik_i) +\beta_i^{-1}a_ib_i\sin[(L_i-1)k_i]\} \tag{6.3a}$$

$$x_i-E_L\equiv 2\beta_i\cos k_i\ , \tag{6.3b}$$

$$l,l'=1,2,\ldots,L_i\ . \tag{6.3c}$$

The formulas (6.2) provide the key to the *strict* characteristics (the eigenvalues and eigenfunctions) of our triple-layer system. In practice, its application is trivial; however, one should keep in mind that the indices (6.3c) have to be shifted as follows: by N for the subsystem $i=2$, and by $N+R$ for $i=3$. In paper I of this series,[3] we have given some simple examples of the application of Eq. (6.2) to calculations of bilayer structure characteristics; they should enable the reader to carry out the appropriate calculations easily for triple-layer systems as well.

By the way, an alternative approach to the eigenproblem of composite systems proposed by us earlier[2] may be worth mentioning. It, too, is based on the concept of rescaling parameters: within the system as a whole, one subsystem (referred to by us as the "nucleus") is selected and the total rescaling parameters for its two boundaries are calculated. The rescaling procedure makes the surface parameters of the nucleus go over into "effective" parameters:

$$a\rightarrow a_{\mathrm{eff}}=a+R(\text{left subsystems}\,|\!\!\!\top H_{\mathrm{nucleus}})\ ,$$

$$b\rightarrow b_{\mathrm{eff}}=b+R(H_{\mathrm{nucleus}}\,\top\!\!\!|\,\text{right subsystem})\ .$$

On solving the eigenvalue problem of the nucleus with the effective (rescaled) surface parameters we get the solution of the eigenvalue problem of the system as a whole (for the details, see Ref. 2).

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