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Wavelength dependence of static intensity correlation functions

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We develop a real-space theory for the wavelength dependence of the intensity correlation function $C(\Delta\lambda)$ in the framework of the diffusion approximation. We calculate $C(\Delta\lambda)$ for various practical geometries and boundary conditions and find that they play a crucial role in determining the functional form.

The propagation of optical waves in random media has recently aroused much interest.[1-7] The multiple scattering of the wave causes new weak localization effects, which result in a narrow coherent backscattered peak, that were observed[3-7] for disordered media. For random solids, in addition to the coherent backscattered peak one gets[6,7] intensity fluctuations. These fluctuations are closely related to the universal conductance fluctuations.[8,9] The time dependence of the backscattered intensity fluctuations were studied experimentally[10,11] and related theoretically[10,11] to the dynamic intensity-intensity correlation function $C(\Delta t)$ by the concept of light trajectories caused by multiple scattering.

These effects were also studied[12] by diagrammatic techniques for various geometries and recently shown[13] to coincide with the real-space method.[10,11] The sensitivity of the transmitted speckle to the wavelength of the wave was demonstrated experimentally.[14] The static autocorrelation function $C(\Delta\lambda)$ for a point source was studied diagrammatically by Shapiro,[15] and Stephen and Cwilich[16] have shown that Shapiro's result is correct only for two points which are apart less than the transport mean free path l. For large distances long-range correlations exist.[16,17]

In this Rapid Communication we develop a real-space theory for calculating $C(\Delta\lambda)$ for various geometries. We show how $C(\Delta\lambda)$ depends on the phase acquired by the real-space photon trajectory and its sensitivity to the source wavelength change $\Delta\lambda$. Larger photon trajectories are more sensitive to $\Delta\lambda$, and therefore are more strongly affected by the surrounding boundaries which act as a cutoff for long photon trajectories. We derive analytical expressions for $C(\Delta\lambda)$ for various practical geometries which may be studied in future experiments.

The motion of an optical wave undergoing multiple elastic scattering performs a random walk in a random media where each step is l, the transport mean free path. This picture has led to a very successful model[18] which accounts[19,20] for the coherent backscattered peak and which was found to be in agreement with rigorous diagrammatic approaches.[16,21,22] The concept of photon trajectories was also used successfully to calculate $C(\Delta t)$, the dynamic correlation functions.[10,11,13,23] Here we use the random-walk theory[13,24] for each geometry for the probability W_N for performing N random steps. The phase ϕ_N acquired by the wave-traveled N steps depends explicitly on λ and is given by[10]

$$\phi_N = (2\pi/\lambda) L_N , \tag{1}$$

where L_N is the length of the trajectory of N steps. On the average, $\langle L_N \rangle = Nl$. The intensity-intensity correlation function at a given point in the diffusion approximation is factorized to yield

$$C(\Delta\lambda) = \left| \sum_{N,N'} P_N P_{N'} \langle e^{i[\phi_N(\lambda) - \phi_{N'}(\lambda+\Delta\lambda)]} \rangle \right|^2 , \tag{2}$$

where $\langle\ \rangle$ denotes an ensemble average and P_N is the amplitude probability[7] ($P_N^2 = W_N$). The sensitivity to $\Delta\lambda$ enters through the optical phases in (2). The random-walk amplitudes P_N are not sensitive to small changes of $\Delta\lambda$. The change in phase of a photon trajectory of N steps is given by

$$\phi_N(\lambda) - \phi_N(\lambda+\Delta\lambda) = (2\pi/\lambda)(\Delta\lambda/\lambda) L_N \tag{3a}$$

and is proportional to the actual trajectory length. Thus, longer trajectories are more sensitive to the wavelength changes. The correlation function $C(\Delta\lambda)$ will depend strongly on the particular geometry through P_N in (2). Smaller values for W_N for large values of N will correspond to a *broader* fall-off of $C(\Delta\lambda)$.

We perform the ensemble average in (2) by using the equality

$$\langle \exp[i(2\pi/\lambda)(\Delta\lambda/\lambda) L_N] \rangle = \exp[i(2\pi/\lambda)(\Delta\lambda/\lambda) Nl] \tag{3b}$$

and neglecting higher moments of L_N which leads to another factor $\exp[-\frac{1}{2}(2\pi/\lambda)^2 \langle (\delta L_N)^2 \rangle]$ in (3b), where δL_N is a fluctuation in the length of a trajectory of N steps. We find that this approximation is almost always justified.[25] Using (3a) and (3b) in (2), we get a *normalized* $C(\Delta\lambda)$ [namely, we divide all our $C(\Delta\lambda)$ by $C(0)$]:

$$C(\Delta\lambda) = \left| \sum_N W_N \exp[i(2\pi/\lambda)(\Delta\lambda/\lambda) lN] \right|^2 , \tag{4}$$

where W_N is the probability of performing N random steps. Equation (4) is our key result and shows how $C(\Delta\lambda)$ depends explicitly on $\Delta\lambda$.

We now derive $C(\Delta\lambda)$ for the following geometries: (i) a point source in an infinite medium (no boundaries); (ii) an injected point source outside a "half-infinite" space geometry (one boundary) with point detection inside the medium; (iii) an infinitely wide light source injected from outside a "half-infinite" space geometry (one boundary) and detection on the boundary; (iv) the same as (iii) but from a slab width S (two boundaries); and (v) transmitted light through a slab.

These geometries were recently used by Edrei and Kaveh[13] to calculate the dynamic correlation function $C(\Delta t)$. Some of these geometries were also used for measuring[10,11,23] $C(\Delta t)$. The explicit experimental form of $C(\Delta\lambda)$ for these geometries was not yet determined. Nevertheless, the half-width of $C(\Delta\lambda)$ for transmission through a slab was measured[14] and our results are in agreement with these data. The above geometries differ by their different forms for W_N which result from solving the diffusion equation for the particular boundary conditions. For a point source in an infinite random medium where the light is collected at a point R from the source, we use for W_N the standard well-known random-walk solution with discrete time[10] $t=Nl/c$, where c is the velocity of light in the medium:

$$W_N=C_1N^{-3/2}\exp[-\tfrac{3}{4}(R/l)^2/N]\,, \tag{5}$$

where $C_1=(4\pi l^2/3)^{-3/2}$.

Inserting (5) in (4) yields, for $C(\Delta\lambda)$,

$$C(\Delta\lambda)=\exp[-(R/l)\sqrt{12\pi l\Delta\lambda/\lambda^2}] \tag{6}$$

in precise agreement with the result first obtained by Shapiro[15] by diagrammatic methods. Thus, the real-space method which leads to (4) is capable of accounting for the correlations between different trajectories to the same accuracy of the Shapiro approach which is believed[16,17] to yield correct results for correlations at a given point from the source. For correlations at *different* points one needs the Stephen-Cwilich (Ref. 16) type diagrams.[17] The stretched exponential in (6) comes about because there is no characteristic length scale in the problem. The averaged diffusive trajectory of the optical wave between the source and a point at distance R is $\langle L\rangle=\sum W_N Nl$ and *diverges.* This leads to the extra sensitivity of $C(\Delta\lambda)$ to small $\Delta\lambda$ which corresponds to large photon trajectories. Indeed, the derivative $dC(\Delta\lambda)/d(\Delta\lambda)\to\infty$ as $\Delta\lambda\to 0$.

We now calculate $C(\Delta\lambda)$ for the geometry of an injected point source on the surface of a "half-infinite" medium and collected at a point with distance $|R|=(z^2+\rho^2)^{1/2}$ from it, where $\rho=(x^2+y^2)^{1/2}$. In this case, W_N can be obtained by using the image method[7,10,18,23,24] in which W_N is given by subtracting from Eq. (5) an image term, and is given by

$$W_N=C_1N^{-3/2}\{\exp[-\tfrac{3}{4}(\bar R-\bar d)^2/l^2N]-\exp[-\tfrac{3}{4}(\bar R+\bar d)^2/l^2N]\}\,, \tag{7}$$

where $\bar d=d\hat{\mathbf z}$ ($\hat{\mathbf z}$ is the direction perpendicular to the boundary) and $d=1.7l$. Inserting W_N as given by (7) in (4), we obtain

$$C(\Delta\lambda)=[AB/(B-A)]^2\{A^{-2}\exp[-(\rho/l)XA]+B^{-2}\exp[-(\rho/l)XB]-2(AB)^{-1}\exp[-(\rho/l)X(A+B)/2]\cos(\rho/l)X(A-B)/2\}\,, \tag{8}$$

where $A=[1+(z+d)^2/\rho^2]^{1/2}$, $B=[1+(z-d)^2/\rho^2]^{1/2}$, and $X=(12\pi l\Delta\lambda/\lambda^2)^{1/2}$. The effect of the boundary is to reduce the probability for large trajectories. Here the asymptotic form of W_N is $N^{-5/2}$ instead of [as follows from (3)] $N^{-3/2}$. The reduction in W_N for large loops causes a *broadening* of $C(\Delta\lambda)$. In Fig. 1, we plot $C(\Delta\lambda)$ as given by (8) [curve (b)] for $z=d$ and $\rho=20l$, and compare the result with a point source in an infinite medium (no boundaries) for $R=20l$ [curve (a)]. We see that the effect of the boundary is to cut the large loops and therefore to broaden $C(\Delta\lambda)$.

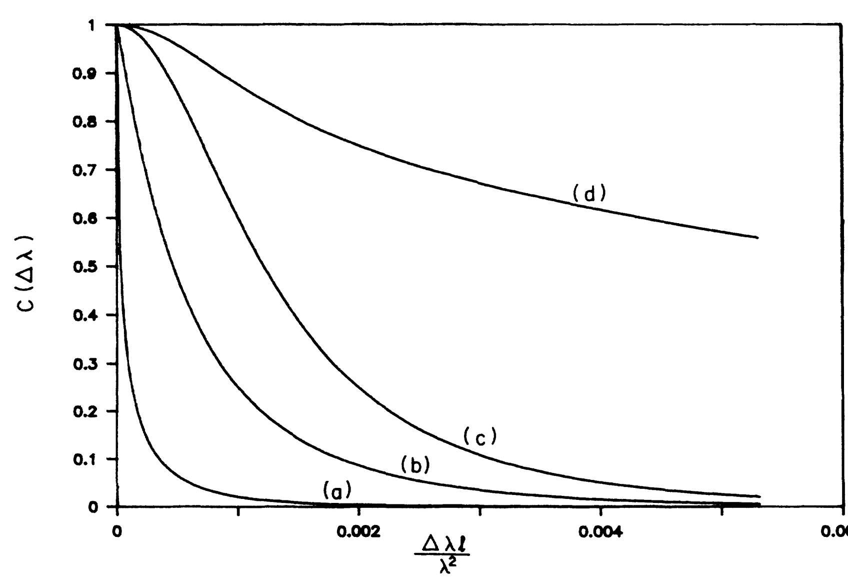

FIG. 1. $C(\Delta\lambda)$ as a function of $\Delta\lambda l/\lambda^2$ for (a) point source in infinite medium for $R/l=20$; (b) point source with one boundary for $\rho/l=20$; (c) transmitted intensity from a slab with $S/l=20$; and (d) backscattered intensity from a slab with $S/l=20$.

For the case where $\rho \gg z$ and for $(\rho/l)x \ll 1$, expanding Eq. (8) and reexponenting it yields a simplified expression for $C(\Delta\lambda)$:

$$C(\Delta\lambda \to 0) \simeq \exp[-1/6(\rho/l)^3(12\pi\Delta\lambda l/\lambda^2)^{3/2}] . \quad (9)$$

This expression is a good approximation for $C(\Delta\lambda)$ for the backscattered intensity-intensity correlation function (where the light is collected at $z = d$).

We now turn to a plane source. In this case, ρ disappears from W_N and therefore $C(\Delta\lambda)$. For a "half-infinite" space (one boundary) W_N is now obtained by integrating Eq. (7) over ρ, which yields

$$W_N = (4\pi l^2/3)^{-1/2} N^{-1/2} \{\exp[-\tfrac{3}{4}(z-d)^2/l^2 N] - \exp[-\tfrac{3}{4}(z+d)^2/l^2 N]\} . \quad (10)$$

For the correlation function $C(\Delta\lambda)$ for a given point z inside the medium, we find

$$C(\Delta\lambda) = \tfrac{1}{2}(l/dX)^2 \{\exp[-(z-d)X/l] + \exp[-(z+d)X/l] - 2\exp(-zX/l)\cos(dX/l)\} , \quad (11)$$

and again $X = (12\pi l\Delta\lambda/\lambda^2)^{1/2}$.

The most practical case is to measure $C(\Delta\lambda)$ for the *backscattered* intensity. Here we make the usual assumption that the intensity at $z = d$ is actually the measured intensity. Inserting $z = d$ in (11) and expanding for $\Delta\lambda \to 0$, we expect to obtain the same result as (6) for a point source but with R replaced by d. The reason for this lies in the fact that for $z = d$ the asymptotic form of W_N is $N^{-3/2}$ exactly as for a point source. Thus, our result for $C(\Delta\lambda)$ for backscattered light from a plane source injected from outside an infinite "half-space" random medium is

$$C(\Delta\lambda) = \exp[-(d/l)\sqrt{12\pi l\Delta\lambda/\lambda^2}] . \quad (12)$$

This result coincides with (11) for $X < 1$, up to 2%.

We have also calculated $C(\Delta\lambda)$ for the backscattering from a slab of width S. In this case W_N is obtained by solving the diffusion equation for the geometry of a slab as done in Ref. 24 and is given by

$$W_N = S^{-1} \sum_k \sin(k\pi d/S)\sin(k\pi z/S) \times \exp[-(k\pi l/S)^2 N/3] . \quad (13)$$

For backscattering correlations we have to take $z = d$, and for transmission $z = S - d$. The correlation function $C(\Delta\lambda)$ for backscattering is given below by Eq. (14). However, for large values of S/l, we find that the result coincides with Eq. (12) (see Fig. 2). The effect of small values of S/l is to cut large trajectories. In particular, trajectories with large number of steps for which $N > (S/l)^2$ will be cut off, causing a rounding of $C(\Delta\lambda)$ for $\Delta\lambda \to 0$. This is demonstrated in Fig. 2 for $S/l = 5$, 10, 20, and $S/l = \infty$. We see a systematic broadening of $C(\Delta\lambda)$ as S/l becomes smaller, and how it rounds off for $\Delta\lambda \to 0$. The full expression for the correlation function $C(\Delta\lambda)$ for backscattering is given by

$$C(\Delta\lambda) = X_1^{-2}(1-d/S)^{-2}[F(X_1)/F(X_2)] \times [F(X_1)G(X_2) + G(X_1)F(X_2) - F(X_1+X_2) + F(X_2-X)] , \quad (14)$$

where $X_1 = Xd/l$, $X_2 = Xs/l$, $F(y) = \cosh y - \cos y$, and $G(y) = \cosh y + \cos y$. In contrast to (12), $C(\Delta\lambda)$ due to finite values of S/l as given by (14) is very flat for $\Delta\lambda \to 0$.

We now provide our final result for $C(\Delta\lambda)$ for the transmitted light, a case studied experimentally by Genack.[14] Inserting W_N as given by (10) with $z = S - d$ in (4) yields

$$C(\Delta\lambda) = (S/dX_1)^2 \times \frac{[1-\cos(X_1)\cosh(X_1)]^2 + \sin^2 X_1 \sinh^2 X_1}{\cosh X_2 - \cos X_2} . \quad (15)$$

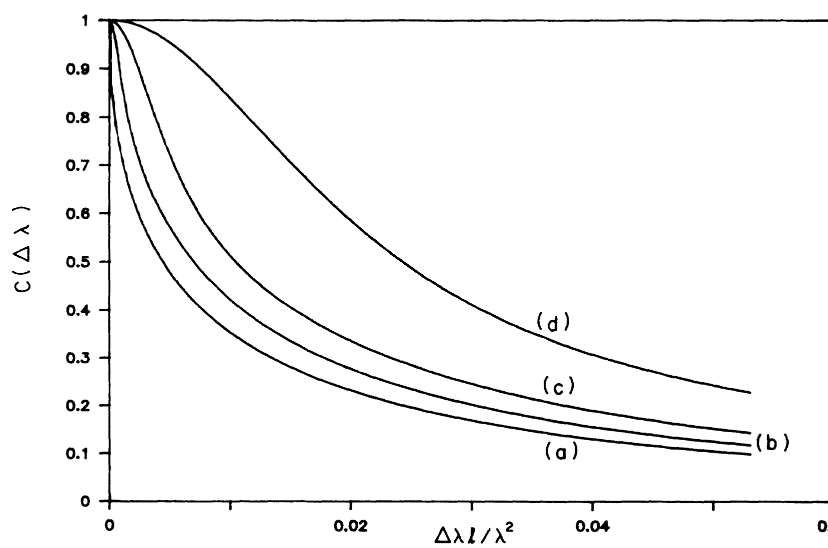

FIG. 2. $C(\Delta\lambda)$ for backscattered intensity from a plane wave from a slab for (a) $S/l = \infty$, (b) $S/l = 20$, (c) $S/l = 10$, and (d) $S/l = 5$.

In Fig. 1 [curve (c)], we plot $C(\Delta\lambda)$ for $S/l=20$. We see that it decays much faster than $C(\Delta\lambda)$, which corresponds to backscattering from a slab with $S/l=20$ [curve (d)]. This follows from the fact that for backscattering the other boundary has little effect on W_N. Thus, Fig. 1 demonstrates clearly the role of the boundaries on $C(\Delta\lambda)$ for the different geometries. From (15) we find that the half-width for $C(\Delta\lambda)$ is proportional to $(l/S)^2$ in agreement with experiment.[14] For $X_2 \gg 1$ and $X_1 \ll 1$, Eq. (15) leads to $C(\Delta\lambda) \simeq \exp[-(S/l)X]$, which coincides with the point-source result as given by (6) but with R replaced by S.

In summary, we have calculated the wavelength dependence of the intensity-intensity correlation functions $C(\Delta\lambda)$ in the diffusion approximation [which leads to Eq. (4) for various geometries]. Our main results are demonstrated by Figs. 1 and 2, where we see clearly the effect of the boundaries on $C(\Delta\lambda)$. The sharpest decay of $C(\Delta\lambda)$ is for a point source in an infinite medium. When a boundary is introduced, $C(\Delta\lambda)$ decays more slowly. The slowest decay of $C(\Delta\lambda)$ corresponds to backscattering from a plane source where only short light trajectories are effective.

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[1]S. John, Phys. Rev. Lett. **53**, 2169 (1984).

[2]P. W. Anderson, Philos. Mag. B **52**, 505 (1985).

[3]Y. Kuga and A. Ishimaru, J. Opt. Soc. Am. A **1**, 831 (1984).

[4]M. P. Van Albada and A. Lagendijk, Phys. Rev. Lett. **55**, 2692 (1986).

[5]P. E. Wolf and G. Maret, Phys. Rev. Lett. **55**, 2696 (1985).

[6]S. Etemad, R. Thompson, and M. J. Andrejco, Phys. Rev. Lett. **57**, 575 (1986).

[7]M. Kaveh, M. Rosenbluh, I. Edrei, and I. Freund, Phys. Rev. Lett. **57**, 2049 (1986).

[8]P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).

[9]B. L. Altshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)]; B. L. Altshuler and D. E. Khmelnitskii, *ibid.* **42**, 291 (1985) [*ibid.* **42**, 359 (1985)].

[10]M. Kaveh, M. Rosenbluh, and I. Freund, Nature **326**, 778 (1987); M. Rosenbluh, M. Hoshen, I. Freund, and M. Kaveh, Phys. Rev. Lett. **58**, 2754 (1987).

[11]G. Maret and P. E. Wolf, Z. Phys. B **65**, 409 (1987).

[12]M. J. Stephen, Phys. Rev. B **37**, 1 (1988).

[13]I. Edrei and M. Kaveh, J. Phys. (Paris) (to be published).

[14]A. Genack, Phys. Rev. Lett. **58**, 2043 (1987).

[15]B. Shapiro, Phys. Rev. Lett. **57**, 2168 (1986).

[16]M. J. Stephen and G. Cwilich, Phys. Rev. Lett. **59**, 285 (1987).

[17]S. Feng, C. Kane, P. A. Lee, and D. Stone (unpublished).

[18]E. Akkermans, P. E. Wolf, and R. Maynard, Phys. Rev. Lett. **56**, 1471 (1986).

[19]M. Rosenbluh, I. Edrei, M. Kaveh, and I. Freund, Phys. Rev. A **35**, 4458 (1987).

[20]E. Akkermans, P. E. Wolf, R. Maynard, and G. Maret, J. Phys. (Paris) **49**, 77 (1988).

[21]M. J. Stephen and G. Cwilich, Phys. Rev. B **34**, 7564 (1986).

[22]D. Schmeltzer and M. Kaveh, J. Phys. C **20**, L175 (1987); Phys. Rev. B **36**, 2251 (1987).

[23]M. Rosenbluh, M. Kaveh, and I. Freund, *Laser Spectroscopy VIII* (Springer-Verlag, Berlin, 1987), p. 158; I. Freund, M. Kaveh, and M. Rosenbluh, Phys. Rev. Lett. **60**, 1130 (1988).

[24]I. Edrei and M. Kaveh, Phys. Rev. B **35**, 6461 (1987).

[25]Only for the slab geometry we find important contributions from $\exp[-\frac{1}{2}(2\pi/\lambda)^2(\Delta\lambda/\lambda)\langle(\delta L_N)^2\rangle]$ in the case where $\langle(\delta b)^2\rangle \simeq l$, where $\langle(\delta b)^2\rangle$ is the averaged square of the fluctuation of the distance between the scatterers.