Hydrodynamics of superfluid ^{4}He in a pseudospin model

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A microscopic theory of nonlinear dynamics in superfluid ^{4}He is formulated using a model in which a system of bosons with hard cores plus attractive nearest-neighbor interactions is described by a pseudospin Hamiltonian on a lattice. In this framework, the superfluid order parameter is the expectation value of the spin-flip operator. Using the spin-coherent-state representation, a nonlinear evolution equation is derived for the order parameter in the lattice model, and calculations are carried out with its continuum version, as is appropriate for the description of a liquid phase. The well-known Gross-Pitaevskii equation for the order parameter (valid for $T \sim T_\lambda$) is recovered in the leading small-amplitude approximation of our equation. Further, a microscopic basis is provided for the nonlinear effects discussed in phenomenological treatments of superfluid ^{4}He films. In bulk ^{4}He, considering a cylindrically symmetric vortex solution, a vortex core of finite thickness and a nonsingular vorticity is obtained. The finiteness of the vortex energy is shown to arise as a natural consequence of the formalism. Unidirectional traveling-wave solutions for the superfluid density with velocity-dependent amplitudes are shown to exist. Finally, for a specific choice of parameters, a static-domain-wall solution is also possible.

I. INTRODUCTION

Theories of liquid ^{4}He have been studied at different levels. The two-fluid theory of London and Tisza,[1] the hydrodynamic theory of Landau,[2] the second quantized formalism of Bogoliubov,[3] and the wave-function description of Feynman[4] have been used to explain several unusual properties of this liquid.

It is well known that in superfluid ^{4}He the manifestation of quantum phenomena occurs on a macroscopic scale. Below the λ point the liquid is characterized by a condensate which is described by a macroscopic wave function. Using the theory of a weakly interacting Bose gas, the hydrodynamics of the superfluid condensate was developed by Gross,[5] leading to a nonlinear evolution equation for the order parameter (the condensate wave function). Vortex solutions and the well-known quantization of circulation[6] were also derived. These results had been obtained earlier in the phenomenological theory of Ginzburg and Pitaevskii.[7] A more formal derivation of the above-mentioned evolution equation, the Gross-Pitaevskii (GP) equation, is also possible. One starts with a second-quantized many-body Hamiltonian for a system of bosons, and postulates that the order parameter is the expectation value of the field operator under the following approximations: (a) The (two-body) interaction is through a purely *repulsive* contact potential, and (b) the Hartree approximation is used in the dynamical equation for the order parameter. The theory then leads to a singular vortex structure, i.e., there is a δ-function singularity in the vorticity. However, a physically realistic vortex core is expected to have a finite thickness and finite vorticity. Determining the detailed structure of a vortex is a problem of considerable interest. Pioneering work on vortices was carried out by Fetter[8] for a weakly interacting Bose gas. Unfortunately, this theory requires an inequality in terms of the superfluid density and the healing length of the vortex to be satisfied; this inequality is known[8] to be violated for liquid ^{4}He. The phenomenological theory of Ginzburg and Pitaevskii[7] predicts that the vortex-healing length must diverge at the λ point. However, no direct experimental evidence has been found for an increased core size near T_λ. In view of these points, it would be desirable to formulate a hydrodynamic theory of superfluid ^{4}He which (a) incorporates a realistic interatomic potential, i.e., a hard-core repulsion together with an attractive interaction, and (b) circumvents the use of the Hartree approximation, if possible, in the evolution equation. Such a theory is presented in this paper.

Superfluidity has also been observed in ^{4}He films.[9] At finite temperatures, a long-wavelength surface density wave (accompanied by a temperature wave), called the third sound, has been reported. Also at low temperatures there are indications of an undistorted pulse propagation in thin films.[10] To explain these observations a phenomenological nonlinear evolution equation for the condensate (based on a generalization of Landau's theory of quantum hydrodynamics) was derived by Rutledge *et al.*[11] It is of interest to identify the microscopic origin of the terms appearing in this equation. This is also done in the present work.

The pseudospin (quantum-lattice gas) model of Matsubara and Matsuda[12] is found to be most suitable for our purpose. This model is derived by starting with a realistic interaction potential between ^{4}He atoms. The hard core in the potential is incorporated by demanding Fermi-like anticommutation relations for the field operators at the same site and Bose-like commutation relations for operators belonging to different sites. The field operators can be shown to behave like $S=\frac{1}{2}$ spin-flip operators, and the many-body Hamiltonian for interacting ^{4}He

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atoms can be represented by an anisotropic Heisenberg exchange Hamiltonian. In this framework, the superfluid order parameter becomes the expectation value of the spin-flip operator.

The ground-state thermodynamic properties and the nature of elementary excitations in the pseudospin model were studied by Whitlock and Zilsel.[13] They showed that in the random-phase approximation the well-known low-density-limit results[14] are reproduced. More recently, Parmenter and Poling[15] have shown that the value of T_λ and the quasiparticle spectrum at $T=0$, are in reasonable agreement with experiments. In view of these results, it would be interesting to investigate the hydrodynamic properties of the pseudospin model, without resorting to the linearization procedure customarily used. As we shall demonstrate, retaining nonlinear terms leads to interesting results.

As the condensate has a very large number of particles occupying the same quantum state, it is most natural to study the order-parameter evolution equation in the spin-coherent-state representation[16] (SCR). The spin Hamiltonian is bilinear in spin operators. This property, combined with some special properties of the SCR, enables us to derive an evolution equation for all $T \leq T_\lambda$ without using a Hartree approximation. The GP equation (valid for $T \sim T_\lambda$) is recovered in the leading small-amplitude approximation. Complete linearization leads to harmonic waves obeying a Bogoliubov-like dispersion relation. A microscopic basis is also provided for the nonlinearities present in the phenomenological treatment of Rutledge *et al.*[11] for thin-^{4}He films. For a vortex solution in a cylindrical geometry we obtain a nonuniform and nonsingular vorticity. We also find a unidirectional traveling-wave solution with a velocity-dependent amplitude, characteristic of a nonlinear excitation, when we consider the flow of the liquid in one direction. Finally, for a specific choice of parameters a static kink solution is found for the condensate density. A preliminary version of this work has been published recently.[17]

II. LATTICE MODEL— A PSEUDOSPIN FORMULATION

Liquid ^{4}He, in its superfluid phase, has a correlation length of the order of a classical fluid (≈ 1 Å). It is well known that lattice models of classical fluids lead to physically relevant results.[18] Siegert[19] has shown that a hard core should be treated more realistically than just considering it as the infinite limit of a repulsive potential. Therefore, a quantum-lattice model of hard-core bosons with attractive interactions would be appropriate for the description of liquid ^{4}He.[12]

The volume Ω of the system is divided into M cubic cells each of volume a^3 so that in the limit $M, \Omega \to \infty$, a remains constant. The centers of the cubical cells are labeled by three integers $(l,m,n)=l$. The corresponding boson field operators are represented by ψ_l with the following commutation relation:

$$[\psi_l, \psi_{l'}]=0, \quad [\psi_l, \hat{\rho}_{l'}]=\psi_l \delta_{ll'} , \tag{2.1}$$

where $\hat{\rho}_l=\psi_l^\dagger \psi_l$ is the number density operator for the lth cell. To incorporate the constraint of the hard core we have

$$\psi_l^2=0, \quad \hat{\rho}_l^2=\hat{\rho}_l, \quad [\psi_l, \psi_{l'}^\dagger]=(1-2\hat{\rho}_l)\delta_{ll'} . \tag{2.2}$$

In other words, the field operators behave like fermion operators within the same cell and like boson operators for different cells.

The microscopic many-body Hamiltonian is

$$H=-\int d\mathbf{r}\psi^\dagger(\mathbf{r})\left[\frac{\hbar^2\nabla^2}{2m}+\mu\right]\psi(\mathbf{r}) +\tfrac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'\,\psi^\dagger(\mathbf{r})\psi^\dagger(\mathbf{r}') \times V(\mathbf{r}-\mathbf{r}')\psi(\mathbf{r}')\psi(\mathbf{r}) , \tag{2.3}$$

where μ is the chemical potential, m is the mass of the ^{4}He atom, and $V(\mathbf{r}-\mathbf{r}')$ is the interaction potential. We discretize the Hamiltonian (2.3) by using the finite difference approximation. Introducing a nearest-neighbor (NN) attractive interaction $-v_0$ $(v_0>0)$ we have

$$H=\frac{\hbar^2}{2ma^2}\sum_{l,\delta}\left[(\hat{\rho}_{l+\delta}+\hat{\rho}_l)-(\psi_{l+\delta}^\dagger\psi_l+\text{H.c.}) -\frac{2ma^2}{\hbar^2}v_0\hat{\rho}_l\hat{\rho}_{l+\delta}\right]-\mu\sum_l\hat{\rho}_l , \tag{2.4}$$

where $\boldsymbol{\delta}$ runs over the NN vectors of the site l.

The commutation relation (2.1) and (2.2) satisfied by ψ_l show that one may equivalently write

$$\psi_l=S_l^x+iS_l^y=S_l^\dagger \text{ and } \hat{\rho}_l=\tfrac{1}{2}-S_l^z , \tag{2.5}$$

where S_l is a spin-$\frac{1}{2}$ operator. Thus Hamiltonian (2.4) may be rewritten as

$$H=-\sum_l\left[(b-\mu)S_l^z+\sum_\delta\left[\frac{\hbar^2}{ma^2}\sum_{\alpha=x,y}S_l^\alpha S_{l+\delta}^\alpha +v_0S_l^zS_{l+\delta}^z\right]\right] , \tag{2.6}$$

where

$$b=D[(\hbar^2/ma^2)-v_0] , \tag{2.7}$$

D being the dimensionality of the lattice. Since $(\hbar^2/ma^2)$ and v_0 are positive, Eq. (2.6) represents an anisotropic, ferromagnetic, Heisenberg NN exchange Hamiltonian in an "external field" given by $(b-\mu)$.

Several thermodynamic properties of this model have been extensively studied in the literature.[12,13,15,20] Within a linearized theory, a qualitatively good fit has been obtained for the excitation spectrum and the ground-state energy. Further, the values obtained for T_λ and the velocity of sound agree reasonably well with the experimental values. It would therefore be interesting to study yet another aspect—the hydrodynamic properties of this model, retaining all its nonlinearities. To this end, we start with the equation of motion for S_l^+,[21]

$$i\hbar\partial_t S_l^+ = [S_l^+, H]$$
$$= (b-\mu)S_l^+ - \sum_\delta (\hbar^2/ma^2) S_l^z S_{l+\delta}^+ + \sum_\delta v_0 S_l^+ S_{l+\delta}^z \; . \quad (2.8)$$

The conversion of this nonlinear operator equation into a c-number equation and its analysis will be carried out in the following sections.

III. SPIN-COHERENT STATES

The existence of an off-diagonal long-range order[22] and the concept of a generalized Bose condensation has been widely used in the context of interacting Bose systems. It is possible to represent the ground-state wave function of a superfluid as a direct product of harmonic oscillator (boson) coherent states[23,24] in momentum space. Using the second-quantized Hamiltonian given in Eq. (2.3), with a purely repulsive contact potential $V(\mathbf{r}-\mathbf{r}')=V\delta(\mathbf{r}-\mathbf{r}')$ to describe a soft-core Bose gas, we write down the equation of motion for the boson annihilation operator ψ as

$$i\hbar\partial_t \psi = -(\hbar^2/2m)\nabla^2\psi + V\psi^\dagger\psi\psi - \mu\psi \; .$$

Denoting the expectation value of ψ in the boson-coherent-state representation by $\alpha(\mathbf{r},t)$, it is readily seen that this equation yields the usual Gross-Pitaevskii (GP) equation:

$$i\hbar\partial_t \alpha = -(\hbar^2/2m)\nabla^2\alpha + V|\alpha|^2\alpha - \mu\alpha \; , \quad (3.1)$$

on using a Hartree approximation. This equation has been frequently used[25] to study superfluid hydrodynamics. In (1+1) dimensions, this equation is just the nonlinear Schrödinger equation,[26] which supports envelope-soliton solutions.

An extension of the boson-coherent-state approach to the case of a hard-core interaction (i.e., $V\rightarrow\infty$) leads to difficulties. As we have discussed, the model of Matsubara and Matsuda incorporates the hard core by using appropriate anticommutation relations between field operators, leading to a pseudospin Hamiltonian [Eq. (2.6)]; $\langle S_l^+\rangle$ represents the condensate order parameter in this model.

In analogy with the boson-coherent states, we shall set up a formalism using spin-coherent states $|\tau_l\rangle$ at a lattice site l, defined by[16]

$$|\tau_l\rangle = (1+|\tau_l|^2)^{-S}\exp(\tau_l S_l^-)|0\rangle_l \; , \quad (3.2a)$$

where S_l^- is the spin-lowering operator and τ_l is a complex quantity. Also, $S_l^z|0\rangle_l = S|0\rangle_l$. For a system of N spins, we work with the direct product

$$|\tau\rangle = \bigotimes_l^N |\tau_l\rangle \; . \quad (3.2b)$$

The normalized states $|\tau_l\rangle$ are nonorthogonal, i.e.,

$$\langle\lambda_l|\tau_l\rangle = (1+\lambda_l^*\tau_l)^{2S}/(1+|\lambda_l|^2)^S(1+|\tau_l|^2)^S \; , \quad (3.3)$$

and overcomplete

$$\pi^{-1}(2S+1)\int d^2\tau_l (1+|\tau_l|^2)^{-2}|\tau_l\rangle\langle\tau_l| = 1 \; . \quad (3.4)$$

In the calculations that follow, the following convenient parametrization of τ_l in terms of polar and azimuthal angles will be used:

$$\tau_l = \tan(\theta_l/2)\exp(i\Phi_l) \quad (3.5)$$

with $0\le\theta_l\le\pi$ and $0\le\Phi_l<2\pi$.

Thus Eqs. (3.2a) and (3.4) become, respectively,

$$|\tau_l\rangle \rightarrow |\theta_l,\Phi_l\rangle = (\cos\tfrac{1}{2}\theta_l)^{2S}\exp \times[\tan(\tfrac{1}{2}\theta_l)\exp(i\Phi_l)S_l^-]|0\rangle_l \quad (3.6)$$

and

$$(4\pi)^{-1}(2S+1)\int d\theta_l \int d\Phi_l \sin\theta_l |\theta_l,\Phi_l\rangle\langle\theta_l,\Phi_l| = 1 \; . \quad (3.7)$$

The advantage of using spin-coherent states is that in this representation, diagonal matrix elements of single-site operators are identical to the corresponding classical expressions. Thus for $S=\frac{1}{2}$, the order parameter η_l is given by

$$\eta_l = \langle S_l^+\rangle = \langle\theta_l,\Phi_l|S_l^+|\theta_l,\Phi_l\rangle = \tfrac{1}{2}\sin\theta_l\exp(i\Phi_l) \quad (3.8)$$

and

$$\langle S_l^z\rangle = \tfrac{1}{2}\cos\theta_l \; . \quad (3.9)$$

When combined, these two equations yield

$$\langle S_l^z\rangle = \pm\tfrac{1}{2}(1-4|\eta_l|^2)^{1/2} \; . \quad (3.10)$$

Using Eq. (2.5), we have

$$\langle\hat\rho_l\rangle = \rho_l = \tfrac{1}{2} - \langle S_l^z\rangle = \sin^2(\theta_l/2) = \tfrac{1}{2}[1\pm(1-4|\eta_l|^2)^{1/2}] \quad (3.11)$$

giving

$$|\eta_l|^2 = \rho_l(1-\rho_l) \; . \quad (3.12)$$

From Eq. (3.8), $|\eta_l|^2\le\frac{1}{4}$. Further, since ρ_l must be an increasing function of $|\eta_l|^2$ for physical relevance, the negative sign in Eq. (3.11) must be chosen, yielding the physical branch $0\le\rho_l<\frac{1}{2}$.

Spin-coherent states have been used to study nonlinear excitations in a quantum *isotropic* ferromagnetic chain,[27] a problem in (1+1) dimensions. In the present context of the superfluid, we are dealing with interacting ^{4}He atoms described by anisotropically interacting spins on a three-dimensional lattice. Our main concern here is to understand the hydrodynamics of a quantum fluid. To study this, we first write down the diagonal matrix elements of the spin-evolution equation [Eq. (2.8)] in the direct product representation $|\tau\rangle$ defined in Eq. (3.2), obtaining the following c-number equation:

$$i\hbar\partial_t\eta_l = (b-\mu)\eta_l - (\hbar^2/ma^2)(1-4|\eta_l|^2)^{1/2}\sum_\delta \eta_{l+\delta} + v_0\eta_l\sum_\delta (1-4|\eta_{l+\delta}|^2)^{1/2} \; . \quad (3.13)$$

In deriving Eq. (3.13) we have used the properties of spin-coherent states, especially those given in Eqs. (3.10) and (3.11). We have *not* used a Hartree approximation in the derivation, and the exact matrix elements have been used. This nonlinear differential-difference equation is now analyzed in its continuum version, as is appropriate for the description of a liquid phase.

IV. ORDER-PARAMETER EVOLUTION EQUATION

In the discrete equation (3.13), we expand $\eta_{l+\delta}$ in a Taylor series in $|\boldsymbol{\delta}|=a$. Retaining terms up to the leading order we have

$$\sum_{\delta}\eta_{l+\delta}=2D\eta+a^2\nabla^2\eta \; . \tag{4.1}$$

Also

$$\sum_{\delta}(1-4|\eta_{l+\delta}|^2)^{1/2}=2(1-4|\eta|^2)^{1/2}\times[D-a^2(1-4|\eta|^2)^{-1}\nabla^2|\eta|^2-2a^2(1-4|\eta|^2)^{-2}\times(\nabla|\eta|^2)^2] \; . \tag{4.2}$$

Substituting Eqs. (4.1) and (4.2) in Eq. (3.13) we obtain the continuum evolution equation for the superfluid order parameter.

$$i\hbar\partial_t\eta=\{b[1-(1-4|\eta|^2)^{1/2}]-\mu\}\eta-(\hbar^2/2m)(1-4|\eta|^2)^{1/2}\nabla^2\eta-v_0a^2\eta(1-4|\eta|^2)^{-1/2}\times[\nabla^2|\eta|^2+2(1-4|\eta|^2)^{-1}(\nabla|\eta|^2)^2] \; . \tag{4.3}$$

This equation is valid for all temperatures below the λ point. Further, all the nonlinearities consistent with Eqs. (4.1) and (4.2) have been retained.

A. Linearization and the Bogoliubov spectrum

Considering a uniform condensate $\eta=\eta_0$ (a constant), Eq. (4.3) yields the following expression for the chemical potential:

$$\mu=b[1-(1-4|\eta|^2)^{1/2}] \; . \tag{4.4}$$

We look for small-amplitude solutions of the form

$$\eta(\mathbf{r},t)=\eta_0+\gamma(\mathbf{r},t)+i\chi(\mathbf{r},t) \tag{4.5}$$

with single-mode expressions $\gamma(\mathbf{r},t)=\gamma_{\mathbf{k}}\sin\zeta_{\mathbf{k}}$ and $\chi(\mathbf{r},t)=\chi_{\mathbf{k}}\cos\zeta_{\mathbf{k}}$, where $\zeta_{\mathbf{k}}=(\mathbf{k}\cdot\mathbf{r}-\omega_k t)$. Substituting Eqs. (4.4) and (4.5) into (4.3) and linearizing the resulting equation, we get to order η_0^2,

$$-\hbar\omega_k(i\gamma_{\mathbf{k}}\cos\zeta_{\mathbf{k}}+\chi_{\mathbf{k}}\sin\zeta_{\mathbf{k}})=[4\eta_0^2b/(1-4\eta_0^2)^{1/2}]\gamma_{\mathbf{k}}\sin\zeta_{\mathbf{k}}+[(\hbar^2/2m)(1-4\eta_0^2)^{1/2}(\gamma_{\mathbf{k}}\sin\zeta_{\mathbf{k}}+i\chi_{\mathbf{k}}\cos\zeta_{\mathbf{k}})+2v_0a^2\eta_0^2(1-4\eta_0^2)^{-1/2}\gamma_{\mathbf{k}}\sin\zeta_{\mathbf{k}}]k^2 \; , \tag{4.6}$$

where we have used

$$(1-4|\eta|^2)^{1/2}\simeq(1-4\eta_0^2)^{1/2}\times\{1-[8\eta_0\gamma_{\mathbf{k}}\sin\zeta_{\mathbf{k}}/(1-4\eta_0^2)]\} \; . \tag{4.7}$$

Equating the real and imaginary parts of Eq. (4.6) yields the following Bogoliubov-like[3] spectrum:

$$\omega_k^2=(k^2/2m)[4\eta_0^2b+(\hbar^2k^2/2m)\times(1-4\eta_0^2+4v_0a^2\eta_0^2m)] \; . \tag{4.8}$$

In the long-wavelength limit, this reduces to

$$\omega_k=ck \tag{4.9a}$$

with a phonon velocity

$$c=(2b/m)^{1/2}\eta_0 \; . \tag{4.9b}$$

This is consistent with the expressions obtained by Matsubara and Matsuda[12] and by Whitlock and Zilsel.[13]

B. Gross-Pitaevskii equation

Since $|\eta|\ll\frac{1}{2}$ for $T\sim T_\lambda$, one is justified in expanding $(1-4|\eta|^2)^{\pm1/2}$ in a power series in $|\eta|^2$ in Eq. (4.3). On neglecting terms involving $|\eta|^{2n}\eta$ $(n>1)$, $|\eta|^{2n}\nabla^2\eta$ $(n>0)$, $\eta\nabla^2|\eta|^2$, and $\eta(\nabla|\eta|^2)^2$, we obtain the Gross-Pitaevskii equation[5,7] [see Eq. (3.1)]

$$i\hbar\partial_t\eta=-(\hbar^2/2m)\nabla^2\eta+2b|\eta|^2\eta-\mu\eta \; , \tag{4.10}$$

where $2b$ is to be identified with V, the strength of the repulsive contact potential used by Gross. The hydrodynamic treatment of this equation is usually carried out by looking for a general solution of the form $\eta=\rho^{1/2}(\mathbf{r},t)\exp[i\Phi(\mathbf{r},t)]$, yielding the equation of continuity

$$\partial_t\rho+\nabla\cdot(\rho\mathbf{v}_{S,\mathrm{GP}})=0 \; , \tag{4.11}$$

where

$$\mathbf{v}_{S,\mathrm{GP}}=(\hbar/m)\nabla\Phi \tag{4.12}$$

is the superfluid velocity. We obtain also the Bernoulli equation

$$-\hbar\partial_t\Phi=\Pi_{\mathrm{GP}}(\rho)+\tfrac{1}{2}mv_{S,\mathrm{GP}}^2 \; , \tag{4.13}$$

where

$$\Pi_{\mathrm{GP}}(\rho)=-\frac{\hbar^2}{2m}\left[\frac{1}{\sqrt{\rho}}\nabla^2\sqrt{\rho}\right]+2b\rho-\mu \; . \tag{4.14}$$

Considering a cylindrically symmetric vortex solution of Eq. (4.10) of the form

$$\eta(\mathbf{r},t)=f_{\mathrm{GP}}(\mathbf{r})\exp(in\phi)\exp(-iE_nt/\hbar) \; , \tag{4.15}$$

where $n=0,1,2,\ldots$ is the winding number, we obtain

$$\mathbf{v}_{S,\mathrm{GP}}=(n\hbar/mr)\hat{e}_\phi \; . \tag{4.16}$$

Hence the circulation is quantized (in units of

$h/m = 9.97\times 10^{-4}\ \mathrm{cm^2\,s^{-1}}$):

$$\Gamma_{\rm GP} = \oint \mathbf{v}_S \cdot dl = nh/m \ , \tag{4.17}$$

and the vorticity can be formally defined as

$$\zeta_{\rm GP} = |\mathrm{curl}\ \mathbf{v}_{S,\rm GP}| \tag{4.18}$$

$$= (n\hbar/mr)\delta(r) \ . \tag{4.19}$$

Thus the vorticity has a singularity at the origin and vanishes everywhere else. Further, the circulation in Eq. (4.17) is independent of the circuit. These unphysical features displayed by the GP equation are presumably due to the contact potential (soft core) assumed in the model.

The function $f_{\rm GP}(r)$ satisfies the following equation:[7]

$$r^2\frac{d^2 f_{\rm GP}}{dr^2} + r\frac{df_{\rm GP}}{dr} - (r^2-n^2)f_{\rm GP} - r^2 f_{\rm GP}^3 = 0 \ . \tag{4.20}$$

The solution to this equation (found numerically) vanishes at the origin and tends to a constant as $r\to\infty$.

C. Two-dimensional superfluid

^{4}He films display certain interesting properties: At finite temperatures a third sound, i.e., a long-wavelength surface density wave accompanied by a temperature wave, is observed. At very low temperatures in monolayer films, an undistorted pulse propagation has been reported.[11] To explain these observations a phenomenological model was proposed by Rutledge *et al.*,[11] leading to the following condensate evolution equation:

$$i\hbar\partial_t\psi = \frac{-\hbar^2}{2m}\nabla^2\psi - \frac{A\psi}{(a_R+|\psi|^2)^3} - B\psi\nabla^2|\psi|^2 - \mu_R\psi \ , \tag{4.21}$$

where A, B, and a_R are phenomenological parameters. μ_R is the "chemical potential" corresponding to the model. Neglecting nonlinear terms in the above equation Rutledge *et al.* derived an expression for the velocity of the third sound mode. Going beyond the linearized theory and incorporating finite amplitude effects, Huberman[10] showed that the superfluid density satisfies a *KdV* equation. More recently Biswas and Warke[28] have derived this result in a more systematic fashion.

Equation (4.21) may be brought into correspondence with the basic Eq. (4.3) as follows. First, let us set $b = 2(\hbar^2/ma^2) - v_0$ (as is appropriate for a two-dimensional film) in Eq. (4.3). Neglecting terms like $(\nabla|\eta|^2)^2\eta$ and expanding the relevant terms in powers of $|\eta|^2$ in both (4.3) and (4.21) appropriately, a close resemblance between these two equations emerges. A comparison shows that

$$\frac{3A}{2a_R^4} = b, \quad B = v_0 a^2, \quad \mu_R + \frac{A}{a_R^3} = \mu \ .$$

These relations express the phenomenological parameters A, B, a_R, and μ_R in terms of v_0, b, and μ, thus providing a microscopic basis for the surface effects described by Eq. (4.21).

V. NONLINEAR DYNAMICS OF THE CONDENSATE

In this section we analyze Eq. (4.3) retaining all the nonlinearities and study the hydrodynamics of the superfluid system. In view of Eq. (3.12) which arises due to the use of spin-$\frac{1}{2}$ coherent states, we may write, in general,

$$\eta = \rho^{1/2}(1-\rho)^{1/2}\exp(i\Phi) \ , \tag{5.1}$$

where ρ and Φ are functions of $\mathbf{r}$ and t. Inserting Eq. (5.1) into (4.3) and equating real and imaginary parts, we obtain the continuity equation for ρ,

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}_S) = 0 \ , \tag{5.2}$$

and the Bernoulli equation

$$-\hbar\partial_t\Phi = \Pi(\rho) + \tfrac{1}{2}m(1-2\rho)(1-\rho)^{-2}v_S^2 \ . \tag{5.3}$$

$\Pi(\rho)$ is defined in Eq. (5.7) below. $\mathbf{v}_S$ is given by

$$\mathbf{v}_S = (\hbar/m)(1-\rho)\nabla\Phi \ . \tag{5.4}$$

Comparing this with Eq. (4.12), we see that in our formalism m is replaced by an effective mass

$$m^* = m(1-\rho)^{-1} \ . \tag{5.5}$$

Here $m^*\ge m$, corresponding to $T\le T_\lambda$. The vorticity is given by

$$\zeta = (\hbar/m)|\nabla\rho\times\nabla\Phi| \ , \tag{5.6}$$

which, in general, is nonvanishing. In Eq. (5.3),

$$\Pi(\rho) = (\hbar^2/8m)\{(1-2\rho)\rho^{-2}(1-\rho)^{-2}(\nabla\rho)^2 - 2[(1-2\rho)\rho^{-1}(1-\rho)^{-1} + v_0 a^2]\nabla^2\rho\} + 2b\rho - \mu \ . \tag{5.7}$$

It is easy to verify that for $\rho \ll 1$ (i.e., $T\sim T_\lambda$) and $v_0=0$, Eq. (5.3) reduces to Eq. (4.13) derived from the GP equation.

Specializing to vortex solutions of the form given in Eq. (4.15), the expression for $\mathbf{v}_S$ in Eq. (5.4) becomes

$$\mathbf{v}_S = (n\hbar/2mr)\{1+[1-4f^2(r)]^{1/2}\}\hat{e}_\phi \tag{5.8}$$

(where n is the winding number of the vortex). Hence

$$\nabla\cdot\mathbf{v}_S = 0 \tag{5.9}$$

and

$$\zeta = \left|(2n\hbar/mr)(1-4f^2)^{-1/2}f\left[\frac{df}{dr}\right]\right| \ . \tag{5.10}$$

Further, Eq. (4.3) yields the following equation for $f(r)$:

$$E_n f = \{b[1-(1-4f^2)^{1/2}] - \mu\}f - (\hbar^2/2m)(1-4f^2)^{1/2}\left[\frac{1}{r}\frac{d}{dr}\left[r\frac{df}{dr}\right] - (n^2/r^2)f\right.$$
$$\left. - v_0 a^2 f(1-4f^2)^{-1/2}\left[\frac{2}{r}f\frac{df}{dr} + 2f\frac{d^2 f}{dr^2} + 2\left[\frac{df}{dr}\right]^2 + 8(1-4f^2)^{-1}f^2\left[\frac{df}{dr}\right]^2\right]\right] \ . \tag{5.11}$$

For $T \sim T_\lambda$, Eq. (5.11) becomes

$$E_n f=(2bf^2-\mu)f-(\hbar^2/2m)(1-2f^2)\left[\frac{1}{r}\frac{d}{dr}\left[r\frac{df}{dr}\right]-(n^2/r^2)f\right]$$
$$-2v_0a^2f(1+2f^2)\left[\frac{1}{r}f\frac{df}{dr}+f\frac{d^2f}{dr^2}+\left[\frac{df}{dr}\right]^2+4(1+4f^2)f^2\left[\frac{df}{dr}\right]^2\right] . \tag{5.12}$$

μ may be estimated by considering a homogeneous solution $f=f_0$; we find

$$\mu=(2bf_0^2-E_n) . \tag{5.13}$$

Linearizing Eq. (5.12) and using Eq. (5.13), we find $f(r)$ satisfies Bessel's equation:

$$(\hbar^2/2m)\left[\frac{1}{r}\frac{d}{dr}\left[r\frac{df}{dr}\right]-(n^2/r^2)f\right]+2bf_0^2f=0 . \tag{5.14}$$

The healing length ξ is the distance beyond which f is essentially constant:

$$\xi=n\hbar/(4mbf_0^2)^{1/2} . \tag{5.15}$$

Equation (5.12) can be solved numerically. For our purposes, it suffices to determine the limiting behavior of $f(r)$ as $r\to0$ and as $r\to\infty$. This is found to be

$$\lim_{r\to0} f(r)\to\alpha(r/\xi)^n, \quad \lim_{r\to\infty} f(r)\to\beta , \tag{5.16}$$

where α,β are constants. Since $f^2=|\eta|^2$, Eq. (5.1) gives $\beta=\rho_S^{1/2}(1-\rho_S)^{1/2}$, where ρ_S is the equilibrium density of the superfluid. Substituting Eq. (5.16) in Eq. (5.10), we obtain for $n=1$.

$$\zeta(r)\simeq(2\hbar\alpha^2/m\xi^2)(1-4\alpha^2r^2/\xi^2)^{-1/2}, \quad r\to0$$
$$\simeq0, \quad r\to\infty . \tag{5.17}$$

Thus $\zeta(0)$ is finite and there is no singularity in the vorticity, in contrast to the behavior predicted by Eq. (4.19).

The circulation Γ calculated from Eq. (5.8) is

$$\Gamma=\oint_c \mathbf{v}_S\cdot dl=(nh/2m)\{1+[1-4f^2(R_0)]^{1/2}\} \tag{5.18}$$

for a circuit c of radius R_0 showing that Γ depends on R_0, so that there is no quantization (in the usual sense) *within* the vortex. However, for $r\gg\xi$, $f(r)\to\beta$. Γ then becomes independent of R_0 and is quantized according to

$$\Gamma=nh/m_S^* \quad (R_0\gg\xi) , \tag{5.19}$$

where

$$m_S^*=2m/[1+(1-4\beta^2)^{1/2}]=m/(1-\rho_S) .$$

It is interesting to note that in the quantization of the circulation, the effective mass m^* ($>m$) occurs, rather than the bare mass m, essentially due to the inclusion of the hard core. The energy per unit length of a vortex of radius R_0 is given by

$$E(R_0)=\pi\int_0^{R_0}\rho v_S^2 r\,dr$$
$$=(\pi\hbar^2/2m^2)$$
$$\times\int_0^{R_0}\frac{dr}{r}f^2(r)\{1+[1-4f^2(r)]^{1/2}\} . \tag{5.20}$$

In a typical calculation, R_0 can be taken to be slightly larger than ξ. In view of the behavior of $f(r)$ given in Eq. (5.16) we get

$$E(R_0)=G_1(\xi)+G_2(\beta)\ln(R_0/\xi) , \tag{5.21}$$

where $G_1(\xi)$ and $G_2(\beta)$ are finite. In a customary calculation to obtain a finite vortex energy, it was found necessary to introduce a nonzero lower limit[4] in the integral given in Eq. (5.20) or a specific model[8] due to the presence of a singularity at the origin. In contrast to this, in the present formalism the finite energy arises in a natural fashion.

VI. UNIDIRECTIONAL FLOWS

In this section we obtain solutions for the superfluid density, corresponding to unidirectional flows in the superfluid. Further, kink solutions are determined in the static case.

Substituting the SCR expression $\eta=(\frac{1}{2})\sin\theta\exp(i\Phi)$ in the continuum equation (4.3) and equating the real and the imaginary parts we obtain

$$\partial_t\theta=(-\hbar/2m)[2\cos\theta(\nabla\theta\cdot\nabla\Phi)+\sin\theta\nabla^2\Phi] , \tag{6.1}$$

$$\hbar\partial_t\Phi=(\mu-b)+b\cos\theta+(\sin\theta)^{-1}[(\hbar^2/2m)\cos^2\theta+(v_0a^2/2)\sin^2\theta]\nabla^2\theta$$
$$-\cos\theta(\nabla\theta)^2(a^2b/2D)-(\hbar^2/2m)\cos\theta(\nabla\Phi)^2 . \tag{6.2}$$

Setting $\cos\theta=p$ and $\Phi=q$ and specializing to flows along the x direction of the bulk liquid, we get

$$\partial_t p=(\hbar/2m)[(1-p^2)\partial_{xx}q-2p(\partial_x p)(\partial_x q)] , \tag{6.3}$$

$$\hbar\partial_t q=(\mu-b)+bp-\tfrac{1}{2}a^2[p(1-p^2)^{-2}(\partial_x p)^2+(1-p^2)^{-1}\partial_{xx}p](bp^2+v_0)$$
$$-(ba^2/2)p(1-p^2)^{-1}(\partial_x p)^2-(\hbar^2/2m)p(\partial_x q)^2 . \tag{6.4}$$

In order to solve the above two coupled nonlinear equations we consider traveling wave solutions of the form

$$q=q(z),\quad p=p(z),\quad z=(x-vt)/a . \tag{6.5}$$

Substituting Eq. (6.5) in Eqs. (6.3) and (6.4) we get

$$U\frac{dp}{dz}=-(1-p^2)\frac{d^2q}{dz^2}+2p\left[\frac{dp}{dz}\right]\left[\frac{dq}{dz}\right]=-\frac{d}{dz}\left[(1-p^2)\frac{dq}{dz}\right] \tag{6.6}$$

and

$$U\frac{dq}{dz}=2(\kappa-R)-2\kappa p+p(1-p^2)^{-2}\left[\frac{dp}{dz}\right]^2+(1-p^2)^{-1}[\kappa p^2+(1-\kappa)]\frac{d^2p}{dz^2}+p\left[\frac{dq}{dz}\right]^2 , \tag{6.7}$$

where

$$U=2vma/\hbar,\quad \kappa=ma^2b/\hbar^2,\quad R=ma^2\mu/\hbar^2 . \tag{6.8}$$

Integrating Eq. (6.6) yields

$$\frac{dq}{dz}=U(p_0-p)(1-p^2)^{-1} , \tag{6.9}$$

p_0 being the constant of integration. Substituting Eq. (6.9) into Eq. (6.7) gives

$$U^2(1-p^2)^{-1}[(p_0-p)-(1-p^2)^{-1}p(p_0-p)^2]+2(R-\kappa)+2\kappa p+\kappa\frac{d^2p}{dz^2}=p(1-p^2)^{-2}\left[\frac{dp}{dz}\right]^2+(1-p^2)^{-1}\frac{d^2p}{dz^2} . \tag{6.10}$$

Substituting the identity

$$\left[p(1-p^2)^{-2}\left[\frac{dp}{dz}\right]^2+(1-p^2)^{-1}\frac{d^2p}{dz^2}\right]=\frac{1}{2}\left[\frac{dp}{dz}\right]^{-1}\frac{d}{dz}\left[(1-p^2)^{-1}\left[\frac{dp}{dz}\right]^2\right] \tag{6.11}$$

on the right-hand side of Eq. (6.10) leads to

$$\left[\frac{dp}{dz}\right]^2=\{-2\kappa p^4+4(\kappa-R)p^3+(c+2\kappa)p^2+[2p_0U^2-4(\kappa-R)]p-[c+U^2(1+p_0^2)]\}(1-\kappa+\kappa p^2)^{-1} , \tag{6.12}$$

where c is a constant of integration.

A. Periodic solutions

Writing the numerator of Eq. (6.12) as $(1-\kappa+\kappa p^2)(A_1p^2+A_2p+A_3)$, we find

$$A_1=-2,\quad A_2=4(\kappa-R)/\kappa,\quad A_3=(c+2)/\kappa . \tag{6.13}$$

Also

$$c=2(\kappa-1)-\kappa U^2(1+p_0^2) , \tag{6.14}$$

$$p_0=2(\kappa-R)/\kappa U^2 . \tag{6.15}$$

Consequently Eq. (6.12) is reduced to

$$\int dp/\{\lambda^2-[p-(1-R/\kappa)^2]\}^{1/2}=\sqrt{2}(z-z_0) , \tag{6.16}$$

where

$$\lambda^2=[(c+2)/2\kappa]+[(\kappa-R)/\kappa]^2 \tag{6.17}$$

and z_0 is a constant of integration. Thus

$$p(z)=[1-(R/\kappa)]+\lambda\sin\sqrt{2}(z-z_0) . \tag{6.18}$$

Using $p=\cos\theta$ and Eqs. (3.9) and (3.11) we have

$$\rho(z)=\tfrac{1}{2}[(R/\kappa)-\lambda\sin\sqrt{2}(z-z_0)] . \tag{6.19}$$

Hence

$$\tfrac{1}{2}[(R/\kappa)-\lambda]\le\rho(z)\le\tfrac{1}{2}[(R/\kappa)+\lambda],\quad \lambda>0 . \tag{6.20}$$

Since $\lambda^2>0$, Eq. (6.17) leads to

$$2<U^2<2[1-(R/\kappa)]^2,\quad (R/\kappa)<0 ,$$
$$2[1-(R/\kappa)]^2<U^2<2,\quad 2>(R/\kappa)>0 . \tag{6.21}$$

Note that the amplitude λ of the traveling wave depends on the velocity, a feature typical of certain nonlinear systems. In this context, it may be noted that a *KdV* equation for the superfluid density fluctuation was obtained by Huberman[10] as an approximate reduction of Eq. (4.21) of Rutledge *et al.*[11] Its hump-shaped soliton solution also displays this feature.[26] Finally, Eq. (6.21) shows that the velocity of the wave cannot be arbitrary.

B. Static solutions

The behavior of superfluid ^{4}He near the λ point in the static case has been discussed by Ginzburg and Pitaevskii,[7] starting from the phenomenological theory of Landau. Restricting their discussion to real solutions ψ_0 of the order parameter, they obtain

$$\frac{d^2\psi_0}{dz^2}=(\psi_0^2-1)\psi_0 , \tag{6.22}$$

where z is a suitably defined dimensionless quantity. Appropriate boundary conditions lead to the following kink solution for the order parameter:

$$\psi_0=\tanh(z/\sqrt{2}) . \tag{6.23}$$

In Eq. (6.12), for $\kappa \ll 1$, expanding the factor $[1+\kappa(p^2-1)]^{-1}$ up to terms linear in κ, we obtain

$$\left[\frac{dp}{dz}\right]^2=-\kappa(c+2)p^4+[4(\kappa-R)-2p_0U^2\kappa]p^3+[(2\kappa+c)+\kappa(2c+U^2+p_0^2U^2)]p^2 +[2p_0U^2-4(\kappa-R)+2p_0U^2\kappa]p+(c+U^2+p_0^2U^2)(1+\kappa) . \tag{6.24}$$

For a static solution, $U=0$. [Equation (6.9) shows that $q=\Phi=$const.] Then Eq. (6.24) can be written in the form

$$\left[\frac{dp}{dz}\right]^2=-2\kappa(\kappa+1)(p^2-1)^2 , \tag{6.25}$$

which can be satisfied for $\kappa<0$ only. The solution is

$$p(z)=\tanh[2|\kappa(\kappa+1)|]^{1/2}z . \tag{6.26}$$

Since $p=\cos\theta$, combining Eqs. (3.9) and (3.11) yields

$$\rho(z)=\tfrac{1}{2}[1-p(z)] =\tfrac{1}{2}\{1-\tanh[2|\kappa(\kappa+1)|]^{1/2}z\} \tag{6.27}$$

with the order parameter η given by the real solution

$$\eta=\rho^{1/2}(1-\rho)^{1/2}=\tfrac{1}{2}\operatorname{sech}[2|\kappa(\kappa+1)|]^{1/2}z . \tag{6.28}$$

For $T\sim T_\lambda$, $\rho\ll 1$, the order parameter $\eta\sim\rho^{1/2}$:

$$\eta\simeq\frac{1}{\sqrt{2}}\{1-\tanh[2|\kappa(\kappa+1)|]^{1/2}z\}^{1/2} . \tag{6.29}$$

This kink solution arises for the case $\kappa<0$ only.

VII. DISCUSSION

The pseudospin model which was introduced by Matsubara and Matsuda[12] as a microscopic model for liquid ^{4}He incorporates both an infinite hard-core and an attractive nearest-neighbor interaction between atoms in a satisfactory manner. In this paper, the model has provided us with a convenient starting point to discuss hydrodynamics in the superfluid. The effective Hamiltonian is an anisotropic ferromagnetic exchange Hamiltonian with a magnetic field along the z direction. In contrast to the magnetism problem, however, this field is not a given external field, but depends on the interaction parameters and μ, which is adjusted to keep the total z component of magnetization (or effectively, the superfluid density ρ) fixed.

In a formalism involving boson operators ψ, one usually defines the order parameter as $\langle\psi\rangle=\rho^{1/2}\exp(i\Phi)$ leading to $\langle\psi^\dagger\rangle\langle\psi\rangle=\rho=\langle\psi^\dagger\psi\rangle$. Thus a Hartree approximation is implicit in this definition, except when boson-coherent states are used to calculate expectation values. In contrast to this, we have seen that in the pseudospin formalism, working with the spin-coherent representation, the corresponding density-order parameter relationship is $\eta=\rho^{1/2}(1-\rho)^{1/2}\exp(i\Phi)$ [see Eq. (5.1)], and in the limit $\rho\ll 1$ $(T\sim T_\lambda)$, the usual definition holds.[29]

The order-parameter evolution equation valid for temperatures $T\le T_\lambda$ [Eq. (4.3)] is highly nonlinear. The phenomenological GP equation for bulk helium, and the equation of Rutledge *et al.* used to explain surface effects in superfluid films, emerge as special cases of Eq. (4.3), when certain nonlinear terms are neglected. An exact analysis of this equation yields a continuity equation for the superfluid density ρ_S. The expression obtained for the superfluid velocity v_S contains an effective mass $m^*=m(1-\rho)^{-1}$ instead of the bare mass [see Eq. (5.5)], leading to the physically relevant result[30] curl $\mathbf{v}_S\neq 0$. This is essentially due to the inclusion of a realistic hard-core interaction between the bosons in the model.

A cylindrically symmetric vortex solution of Eq. (4.3) is analyzed. A comparison of the conventional expression for the vorticity [Eq. (4.19)] with that obtained by us [Eq. (5.17)] shows the resolution of the longstanding problem of obtaining a nonsingular vorticity. Further, the customary need to introduce specific models to avoid obtaining an infinite vortex energy is also eliminated. The quantization of circulation arises only for streamlines far away from the vortex core, as expected physically [see Eq. (5.19)]. The result $\Gamma=nh/m_S^*=nh(1-\rho_S)/m$ implies that Γ depends on temperature, due to the presence of ρ_S. The range $0\le\rho\le\frac{1}{2}$ [see text following Eq. (3.12)] is a general bound which arises due to spin-operator identities, with no reference either to the symmetry of the effective spin Hamiltonian for the problem or the temperature. The ground state will depend on the relative magnitudes of the parameters $\hbar^2/ma^2$ and v_0. Only a detailed statistical mechanical calculation can yield the temperature dependence $\rho_S(T)$ for $0<T<T_\lambda$. However, it is possible to obtain rough quantitative estimates for bulk ^{4}He using phenomenological results: ρ_S has been estimated[31] to attain an extrapolated value of 0.12, within 2% accuracy, at $T=0$. Thus at very low temperatures, Γ has a predicted value [see Eq. (5.19)] which is

$(12\pm 2)\%$ less than the usual value h/m valid at $T \sim T_\lambda$. While this may appear to be a large decrease, it must be noted that in recent direct measurements of circulation such as those carried out by Yarmchuk *et al.*[6] (at ~ 0.1 K when ρ_S is expected[31] to have reached almost its maximum value) Γ has been found to be h/m to within 5% accuracy only. Furthermore, as is well known, early classic experiments typically showed a wide spread in Γ with a pronounced maximum near h/m. Hence more sensitive experiments are required to gauge the role of m_S^*, the effective mass that occurs in the expression for Γ. For $T \sim T_\lambda$, an expression for the core radius is given in Eq. (5.15).

The parameter b [Eq. (2.7)] which represents the difference between the zero-point kinetic energy and the magnitude of the attractive part of the interaction between ^{4}He atoms plays an important role. This is to be expected in any model that is formulated by starting with interacting hard-core bosons. For both signs of b, the superfluid density supports a periodic wave solution with an amplitude-dependent velocity, which feature is essentially due to the inclusion of nonlinear effects. For $b<0$, a static domain wall solution is found, although the physical relevance of this solution is not very clear at present. It would be of interest to explore the possibility of a gauge theoretic description[32] of superfluid ^{4}He in the present formalism to gain a better understanding of the topological features of the problem.

[1]F. London, *Superfluids* (Wiley, New York, 1954); L. Tisza, Nature (London) **141**, 913 (1938).

[2]L. D. Landau, J. Phys. (Moscow) **5**, 71 (1941).

[3]N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).

[4]R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, MA, 1972).

[5]E. P. Gross, J. Math. Phys. **4**, 195 (1962).

[6]L. Onsager, Nuovo Cimento Suppl. 2, **6** 249 (1949); see E. J. Yarmchuk, M. J. V. Gordon, and R. E. Packard, Phys. Rev. Lett. **43**, 214 (1979); and references therein for experimental results.

[7]V. L. Ginsburg and L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **34**, 1240 (1958) [Sov. Phys.—JETP **7**, 858 (1958)]; L. P. Pitaevskii, *ibid.* **40**, 646 (1961) [*ibid.* **13**, 451 (1961)].

[8]A. L. Fetter, in *The Physics of Liquid and Solid Helium,* edited by K. H. Bennemann and J. B. Ketterson (Wiley, New York, 1976), Pt. I, p. 207; Phys. Rev. Lett. **27**, 986 (1971); Ann. Phys. (N.Y.) **70**, 67 (1972).

[9]K. R. Atkins and I. Rudnick, Prog. Low Temp. Phys. **6**, 37 (1970).

[10]B. A. Huberman, Phys. Rev. Lett. **41**, 1389 (1978).

[11]J. E. Rutledge, W. L. McMillan, J. M. Mochel, and T. E. Washburn, Phys. Rev. B **18**, 2155 (1978).

[12]T. Matsubara and H. Matsuda, Prog. Theor. Phys. **16**, 569 (1956).

[13]R. T. Whitlock and P. R. Zilsel, Phys. Rev. **131**, 2409 (1963).

[14]K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1128 (1957).

[15]R. H. Parmenter and D. A. Poling, Collect. Phenom. **2**, 185 (1977).

[16]J. M. Radcliffe, J. Phys. A **4**, 313 (1971).

[17]Radha Balakrishnan, R. Sridhar, and R. Vasudevan, Phys. Lett. **A125**, 469 (1987).

[18]See, for instance, H. E. Stanley, *Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971).

[19]A. J. F. Siegert, Phys. Rev. **116**, 1057 (1959).

[20]P. R. Zilsel, Phys. Rev. Lett. **15**, 476 (1965).

[21]Note that a factor of $\frac{1}{2}$ must be included in the second and third terms on the right-hand side of Eq. (5) of Ref. 17, to agree with Eq. (2.8) of this paper.

[22]O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956); C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).

[23]F. W. Cummings and J. R. Johnston, Phys. Rev. **151**, 105 (1966); A. Coniglio and R. Vasudevan, Nuovo Cimento **70B**, 39 (1970).

[24]J. S. Langer, Phys. Rev. **167**, 183 (1968).

[25]G. V. Chester, in *The Helium Liquids,* edited by J. G. M. Armitage and I. E. Farquhar (Academic, New York, 1975), p. 39.

[26]See, for instance, A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE **61**, 1449 (1973).

[27]Radha Balakrishnan and A. R. Bishop, Phys. Rev. Lett. **55**, 537 (1985); Phys. Rev. B (to be published).

[28]A. C. Biswas and C. S. Warke, Phys. Rev. B **22**, 2581 (1980); Phys. Lett. **A84**, 195 (1981).

[29]Ann Eggington, in *The Helium Liquids,* edited by J. G. M. Armitage and I. E. Farquhar (Academic, New York, 1975), p. 199.

[30]I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, Reading, MA, 1965); see also, Ref. 25, p. 30.

[31]A. C. Olinto, Phys. Rev. B **35**, 4771 (1987).

[32]K. Kawasaki and H. R. Brand, Ann. Phys. (N.Y.) **160**, 420 (1985).